Inaccessibility-Inside Theorem for Point in Polygon

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Abstract. The paper explores the ideology behind the concept of inside a simple or self intersecting polygon by presenting new definitions on Inaccessibility and Inside for a point S related to a polygon P. It further goes on to give a theoretical proof to establish a relation as to when a point is inaccessible and inside a polygon. The proposed analytical solution depicts a novel way of tackling the well known point in polygon problem by employing the properties of epigraphs and hypographs, explicitly. Contrary to the ambiguous solutions given by the cross over for the simple and self intersecting polygons and the solution of a point being multi-ply inside a self intersecting polygon given by the winding number rule, the current solution gives unambiguous and singular result for both kinds of polygons. The solution also deals with the rare and subtle issues of ray crossing a odd or even number of vertices, an edge as well as analytically questions the idea of a point being multi-ply inside a polygon. Finally, the current solution proves to be mathematically correct for simple and self intersecting polygons.

Key words: Inaccessibility, Inside, Point, Polygon, Epigraph, Hypograph, Affine Combination, Affine Set.

1 Introduction

The point in polygon (PiP) problem being a widely researched topic many algorithms have been proposed hitherto. Given a polygon P or the vertices of the polygon, say \((x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)\), it is desired to know whether a sample point \(S(x_0, y_0)\) lies within \(P\). The status of a point \(S\) related to a polygon \(P\) being termed as inside needs to be defined correctly. The definition is very important in order to retrieve unambiguous results for not only simple but intersecting polygons also in a 2D Cartesian plane. In this manuscript, new definition of

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inaccessibility and inside has been proposed in order to accurately specify the meaning behind the inclusion of a point within or without a polygon.

The most commonly used algorithms use the cross over and the winding number rule definitions. Cross Over states that if a semi infinite line drawn from \( S \) cuts the \( P \) odd number of times, then the point is inside the polygon. Three issues arise in this case, i.e. • depending on the orientation of the line from the query point, odd or even values can be obtained, if the line passes through vertices. This gives rise to ambiguous results for the same point with different rays at different orientation. A prevalent solution is the shifting of the ray infinitesimally, but then again solution may change drastically depending on the direction of the shift. Even though this may be a rare case with negligible chance of occurance, the issue persists, leading to ambiguous results. • A second issue is that of repeating the cross over multiple times until the point lies inside the polygon. This leads to non determinism as it is not known how many times the rays need to be shot to get an affirmative answer, if ever it is conducted. • In case of intersecting polygons, areas exist which give a different solution than the winding number rule concept.

Winding Number Rule states that the number of times one loops around \( S \) while traversing \( P \) before reaching the starting point on the polygon shows whether the point is inside the polygon or not. So a number \( \ell \) greater than one can mean that the point is \( \ell \) times inside the polygon. This is an issue because if a point lies inside a polygon once, it lies forever. Thus \( \ell > 1 \) depicts the idea of redundancy. As will be explained later in detail, the current solution looks at these problems afflicting the status of point related to polygon from a different perspective. The manuscript defines the concepts of inaccessibility and inside of a polygon \( P \) while relating to the query point \( S \).

Most of the algorithms proposed in literature, take the polygon as a reference and compute whether the point lies inside the polygon. This is achieved by employing various techniques like decomposition of polygon into different geometrical figures, the winding number rule or the even odd parity rule. The ray crossing method ([5], [6] and [7]), sum of angles [8], swath [9], wedge [8] and the triangle based method [10], to name a few, are techniques widely employed. Brief surveys of the algorithms employed can be found in [4] and [1].

The proposed solution in this paper approaches the problem from a different perspective by taking the sample point \( S \) as the reference and then processes the polygon \( P \) to determine whether the former lies within the latter. The algorithm uses a straight line passing through the point under test, to decompose the polygon into chains.

This basic idea is motivated from [2], where the sample point is treated as a reference point with the introduction of a coordinate system with respect to \( S \). In [2] the algorithm checks the position of the consecutive vertices in the newly defined coordinate system and finally uses the line crossing method that employs a horizontal line from \( S \) to negative infinity for determining the location of the sample point.
A major difference from the work proposed in [2] is that the newly proposed algorithm in this paper uses $S$ as a reference point to draw a line chain through it, that cuts the polygon at different intersection points. These intersection points are subsequently used as markers that divide the polygon into different chains. A chain is termed valid, depending on whether $S$ is in an epigraph or a hypograph with respect to the former, respectively. For a chain marked as invalid, the vice versa holds. Once the invalid chains have been discarded, a vertical line is drawn to intersect the valid chains. If the $y_0$ is an affine combination of a pair of $y$ intersections of the valid chains, then $S$ lies within $\mathcal{P}$.

Major advantages arise from this solution. • The solution is a deterministic answer, whatever be the orientation of the line passing through $S$. • Evaluations for vertices and points lying on edges are in compliance with the new definitions. • It gives robust results for intersecting polygons where cross over and winding number rule contradict each other. • Finally, the solution is only affected by the precision of the computer under use and not by fault of the proposed definition.

Often reliability of the solution to the PiP problem forms an extremely important subject as it deals with the correctness of the solution. Numerous algorithms have been evaluated in both [12] and [13]. The former shows via experimental study of various existing algorithms that most of them give false results to many simple as well as degenerate cases. The latter gives statistics on the PiP regarding the time consumed to generate accuracy.

Again, the generation of accuracy is to be studied more carefully as it requires the presence of ground truth on the first hand. That What is the ground truth and on what basis it is termed as the ground truth? is an important question in this aspect as it is based on the definition of inside and outside. The proposed definition and the corresponding theorem reveals the ideology behind what is a ground truth in case of the point in polygon problem from a new perspective.

The current manuscript presents an analytical proof of the relation between inaccessibility and inside of a polygon related to a query point. The theorem indicates the the correctness of the solution generated via concepts of epigraphs, hypographs, affine combination and affine functions. It also suggests when a point can be considered outside or inside a polygon by binding the winding number rule with stricter conditions. Finally, the problem of cross over where the ray crosses vertices is easily solved based on the proposed solution as it does not take into account the assumption of shifting the query point infinitesimally.

It is claimed that the theoretical proof is not refined to the finest detail but does fit well to suggest that the solution provided is analytically correct and thus sufficient to provide ground truth results for the PiP. Due to the intensive nature of the computation involved in the implementation of the algorithm, the proposed solution is not the fastest algorithm. The computational time complexity of the proposed algorithm is ($= An + B + n \log(n)$) in the worst case scenario, where $n$ is the number of edges in the polygon, $A$ and $B$ are constants. This computation time involves steps like • the traversal along the edges of the polygon boundary that needs to be checked for computing the markers for the
chains, • deciding on their validity and • finally calculating whether the point lies inside the sorted pairs of consecutive valid chains based on convex combination property.

The following section (2) shed light on novel algorithm explained in simple term along with the assumptions involved. Next a theoretical analysis of the solution is given in 3. After the analysis, an algorithmic implementation of the same is presented in section 4. Section 5 helps understand the time complexity of the algorithm is then understood with the steps of the algorithm in the background. A detailed comparison with the crossover as well as the winding number rule along with the proposed algorithm is made in section 6. The statistical significance as well as measures indicating the accuracy of the proposed algorithm are tested with two commonly used concepts of crossing over and winding rule in section 7 followed by a discussion on the same in section 8. Finally, the conclusion is reached in section 9.

2 A Novel Algorithm

The novel algorithm in simple terms can be described as follows. Given the sample point \((x_0, y_0)\), a horizontal line \(y = y_0\) is drawn through \(S\) to cut the \(P\) at \(q\) locations \(\{(x_1^{\text{int}}, y_0), \ldots, (x_q^{\text{int}}, y_0)\}\), thus breaking the polygon into \(q\) chains.

**Definition 1.** A chain is a series of connected edges of the polygon whose starting and ending points lie on the horizontal straight line that passes through \(S\). Mathematically, a chain is a function \(f_c\), with a closed domain defined by the starting and ending points on the horizontal line passing through the point of test \(S\), and a range that is the graph of the currently under investigation connected edges of the polygon.

Each chain is then checked for whether its two endpoints contain the test point between them; if not, the chain is discarded. Discarded chains are termed as invalid chains and those kept for further consideration are referred to as valid chains. The remaining chains are then tested for intersection with a vertical line \(x = x_0\) through \(S\). The intersections found are sorted by height, and paired up. If the test point is not between a pair, it is outside. This criterion of containment is checked via the definitions of affine sets and affine combination as follows:

**Definition 2.** A set \(T \subseteq \mathbb{R}^n\) is an affine set, if for any two points \(x_i, x_j \in T\) \((j > i)\) and \(\theta \in [0, 1]\), \(\theta x_i + (1 - \theta)x_j \in T\).

**Definition 3.** An affine combination of \(x_i, x_j \in \mathbb{R}\) are a set of points of the form \(\theta_i x_i + \theta_j x_j\), where \(\theta_i + \theta_j = 1\).

These definitions and notations and a few others, are adopted from [14]. It is assumed that the vertices of the polygon \(P\) are arranged in order of traversal, starting from one of the vertices. The traversal order can be in any one direction.
Another assumption is that the edges are traversed only once. This is useful in avoiding multiple loops that may occur in cases of intersecting polygons.

If $S$ lies out of the bounding box of the polygon, it is considered outside $P$ and no further processing is done. Lastly, if the sample point is one of the vertices of the polygon, then it is considered to be in the polygon. This final point is assumed as the proposed solution would reach the same conclusion at the expense of computational time. In the theoretical proof, it will be shown that the assumption for implementation issue is correct.

As the algorithm is explained the concepts of epigraph and hypographs will also be used for providing an analytically complete elucidation of the generated solution. The definition of these are as follows:

**Definition 4.** The epigraph of a function (chain) $f_c: \mathbb{R}^n \to \mathbb{R}$ is a set of points that lie on or above the graph under consideration, such that $\text{epi}(f_c) = \{(x, t) : x \in \mathbb{R}^n, t \in \mathbb{R}, f_c(x) \leq t\}$ is a subset of $\mathbb{R}^{n+1}$.

**Definition 5.** The hypograph of a function (chain) $f_c: \mathbb{R}^n \to \mathbb{R}$ is a set of points that lie on or below the graph under consideration, such that $\text{hypo}(f_c) = \{(x, t) : x \in \mathbb{R}^n, t \in \mathbb{R}, f_c(x) \geq t\}$ is a subset of $\mathbb{R}^{n+1}$.

Finally, to decide if the point lies inside or is inaccessible with respect to a polygon under consideration, the definition of nearest chains would be needed. This definition is as follows:

**Definition 6.** Chains $C_i$ and $C_j$ are nearest valid chains if:

- the $\text{epi}(f_{C_i}) \subset \text{epi}(f_{C_u}) \forall u \in 1, ..., q$ chains below $S$ such that $(x_0, y_0) \in \text{epi}(f_{C_u})$.
- the $\text{hypo}(f_{C_j}) \subset \text{hypo}(f_{C_v}) \forall v \in 1, ..., q$ chains above $S$ such that $(x_0, y_0) \in \text{hypo}(f_{C_v})$.

### 3 Inaccessibility-Inside Theorem

Given the new solution, it becomes imperative to prove the correctness of the solution. This follows due to the fact that definitions like the cross over and the winding number rule exist that state the meaning of inside from different perspectives, thus giving contradictory results. New definitions of inside and inaccessibility of a point $S$ related to polygon $P$ are proposed and a relation between inaccessibility and inside is proved. This proof shows that consistent results can be obtained if the meaning of the inaccessibility and inside of a polygon related to a point is are framed correctly in an abstract sense.

It must be noted that the points that lie on vertices and edges are special cases and the definitions of inside and inaccessibility get slightly modified. But this does not mean that the meaning of inaccessibility and inside get twisted or modified from an abstract sense. Two cases are presented, one that is a point lying on a vertex and the other for the general case where it lies either on the edge or anywhere else.
3.1 Point on Vertex of Polygon

The definitions of inaccessibility and inside are proposed for the case of a point lying on a vertex. The essence of abstract meaning of the same gets carried over to points not on vertex also but the definitions are slightly modified.

**Definition 7.** The inaccessibility \( \text{Inacc}_P(S) \) of a point \( S \) related to a polygon \( P \), is the number of valid chains that need to be broken.

\[
\text{Inacc}_P(S) = \begin{cases} 
N, & N \neq 0 \text{ valid chains to be broken} \\
0, & \text{otherwise}
\end{cases}
\]

**Definition 8.** The status of a point \( S \) related to a polygon \( P \), that is \( \text{Inside}_P(S) \), is the existence of a chain \( C \) such that \( S \in \text{epi}(f_C) \) or \( S \in \text{hypo}(f_C) \).

\[
\text{Inside}_P(S) = \begin{cases} 
1, & \text{if } S \in \text{epi}(f_C) \text{ or } S \in \text{hypo}(f_C) \\
0, & \text{otherwise}
\end{cases}
\]

Based on these two definitions, two theorems need to be proved regarding the relationship of inaccessibility of a point as well as the status of a point whether it is inside with respect to the polygon.

**Theorem 1.** A point \( S \) related to polygon \( P \) is inside as well as inaccessible when: \( \text{Inside}_P(S) \in \{1\} \) iff \( \text{Inacc}_P(S) \in \{N\} \)

**Proof.** (a) If \( \text{Inacc}_P(S) \in \{N\} \) then \( \text{Inside}_P(S) \in \{1\} \)

Given that \( \text{Inacc}_P(S) = N \), there exists \( N \) valid chains that need to be broken according to definition 7. It is known that a chain is valid when either its epigraph or hypograph contains \( S \). This existence of \( N \) valid chains imply that \( S \in \{\text{epi}(f_{C_k}), \text{hypo}(f_{C_k})\} \forall k \in \{1, \ldots, N\} \). But this is the definition of status of \( S \) related to \( P \), i.e. \( \text{Inside}_P(S) = 1 \) or \( \text{Inside}_P(S) \in \{1\} \).

(b) If \( \text{Inside}_P(S) \in \{1\} \) then \( \text{Inacc}_P(S) \in \{N\} \)

Given \( \text{Inside}_P(S) = 1 \) implies that \( S \in \{\text{epi}(f_{C_k}), \text{hypo}(f_{C_k})\} \) for a chain \( C \) in \( f \). Thus chain \( C \) is a valid chain, as it contains the point \( S \). In order for \( S \) to be inaccessible, there must exist atleast 1 valid chain in \( P \) that needs to be broken. Since \( C \) is one such chain and the only chain that contains \( S \), the inaccessibility order of \( S \) related to \( P \) in \( \text{Inacc}_P(S) = 1 \) or \( \text{Inacc}_P(S) \in \{1\} \).

If \( S \) is a vertex such that it is an intersection point of two or more lines of a polygon, then all chains that have their epigraph or hypograph contain \( S \), are valid. Since it requires \( N \) (if \( N \) is the number of valid chains) chains to be broken.
Cases need to be shown pictorially to get a feel of what the theorem is suggesting about. Figure 1 shows three different polygons with $S$ as the point under consideration. The polygon in figure 1.(A) has four chains that contain $S$ namely (a) $STU$ (b) $UVS$ (c) $SWX$ and (d) $XYS$, which are valid. Thus by theorem 1, $\text{Inside}_P(S) = 1$ and $\text{Inacc}_P(S) = 4$. Thus $S$ lies inside the polygon. Similarly, for figure 1.(B) there is one chain $STUS$ which is valid as it contains the point $S$. Thus $\text{Inside}_P(S) = 1$ and $\text{Inacc}_P(S) = 4$. For the case of figure 1.(C) there exists two chains that contain $S$ i.e. (a) $STUS$ and $SVWS$ which are valid. So $\text{Inside}_P(S) = 1$ and $\text{Inacc}_P(S) = 2$.

Note that since this holds true always when $S$ lies on the vertex of a polygon, it is obvious and correct to assume that the point is in the polygon by first checking if $S$ is any one of the vertices in the polygon. This helps to avoid the implementation hurdle of checking the theorem. But again it is stressed that first the point needs to be checked against vertices of the polygon, in order to know if they belong to $P$.

**Theorem 2.** A point $S$ related to polygon $P$ is not inside as well as not inaccessible when: $\text{Inside}_P(S) \in \{0\}$ iff $\text{Inacc}_P(S) \in \{0\}$.
Proof. (a) If $\text{Inacc}_P(S) \in \{0\}$ then $\text{Inside}_P(S) \in \{0\}$

Given that $\text{Inacc}_P(S) = 0$, there exists no valid chains that need to be broken according to definition 7. This means that $S \notin \{\text{epi}(f_{C_k}), \text{hypo}(f_{C_k})\} \forall k$ valid chains in $P$. Since no chain exists whose epigraph or hypograph contains $S$, the status of $S$ related to $P$ is $\text{Inside}_P(S) = 0$ or $\text{Inside}_P(S) \in \{0\}$.

(b) If $\text{Inside}_P(S) \in \{0\}$ then $\text{Inacc}_P(S) \in \{0\}$

Given $\text{Inside}_P(S) = 0$ implies that $S \notin \{\text{epi}(f_{C_k}), \text{hypo}(f_{C_k})\} \forall k$ chains in $P$. This means no valid chains exist in $P$ that need to be broken. Thus the inaccessibility of $S$ related to $P$ is zero, i.e. $\text{Inacc}_P(S) \in \{0\}$, which is the desired result.

Cases for theorem 2 are simple and depicted in figure 2. Figure 2 shows two different polygons with $S$ as the point under consideration. The polygon in figure 2.(A) has four chains that do not contain $S$ namely (a) RTU (b) UVR (c) RWX and (d) XYR, which are invalid. Thus by theorem 2, $\text{Inside}_P(S) = 0$ and $\text{Inacc}_P(S) = 0$. Thus $S$ lies outside the polygon. Similarly, for figure 2.(B) there exists two chains that do not contain $S$ i.e. (a) RTUR and RVWR, which are invalid. So $\text{Inside}_P(S) = 0$ and $\text{Inacc}_P(S) = 0$.

3.2 Point not on Vertex of Polygon

Now for the general case of point in polygon, the definition of inaccessibility and inside evolve slightly while preserving the abstract essence of the idea. Again the theorems will be proved which find a relation between when a point is inaccessi-bile as well as inside the polygon.

Definition 9. The inaccessibility $\text{Inacc}_P(S)$ of a point $S$ related to a polygon $P$, is the number of valid chains that need to be broken and/or ignored.

$$\text{Inacc}_P(S) = \begin{cases} 1, & \text{a pair of chains need to be broken} \\ N, & N \neq 0 \text{ pairs of chains to be ignored} \\ 1 + N, & \text{a pair to be broken and} \\ N \text{ pairs to be ignored} \end{cases}$$

Definition 10. The status of a point $S$ related to a polygon $P$, that is $\text{Inside}_P(S)$, is the existence of a pair of chains $C_i$ and $C_j$ such that $S \in \text{epi}(f_{C_i})$ and $S \in \text{hypo}(f_{C_j})$.

$$\text{Inside}_P(S) = \begin{cases} 1, & \text{pairs of chains } C_i \text{ and } C_j, \\ s.t. S \in \text{epi}(f_{C_i}) \text{ and } S \in \text{hypo}(f_{C_j}) \end{cases}$$

$$0, \text{ otherwise}$$
Again the relation between inaccessibility and inside of a polygon is proved via two theorems. The theorems are as follows:

**Theorem 3.** A point $S$ related to a polygon $P$ is inside as well as inaccessible when: $\text{Inside}_P(S) \in \{1\}$ iff $\text{Inacc}_P(S) \in \{1, 1 + \mathcal{N}\}$

**Proof.** (a) If $\text{Inacc}_P(S) \in \{1, 1 + \mathcal{N}\}$ then $\text{Inside}_P(S) \in \{1\}$.

Given $\text{Inacc}_P(S) \in \{1, 1 + \mathcal{N}\}$ imples that there exist a pair of valid chain in $\mathcal{P}$ that need to be broken and/or $\mathcal{N}$ pairs of valid chains that need to be ignored. A valid chain by definition is one whoes epigraph or hypograph contains $S$. Taking the general case of $1 + \mathcal{N}$ (if $\mathcal{N} = 0$, $1 + \mathcal{N}$ collapses to 1), there are $2 \times (1 + \mathcal{N})$ valid chains such that half lie above/on $S$ and the rest half lie below/on $S$. If a vertical line passing through $x = x_0$ is drawn such that it cuts the valid chains and $S$, then the chains can be sorted according to the value of intersection points in $x = x_0$. Let $\mathcal{C}_1, \ldots, \mathcal{C}_{2 \times (1 + \mathcal{N}) - 1}, \mathcal{C}_{2 \times (1 + \mathcal{N})}$ be the sorted order of chains from bottom to top. Taking consecutive pairs of these valid chains i.e. $(\mathcal{C}_1, \mathcal{C}_2), (\mathcal{C}_3, \mathcal{C}_4), \ldots, (\mathcal{C}_i, \mathcal{C}_{i+1}), \ldots, (\mathcal{C}_{2 \times (1 + \mathcal{N}) - 1}, \mathcal{C}_{2 \times (1 + \mathcal{N})})$, it is easy to know whether $S$ is an affine combination of $(x_0, y^u_{i+1})$ and $(x_0, y^l_{i+1})$, $\forall \in \{1, 3, 5, \ldots, 2 \times (1 + \mathcal{N}) - 1\}$.
Here \( y_{C_k}^{int} \) and \( y_{C_{k+1}}^{int} \) are the intersection points on the chains \( k \) and \( k+1 \) due to the line \( x = x_0 \).

Since it is known that at least one pair of chains need to be broken, a pair of points \( (x_0, y_{C_k}^{int}) \) and \( (x_0, y_{C_{k+1}}^{int}) \) exists for which \( S(x_0, y_0) \equiv (x_0, \theta y_{C_k}^{int} + (1 - \theta)y_{C_{k+1}}^{int}) \) for \( 0 \leq \theta \leq 1 \). This implies that \( C_i \) is the nearest chain below/on \( S \) and \( C_{i+1} \) is the nearest chain above/on \( S \), otherwise \( S \) won’t be an affine combination of \( (x_0, y_{C_k}^{int}) \) and \( (x_0, y_{C_{k+1}}^{int}) \). For the rest of the \( N \) pairs, since \( S \) is not an affine combination of \( (x_0, y_{C_k}^{int}) \) and \( (x_0, y_{C_{k+1}}^{int}) \) \( \forall k \in \{1, 3, ..., 2 \times (1+N)-1\} - \{i\} \), these \( N \) pairs of chains can be ignored from further processing or consideration. Since these nearest chains \( C_i \) and \( C_{i+1} \) are also valid, their epigraph and hypograph contain \( S \), respectively. This existence of a pair of a valid chains which has to be broken such that \( S \in epi(f_{C_i}) \) and \( S \in hypo(f_{C_{i+1}}) \) implies \( \text{Inside}_P(S) = 1 \), the status of \( S \) related to \( P \).

(b) If \( \text{Inside}_P(S) \in \{1\} \) then \( \text{Inacc}_P(S) \in \{1, 1+N\} \).

Let \( P \) be a polygon such that \( \text{Inside}_P(S) = 1 \) implies there exists a pair of chains \( C_i \) and \( C_j \) such that \( S \in epi(f_{C_i}) \) and \( S \in hypo(f_{C_j}) \). Given only these two chains, it is evident that both of them are nearest chains to \( S \). Let the starting and ending points of \( C_i \) and \( C_j \) be \( \{(x_{C_i}^{int}, y_0), (x_{C_i}^{int}, y_0)\} \) and \( \{(x_{C_j}^{int}, y_0), (x_{C_j}^{int}, y_0)\} \), respectively. If a vertical line \( x = x_0 \) is drawn through \( (x_0, y_0) \) it would interest the chains \( C_i \) and \( C_j \) at \((x_0, y_{C_i}^{int})\) and \((x_0, y_{C_j}^{int})\), respectively. Since \( (x_0, y_{C_i}^{int}) \) lies below \( (x_0, y_0) \) and \( (x_0, y_{C_j}^{int}) \) lies above \( (x_0, y_0) \), \( S \) is an affine combination of \( (x_0, y_{C_i}^{int}) \) and \( (x_0, y_{C_j}^{int}) \). Thus \( S \) lies between \( C_i \) and \( C_j \). Now, if an end point of \( C_i \) is joined with an end point of \( C_j \) and another end point of the former joined to the remaining end point of the latter, then a closed loop is formed such that traversing once from any one point, lead to the same point in the end. Let this loop be \( P' \).

As long as the winding number of \( P' \) is around \( S \) is same as that of \( P \) around \( S \), \( P' \) can be deformed into \( P \), by Hopf’s degree theorem [4]. Since \( P \) is formed from the two necessary chains \( C_i \) and \( C_j \) which are valid, a pair exists in polygon \( P \) that needs to be broken. Thus the minimum inaccessibility of \( S \) related to \( P \) is \( \text{Inacc}_P(S) = 1 \). If there exists extra pairs of valid chains, then they would be ignored from consideration, while checking for the affine combination criteria of \( S \) with respect to the sorted pairs of chains on \( x = x_0 \). If \( N \) is the minimum number of pairs of valid chains that are ignored, then the inaccessibility of \( S \) related to \( P \) is \( \text{Inacc}_{P'}(S) = 1 + N \). Thus \( \text{Inacc}_P(S) = \{1, 1+N\} \).

Now, the cases of theorem 3 are presented with visual representations in figures 3 and 4. In figure 3.(A), 3 chains exist of which two are valid. The valid chains are (a) UV and (b) WU. The chain VW is invalid. Thus \( \text{Inside}_P(S) = 1 \) and \( \text{Inacc}_P(S) = 1 \). In figure 3.(B) the point lies on the edge and the polygon can be divided into 3 chains of which only two contain \( S \). These valid chains are (a) SU and (b) VS. Thus \( \text{Inside}_P(S) = 1 \) and \( \text{Inacc}_P(S) = 1 \). In part (C) of figure 3, the horizontal line cuts through two edges and touches two vertices. Thus there
exists 4 chains of which two are valid, namely (a) UV and (b) XU which contain S. Thus $\text{Inside}_P(S) = 1$ and $\text{Inacc}_P(S) = 1$. Lastly, in part (D) of the same figure, S lies on an edge and the horizontal line passes through a vertex. In this case, there exist 5 chains of which two contain the point of test and are thus valid. They are (a) VW and (b) YU. Thus $\text{Inside}_P(S) = 1$ and $\text{Inacc}_P(S) = 1$.

Next, in figure 4.(A) 8 chains exist namely, (a) YZ (b) ZW (c) WB (d) BU (e) UV (f) VA (g) AX and (h) XY, of which 6 are valid except UV and XY. These two do not contain S. Now, the pair that needs to be broken is YZ and ZW, while the pairs (WB, AX) and (BU, VA) are to be ignored from consideration. Thus, $\text{Inside}_P(S) = 1$ and $\text{Inacc}_P(S) = 1 + 2$ (i.e one pair requires to be broken and two need to be ignored from consideration). Finally, in figure 4.(B) two chains exist of which both are valid, i.e. (a) UV and (b) VWWU. Thus $\text{Inside}_P(S) = 1$ and $\text{Inacc}_P(S) = 1$.

**Theorem 4.** A point $S$ related to a polygon $P$ is not inside as well as inaccessible when: $\text{Inside}_P(S) \in \{0\}$ iff $\text{Inacc}_P(S) \in \{N\}$

**Proof.** (a) If $\text{Inacc}_P(S) \in \{N\}$ then $\text{Inside}_P(S) \in \{0\}$. 
Given $\text{Inacc}_P(S) = N$ implies pairs of valid chains need to be ignored. Again, a valid chain by definition is one whose epigraph and hypograph contain $S$. If a vertical line passing through $x = x_0$ is drawn such that it cuts the valid chains and $S$, then the chains can be sorted according to the value of intersection points in $x = x_0$. Let $C_1, ..., C_{2 \times N}$ be the sorted order of chains from bottom to top. Taking consecutive pairs of these valid chains, i.e. $(C_1, C_2), (C_3, C_4), ..., (C_i, C_j), ..., (C_{2 \times N-1}, C_{2 \times N})$, it is easy to know whether $S$ is an affine combination of $(x_0, y_{int}^{C_k})$ and $(x_0, y_{int}^{C_{k+1}}) \forall k \in \{1, 3, ..., 2 \times N - 1\}$. Since $N$ pairs need to be ignored, it is evident that $S$ is not an affine combination of any of the above pairs. This suggests that there does not exist a pair such that $S \in \text{epi}(f_{C_k})$ and $S \in \text{hypo}(f_{C_{k+1}})$, that can be broken. Since no such pair exists, the status of $S$ related to $P$ is $\text{Inside}_P(S) = 0$, which is the desired result.

(b) If $\text{Inside}_P(S) \in \{0\}$ then $\text{Inacc}_P(S) \in \{N\}$.

Given $\text{Inside}_P(S) = 0$ implies there does not exist a pair of chains $C_i$ and $C_j$ such that $S \in \text{epi}(f_{C_i})$ and $S \in \text{hypo}(f_{C_j})$. Thus it is difficult to proceed with the proof. Instead, by proving its contrapositive the above statement will hold.
Thus if $\text{Inacc}_P(S) \notin \{N\}$ then $\text{Inside}_P(S) \notin \{0\}$. Since $\text{Inacc}_P(S) \notin \{N\}$, it implies that $\text{Inacc}_P(S) \in \{1, 1+N\}$. It has been proved in part (a) above that if $\text{Inacc}_P(S) \in \{1, 1+N\}$ then $\text{Inside}_P(S) \in \{1\}$. But $\text{Inside}_P(S) \in \{1\}$ also means that $\text{Inside}_P(S) \notin \{0\}$. Thus $\text{Inacc}_P(S) \notin \{N\}$ implies that $\text{Inside}_P(S) \notin \{0\}$. Since the contrapositive holds, so does the original statement.

Finally, two cases for theorem 4 are depicted in figure 5. In figure 5.(A) 8 chains exist namely, (a) UV (b) VA (c) AX (d) XY (e) YZ (f) ZW (g) WB and (h) BU. Of these, 3 pairs exist, all of which are valid chains but need to be ignored as non of them enclose $S$ as an affine combination of two intersection points on $x = x_0$. These pairs are (WB, AX), (XY, ZW) and (VA, BU). Thus $\text{Inside}_P(S) = 0$ and $\text{Inacc}_P(S) = 3$. For the case of intersecting polygon in figure 5 (B) 6 chains exist namely, (a) ZU (b) UY (c) YW (d) WX (e) VX and (f) XZ, of which WY and XZ are not valid. Remaining pairs of valid chains need to be ignored and thus $\text{Inside}_P(S) = 0$ and $\text{Inacc}_P(S) = 2$. 
Fig. 6. Query point location in (A) closed and (B) intersecting polygon.

4 Algorithm Implementation

The implemented algorithm will be illustrated with each step explained with a pictorial representation of result of the execution of that particular step. Examples include closed, intersecting and non intersecting polygons. Figures 6 and 7 show the polygons with the sample point being tested at different locations.

4.1 Intersecting the $P$

$P$ is an ordered series of vertices $(x_1, y_1)$, $(x_2, y_2)$, ..., $(x_n, y_n)$, starting from $(x_1, y_1)$ that defines the polygon. Given $P$, the first step is to draw a straight line through the sample point $S(x_0, y_0)$, such that it intersects the polygon at certain points. For simplicity, the horizontal line $y = y_0$ is considered without loss of generality.

The intersection points are obtained via computing the values of the $x_{int}$ coordinates between the $y = y_0$ and the straight line, extending from $(x_i, y_i)$ to $(x_{i+1}, y_{i+1})$. The slope and the constant of the former is $0$ and $y_0$, and that of latter is $m_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$ and $c_i = y_i - m_i x_i$. Here $i$ and $i+1$ are consecutive points on $P$. Solving the algebraic equation between two straight line gives:
Inaccessibility-Inside Theorem for Point in Polygon

Fig. 7. Query point location in (A) closed and (B) intersecting polygon.

\[ x^{int} = \frac{y_0 - c_i}{m_i} \]  \hspace{1cm} (1)

Once an intersecting point with coordinate \((x^{int}, y_0)\) is obtained, the algorithm checks if this point lies in between \((x_i, y_i)\) and \((x_{i+1}, y_{i+1})\), on the line. This is achieved via affine combination property in definition 3. Three different cases arise depending on the slope of the line:

1. \(m_i = \pm \infty\): If the edge is a vertical line, the point \((x^{int}, y_0)\) lies on the line. This is because \(x^{int} = x_i = x_{i+1}\).

2. \(m_i = 0\): If edge is a horizontal line, the intersection point \((x^{int}, y_0)\) is considered to lie outside the range of \((x_i, y_i)\) and \((x_{i+1}, y_{i+1})\). This is because, if it lies within \((x_i, y_0)\) and \((x_{i+1}, y_0)\), then there are infinitely many points that could be considered. To save from randomly selecting any point within the given range, it is thus considered that the point \((x^{int}, y_0)\) outside the range. Also, if \((x_0, y_0)\) lie on a horizontal edge, it is still considered outside the range for further processing.

3. \(m_i \in \mathbb{R} - \{0, \pm \infty\}\): Finally, this being the simplest case, it is easy to compute whether \((x^{int}, y_0)\) lie on the line between the given points using definition 3.
Figures 8 and 9 show the horizontal line \( y = y_0 \) passing through \( S \) and intersecting the polygon at different edges. The sample point is indicated via the red arrow while the vertices at the intersection of \( y = y_0 \) with edges of the polygon are pointed to by the blue arrows. Once the vertices of the intersecting points are computed, they are added to the list of pre-existing vertices of the polygon under consideration. The newly created intersecting vertices are appended in such a manner that the traversal order is not affected.

It may happen that some of the newly added vertices with their \( y \) coordinate as \( y_0 \) are very close to other pre-existing vertices in the tolerance range of \( \pm 10^{-5} \) or smaller. The algorithm removes the vertices from the list that lie in such close range, but keeps the newly added vertices with coordinates \((x^{\text{int}}, y_0)\). Two reasons arise for executing this step:

1. To avoid further computations that may involve floating point precision of the order smaller than or equal to \( \pm 10^{-5} \).
2. To retain newly appended vertices with coordinates \((x^{\text{int}}, y_0)\) that will be later used for searching chains whose epigraph or hypograph may contain \( S \).

It must be noted that this insertion of vertices does not affect the solution to the PiP on paper, but may affect solution practically on a computer due to the floating point representation and operations on it. Also, note that the tolerance range is not the issue investigated in this work.

### 4.2 Decomposition of Polygon into Valid and Invalid Chains

The new vertices with coordinates \((x_i^{\text{int}}, y_0)\) were \( \{j \in \{1, \ldots, m\} \) and the sample point \((x_0, y_0)\) become the basis for the next steps. From definition 4 it is known that a point belongs to the epigraph (hypograph) if it lies on or above (below) the graph under consideration.

To use the mentioned properties, the polygon \( P \) needs to be decomposed into the chains. These chains would then be tested for convexity or concavity. The algorithm achieves this in the following way. A vertex on \( P \), especially with the \( y \) coordinate being \( y_0 \) is picked up as the starting vertex. A traversal order is chosen randomly and is followed until the starting point is reached again.

Vertices lying on a path between any two consecutive \((x_i^{\text{int}}, y_0)\) and \((x_j^{\text{int}}, y_0)\) points on \( P \), including the intersecting points themselves, form a chain. Thus as the traversal is done from one intersecting vertex to the other, the polygon gets decomposed into chains, that lie either above or below the horizontal line \( y = y_0 \).

Next the chains are tested for validity using the definitions in 4 and 5. In non-mathematical terms, if the starting and ending vertices of a chain are on either side of the \( S \) on the line \( y = y_0 \) and not on the same side with respect to the sample point, then the chain is a valid one. The valid chains are stored with their starting and ending vertices along with coordinates of points on the chain.

Figures 10 and 11 show the valid and invalid chains pointed by the green and blue arrows respectively. The solid and dotted lines also demarcate the chains.
Fig. 8. Finding intersecting points on the polygon. Sample point indicated via red arrow lies in (A) closed and (B) intersecting polygon. Blue arrows point to newly found intersection points. The green line depicts $y = y_0$.

The solid lines represent the valid chains and the dotted lines represent the invalid chains. The red arrow indicates the sample point’s location in each of the figures.

4.3 Chain Intersection

Hitherto, it is known that the $x$ coordinate of $S$ lies in the epigraph or the hypograph of the valid chains. To know that the sample point lies within the polygon, finally it needs to be tested whether the $y$ coordinate of $S$ lies within any two nearest valid chains. The rationale behind doing these steps will be elucidated a little later, but before that it is important to define what the nearest valid chains mean:

After the valid chains have been obtained, a similar procedure of intersecting the chains is done using the vertical line $x = x_0$. For each chain, defined by a set of vertices $(x_i, y_i)$, points of intersection are computed between the edges of the chain and the vertical line. The slope of the straight line joining two vertices on
Fig. 9. Finding intersecting points on the polygon. Sample point indicated via red arrow lies out of (A) closed and (B) intersecting polygon. Blue arrows point to the newly found intersection points. The green line depicts $y = y_0$.

the chain is $m_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$ and the constant is $c_i = y_i - m_i x_i$. Now, $i$ and $i + 1$ are consecutive points on the valid chain. The value of $y$ coordinate of the point of intersection are computed as follows:

$$y^{int} = m_i x_0 + c_i$$

(2)

Once the coordinate of the intersection point $(x_0, y^{int})$ is obtained, a test is conducted to find whether the intersection point lies in between $(x_i, y_i)$ and $(x_{i+1}, y_{i+1})$. This is achieved using the affine combination property in definition 3. Three cases may arise, that need to be considered:

1. $m_i = \pm \infty$: If edge is a vertical line, the intersecting point $(x_0, y^{int})$ is considered to lie inside the range of $(x_0, y_i)$ and $(x_0, y_{i+1})$. This is because, if the chain crosses $x = x_0$ many times before going from left of $S$ to the right of $S$ or vice versa, then there can be infinitely many points on one chain that may be considered as intersection points. That is, it may require the algorithm to store many intersection points for just one chain. To avoid the
presence of many intersection points on a single chain, the algorithm stores
the vertex of one of the vertical edges in a chain.
2. \( m_i = 0 \): If the edge is a horizontal line, the intersecting point \((x_0, y^{\text{int}})\) lies
on the line. This is because \(x^{\text{int}} = x_0\) and \(y^{\text{int}} = y_i = y_{i+1}\).
3. \( m_i \in \mathbb{R} - \{0, \pm \infty\} \): Finally, this being the simplest case, it is easy to compute
whether \((x_0, y^{\text{int}})\) lie on the line between the given points using definition 3.

The pictorial representation of the line \(x = x_0\) intersecting the valid chains are
shown in figures 12 and 13. The green arrows indicate the points of intersection
on the valid chains that are now pointed to by the blue arrows. The sample point
is indicated via the red arrow. As mentioned earlier, the invalid dotted chains
have been removed by the algorithm, in its final stage of processing.

### 4.4 Point Inclusion Test

Each of the valid chains have a point of intersection with \(x = x_0\). Next, the
algorithm sorts the chains according to the \(y^{\text{int}}\). This is done so as to arrange
Fig. 11. Polygon decomposition into chains. (A) closed and (B) intersecting polygon decomposed into chains. Green and blue arrows indicate the valid and invalid chains, respectively. Location of query point w.r.t the polygon and is shown via the red arrow.

all the intersecting points \((x_0, y^{\text{int}})\) in the different chains in an ascending order, with the sample point \((x_0, y_0)\) somewhere in between (if the case is).

Finally, considering a pair of \(y^{\text{int}}\)'s at a time, i.e a pair of chains at a time, it is tested whether \((x_0, y_0)\) is an affine combination of \((x_0, y_i^{\text{int}})\) and \((x_0, y_j^{\text{int}})\) (for \(i < j\)). If such a pair is found, then the point lies inside the polygon, else it is outside. The only care that the algorithm takes while executing this step is that, the pair of chains or the pair of intersecting \(y\) coordinates are mutually exclusive. This means that if there are \(n/2\) pairs, with \(n\) being an even number and \(y_1^{\text{int}}, y_2^{\text{int}}, \ldots, y_n^{\text{int}}\) are the intersecting \(y\) values in order, then \((y_1^{\text{int}}, y_2^{\text{int}}), (y_3^{\text{int}}, y_4^{\text{int}}), \ldots, (y_{n-1}^{\text{int}}, y_n^{\text{int}})\) are mutually exclusive in the sense that element of one pair cannot be included in any other pair.

The fundamental idea behind such a rule is that, in a pair, if the path of traversal in a chain is moving from left side of \(S\) to the right side (say), then the traversal in the other chain must move from right side \(S\) to the left. Thus, any pair shall not contain chains from any other pair.
Fig. 12. Point inclusion test. Valid chains of (A) closed and (B) intersecting polygon indicated by blue arrows. Green arrows depict the newly found intersection points with the vertical line $x = x_0$. The red arrow shows the point lies inside the polygon.

These ideas can be seen in figures 12 and 13. The green arrows mark the intersecting points on the valid chains. The test of affine combination with respect to the sample point $S$, indicates whether the point lies within the polygon or without.

5 Time Complexity

The computational complexity of the algorithm in terms of time needs to be addressed. With $n$ edges, the algorithm takes a constant time of $C_1$ to process the first step mentioned in section 4.1. $C_1$ is the constant number of steps used in computing the intersection of the horizontal line with each of the edges and deciding whether the intersection point lies on the edge, between the vertices of the edge. This process thus takes up $C_1 n$ steps. The newly found intersection points are added to the vertex list of the polygon while some vertices close to the intersecting points in the range of $\pm 10^{-5}$ or less are removed to address the
Fig. 13. Point inclusion test. Valid chains of (A) closed and (B) intersecting polygon indicated by blue and green arrows, respectively. Green arrows also depict the newly found intersection points with the vertical line $x = x_0$. The red arrow shows the point lies outside the polygon.

floating point precision. This execution step on the whole increases the number of vertices by a fraction, say $f_1$ were $0 \leq f_1 < 1$.

Thus the total number of vertices on the polygon now amount to $(1 + f_1)n$. The decomposition of the polygon into different chains requires the processing of all $(1 + f_1)n$ edges, as has been described in section 4.2. This processing is of the order of $(1 + f_1)n$ with some constant $C_2$ required for checking if the chains are valid or not.

In section 4.3, as in section 4.1 the points of intersection are computed for the vertical line and one of the edges in each of the chains. Since the number of edges belonging to the chains is a fraction $f_2$ ($0 \leq f_2 < 1$) of the total number of edges of the polygon, the execution of this step requires $C_1f_2(1 + f_1)n$ units of cpu time. Finally, after sorting of the chains which requires $m\log(m)$ where $m = f_2(1 + f_1)n$ and checking whether the sample point is an affine combination, requires a constant time of $C_3$. This procedure is mentioned in the previous section.
Summing up the total time of execution, the algorithm works in \((C_1n + (1 + f_1)n + C_2 + C_1f_2(1 + f_1)n) + C_3 + m\log(m) = (C_1 + 1 + f_1 + C_1f_2 + C_1f_2f_1)n + C_2 + C_3 + m\log(m)\). Let \(A = (C_1 + 1 + f_1 + C_1f_2 + C_1f_2f_1)\) and \(B = C_2 + C_3\), then the computational time complexity of the algorithm is \(An + B + m\log(m) \to O(m\log n)\). This time in computation is the worst case scenario, where all the edges have to be processed in for all the four main steps described above.

6 Comparison with Crossover and Winding Number Rule

It becomes necessary to understand how the proposed solution behaves with respect to the existing solutions given by the crossover as well as the winding number rule. For simple polygons, the results are same, but ambiguity arises when the intersecting polygons are under consideration. For the current section, let EH (epi/hypo-graph method) denote the proposed method.

6.1 Crossover vs EH

Crossover (Cr) states that a line drawn from a point \(S\) in a direction, if it cuts the polygon \(\mathcal{P}\) odd number of times, implies that \(S\) is inside \(\mathcal{P}\), i.e.

\[
\text{Inside}_{CR}^{\mathcal{P}}(S) = \begin{cases} 
1, \text{ odd intersections} \\
0, \text{ even intersections}
\end{cases}
\]

For the case of vertices, the problem is solved by shifting the line infinitesimally. Two issues arise when a line is shot from \(S\) and it intersects a vertex. (1) There can be two solutions, if the line is not shifted slightly. (2) If the crossover has to be repeated several times until it finds an odd number of intersections, then it is a nondeterministic problem, in case the line is shot randomly.

By (1) ambiguity arises on the way a ray or line is shot from \(S\) and by (2) nondeterminism arises due to repeatedness because of the line being shot randomly. The following figures will illustrate these issues in detail. In contrast to the Cr, by checking through theorems 1 and 3, the EH method can easily determine if \(S\) lies in \(\mathcal{P}\) or not, deterministically. This is because whichever way a line is drawn through \(S\), if it cuts the polygon, then it will dismember \(\mathcal{P}\) into a finite number of countable chains. If it doesn’t cut the polygon and \(S\) is a vertex, then also there exist at least one chain that contains \(S\). Searching for these valid chains and then locating which of those need to be broken is deterministic.

The figure 14 shows the different cases under consideration for the comparison of CR and EH method for the same point of investigation. In figure 14.(A), if the a horizontal line is drawn to the left of the \(S\), then it intersects at two points \(U\) and \(V\) and if it drawn to the right, it intersects at the point \(W\). According to CR, when the line is drawn to the right of the \(S\), then \(S\) is inside the polygon. If the line is drawn to the left of \(S\), then the point is outside the polygon. This is definitely a case of ambiguity. Also, the outcome of the CR depends on the
direction of the ray that is shot from $S$. This makes the outcome of the test nondeterministic in the sense that it is not known which ray would give the correct result, if the rays are shot randomly.

The EH method overcomes this problem by segmenting the polygon into finitely countable chains. The searching for an affine combination of valid chains that may contain $S$ is deterministic as there are only limited number of chains available for checking. Thus the outcome of the EH method is final and deterministic. If two perpendicular rays with its intersection point as $S$ are drawn at a different orientation, thus intersecting the polygon at different places, even then by rotating the oriented axis and the polygon to horizontal vertical frame, the solution remains the same. Thus randomness of the rays do not affect the outcome of the point inclusion test for $S$. For part (A) in the figure 14, by CR $Inside_{CR}^{P}(S) = (0, 1)$ depending on the number of intersections that is $(2, 1)$. By EH method, $Inside_{EH}^{P}(S) = 1$ and $Inacc_{EH}^{P}(S) = 3$ by theorem 1. It must be noted that the inaccessibility of the point related to the polygon may change but the status of $S$ related to $P$ captured by the definition of Inside will not change if the point is inside the polygon.
Similarly, for the part (B) and (C) in figure 14, by CR the $Inside_{CR}(S) = (1, 0)$ depending on the number of intersections based on the direction of the ray which is $(1, 2)$. Finally, in figure 14.(D), for point $S_1$ four valid chains exist namely, (a) VW (b) XU (c) US and (d) $S_2 V$, none of which need to be broken or ignored. Thus by theorem 3 $Inside_{EH}(S_1) = 0$ and $Inacc_{EH}(S_1) = 2$. By CR the outcome of the inclusion test changes, that is $Inside_{CR}(S_1) = (0, 1)$ depending on the intersections obtained by the direction of the ray that is $(2, 3)$. For the point $S_2$, two valid chains exist namely (a) $S_2 V$ and (b) VWUXS. Thus by theorem 1, $Inside_{EH}(S_2) = 1$ and $Inacc_{EH}(S_2) = 2$. By CR $Inside_{EH}(S_2) = 0$ as the number of intersections is 4.

6.2 Winding Number Rule vs EH

The winding number rule (WNR) states that the number of times one loops round the point $S$ before reaching the starting point on polygon $P$, decides the number of times whether $S$ lies inside $P$ or not. Thus

$$Inside_{WNR}^{P}(S) = \begin{cases} N, & \text{N loops around } S \\ 0, & \text{zero loops around } S \end{cases}$$

An analogy of a prison wall shown in figure 15 is taken into account to explain the differences. Figure 15.(A) is the initial structure of the prison and then the final structure is shown in the part (B) of the same figure. Initially, via the WNR, $S_1$ was lying outside and $S_2$ inside the prison wall. Same is the verdict by the new method. Next a portion of the prison wall is extended and the final structure looks like that in figure 15.(B). Note that $S_1$ and $S_2$ are still outside the new prison via the new definition, as the areas in which $S_1$ and $S_2$ lie, are not reachable from prison’s perspective. This is because two pairs of walls have to be ignored and not broken in each case. From this point of view both $S_1$ and $S_2$ are outside $P$, in figure 15.(B). Also, even though WNR = 2 for $S_2$ in new prison in part (B) of the same figure, implies the point lies twice inside, it does not make sense. It can be stated that if a point lies inside once, then it lies forever. There does not arise the idea of point lying inside $N$ times. Thus a point lying inside $N$ times, is the same as point lying inside once. If it does not lie inside, then it won’t lie forever. In this way the new definitions and the accompanying theorem are definitive in producing a concrete answer via means of epigraph-hypograph method.

7 Results

With the proof that the proposed algorithm gives theoretically reliable results, it would of interest to know how algorithms based on the two widely used crossing over and winding number rule concepts, fair. One of the measures of fairness is the test of significance of results obtained from the algorithms.

Artificial polygons with number of vertices in the set $\{4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048\}$ were generated. For a particular vertex number, 10000 polygons were generated.
Fig. 15. Cases for Winding Number Rule and EH method.

Fig. 16. Check of statistical significance using McNemar’s Test for polygons with increasing vertices. Red, blue and green lines indicate the significance of P vs Cr, P vs Wi and Wi vs Cr, respectively. The black line represents the standard $\chi^2$ value of 3.85 with a $p$ value of 0.05. The vertical axis is the $\log_{10}$ representation of the test values.
and stored. Each polygon was generated by randomly generating the coordinates and joining the vertices in order. The last vertex was finally joined with the first vertex. This created a series of intersecting as well as non intersecting polygons. All polygons were generated in the bounding box of \([0, 1]\).

For each of the 10000 polygons having a particular number of vertices, a random test point was generated in the bounding box. McNemar’s Test [11] was employed to check the statistical significance of one algorithm against another. In a 2 \times 2 contingency table, the McNemar’s test gives a statistic similar to the chi squared statistic which is formulated as:

\[
\chi^2 = \frac{(n_{01} - n_{10})^2}{n_{01} + n_{10}}
\]

(3)

\[
\chi^2 = \frac{(|n_{01} - n_{10}| - 1)^2}{n_{01} + n_{10}}
\]

(4)
where, \( n_{01} \) and \( n_{10} \) are the false positives and the false negatives, respectively. Equation 4 was used as it contains the correction for discontinuity. In this study a \( \chi^2 \) value of 3.84 and above was set as a standard to account for the significant difference of one algorithm against another. Thus with \( p \leq 0.05 \), it is highly unlikely, that the algorithm may be significant from the one that it is being compared with.

In the present scenario, by the proof of the above theorems, it is known that the proposed algorithm give exact results. Thus it is expected that the results obtained from the implementation of the same theoretical idea shall approach ground truth with a margin of error that is completely due to the precision format of the computer. Let \( (P) \) be the proposed algorithm, \( (Cr) \) the Crossing over algorithm and \( (Wi) \) the Winding number rule algorithm. A matlab version of the crossing over algorithm was adopted from [16]. Matlab’s [15] inpolygon algorithm was taken as an implementation of the winding number rule algorithm.

A general analysis is presented relating to the overall behaviour of the algorithms apropos to McNemar’s Test, precision, recall, false positive rate (Fpr) and error rate, as the number of vertices increases. All artificially generated polygons have been stored and can be reused with a new set of random test points to generate similar results for checking.

8 Discussion

In order to evaluate the degree of fairness of the crossing over and the winding number rule based algorithms w.r.t the proposed algorithm, McNemar’s test was performed on polygons with increasing number of vertices with as sample size of 10000. Final results in figure 16 show the statistical significance in the performance of the algorithms against each other. Clearly, it can be seen that the results generated by Cr is significantly similar to those generated via P (red graph in the figure). While comparing the results of Wi with those of Cr as well
Recall of Cr against P, Wi against P and Cr against Wi.

as P, it can be seen that the results are significantly different as the number of vertices increases (blue and green graphs in the figure).

This apparent statistical difference is due to the fact that if the winding number is \( \ell \neq 0 \), then the point lies \( \ell \) times inside the polygon. This is not true with the Cr which states that even cross overs imply that the points are outside the polygons. P, on the other hand sets a bound on the winding number rule, stating that the if \( \ell \neq 0 \), then the point lies inside the polygon if and only if the criterion of having a pair of chains (one above and other below the point) is true (via proved theorems above). Thus even if the winding number evaluates to say 2, in case of the star polygon for the point in figure 9, P will always give the result that the point lies outside the polygon and not twice inside the polygon.

Figure 17 shows the accuracy of the results obtained by the Cr vs P, Wi vs P and Wi vs Cr. The behaviour of the error rate of Wi with respect to P and Cr, again can be attributed to the explanation in the foregoing paragraph. Even though the results of the Cr have been shown not to be statistically significant to that of P, with increasing number of vertices (on a sample size of 10000), the error rate creeps up. The accuracy by itself does not always suffice to give the measure of correctness and thus is aided via means of the false positive rate, precision and recall. The graphs for the same have been depicted in figures 18, 19 and 20.

The false positive rate tells how much one algorithm generated falsely true results given that another algorithm had labeled the results as false. Thus comparing results of Cr against P, Wi against P and Cr against Wi, the false positive rate was generated over increasing number of vertices and calculated on a sample size of 10000 polygons. The graphs suggest that both the Cr and Wi start contributing to the false positives with respect to P as the number of vertices increases. Wi also gives greater false positives with respect to Cr.

The Cr which works on the fundamental idea of drawing a semi infinite line from the sample point thus crossing the edges/vertices of polygon can give different results depending on where the line passes through. A different line
Fig. 21. Average time in seconds on a $\log_{10}$ scale for polygons with increasing number of vertices. Sample size of 10000 for each number of vertices.

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drawn through the same sample point may lead to a contrasting result as has been discussed in the previous section 6.1. Another case that happens in $Cr$ is that if the line passes through say one edge and a vertex (or two vertices), then the results based on even odd rule, change. It has often been pointed out that such cases are very rare and can be dealt by infinitesimally shifting the sample point and the line slightly. First, the the idea of being a rare case does not imply that the problem is solved. Secondly, the solution of shifting the sample point does work, but again it is argued that would it give theoretically correct solution, if the sample point was on another vertex and a slight shift would put the point outside the polygon itself.

With the proof of the proposed solution in the manuscript, it can be shown that even if the polygon is intersected by two perpendicular lines passing through the sample point at angles $\theta$ and $\theta \pm 90$, the lines after rotation to horizontal-vertical frame, will unarguably give correct results. Thus in rare cases also, the proposed solution will work and give theoretical ground truths. Coming back to the false positive rates, with a large sample, increasing size and complexity of the polygons, the $Cr$ and $Wi$ are bound to give false positives with respect to $P$. Finally, the precision and the recall graphs in figures 19 and 20, suggest the concentration and the retrieval of the results.

Lastly, the average time in seconds on a log scale has been plotted in figure 21 to depict the time consumption by the algorithms based on the concepts of the proposed solution, the crossing over and the winding number rule. The proposed solution apparently takes more time than the crossing over which in turn consumes more time than the winding number. Also, in figure 22, for polygons with the number of the vertices 2048, and an increasing sample size, the cumulative time in seconds is plotted on the log scale. These graphs suggest that the reliability of the proposed solution comes at a computational cost. But apart from that, the correctness of the proposed solution is not affected, which forms the kernel of the manuscript.
9 Conclusion

A theoretical and an analytically correct solution to the point in polygon problem is proposed with a description of algorithmic implementation in detail. The proof for the same is given by building a relationship between two new concepts of inaccessibility as well as inside. It is proved that the solution is good interpretation of inside for both the simple and self intersecting polygons. Finally it is shown via McNemars test, that the proposed solution is significantly different from the winding number rule but not from the crossing over. It is also shown by rare examples that the proposed solution differs drastically on subtle issues of cross over. The error rate complemented with the false positive rate, precision and recall bolster the claim of the significance as well as the theoretical correctness of the proposed solution. This reliability and robustness of the proposed solution comes at the cost of computational intensiveness in terms of time complexity. Nevertheless, the theoretical reliability is confirmed, which forms the crux of the manuscript.

References