About the sensitivity of FB modulations to timing offsets

Th. Sartenaer and L. Vandendorpe
Communications and Remote Sensing Laboratory, Université catholique de Louvain,
Place du Levant, 2 - 1348 Louvain-la-Neuve, Belgium

Abstract—This paper studies the impact of timing errors in the receiver for baseband filter bank (FB) modulation techniques, including DMT and multicode modulations. Matched filtering detectors are first investigated, followed by MMSE linear and DF detectors.

I. INTRODUCTION

Timing synchronization plays a crucial role in the performance of digital communication systems. In modern systems, a digital clock synchronizer is implemented in the receiver to estimate the best sampling instant of the received signal [1]. In multiple signal transmission systems, the residual timing error is responsible for an increased interference between adjacent symbols and between parallel waveforms. In [2] the performance degradation on a digital filter bank (FB) based multiple signal system was considered.

The present paper extends these results by considering parametric filters and assuming an ideal half Nyquist pulse shaping filter in the transmitter, a static frequency selective channel, and various IIR receivers. The independence of the average timing sensitivity to the selected FB is demonstrated. The same independence has been shown concerning the Cramér-Rao lower bound on the timing estimator in [3]. This restricts a possible optimization towards the pulse shaping filter design only [4]. Many recent works [5][6][7] investigate a wavelet-based multiple signal system and propose wavelet design techniques to decrease the timing sensitivity. In these cases, the design is focused on the analog wavelet prototype.

II. THE BASEBAND FB TRANSMISSION SCHEME

In a baseband filter bank (FB) transmission scheme, N parallel real symbol streams \( I_p(n) \) with \( p \in [0 \cdots N-1] \) are produced by the transmitter at a baud rate \( 1/NT \). The symbol variance is denoted by \( \sigma^2_I \). These symbols are used to modulate a set of \( N \) synthesis filters \( s_p(n) \) defined at a rate \( 1/T \). The samples produced by the synthesis operation are given by

\[
p(n) = \sum_{p=0}^{N-1} \sum_{m=-\infty}^{+\infty} I_p(m) s_p(n - mN)
\]  

(1)

Splitting this sequence into its \( N \) type 1 polyphase components, we get

\[
p_p(n) = p(nN + \rho) = \sum_{p=0}^{N-1} \sum_{m=-\infty}^{+\infty} I_p(m) s_{p,\rho}(n - m)
\]  

(2)

where \( s_{p,\rho}(m) = s_p(mN + \rho) \). Using \( z \)-transforms of the above quantities, we write:

\[
P(z) = \tilde{S}(z) I(z)
\]  

(3)

In the sequel, we focus on a specific type of synthesis filter banks that fulfill the paraunitary property:

\[
S(z) S^H(1/z^*) = F_N
\]  

(4)

where \( F_N \) stands for the unit matrix of size \( N \) and \( ^H \) means transposition and complex conjugation. This family of FBs includes the well-known DMT and Hadamard modulations. In these two examples, the filter impulse responses are restricted to a length \( N \), i.e. one symbol period. The definition is however not restricted to such filters. Another interesting example is obtained by letting \( S(z) = F_N \), which means that we just have a simple baseband PAM transmission described by a parallel system.

We denote by \( h(t) = f(t) \odot c(t) \) the equivalent channel impulse response resulting from the pulse shaping filter \( f(t) \) and the physical channel \( c(t) \). The transmit filter is of the half-root Nyquist type with rolloff factor \( 0 < \alpha < 1 \). An AWGN \( n(t) \) with two-sided power spectral density \( \sigma_n^2 \) is assumed. The received signal is given by

\[
r_a(t) = \sum_{m=-\infty}^{+\infty} I_p(m) h_p(t - mNT) + n(t)
\]  

(5)

where the \( N \) equivalent channels \( h_p(t) \) are defined as \( h_p(t) = \sum_{n=-\infty}^{+\infty} h(t - nT) s_p(n) \).

We now introduce two useful parameters to characterize the \( N \) received symbol streams:

- The received symbol energy on stream \( p \):
  \[
  E_p = \sigma^2_I \int_{-\infty}^{+\infty} |h_p(t)|^2 dt = \sigma^2_I \int_{-\infty}^{+\infty} |H_p(f)|^2 df
  \]  

(6)

- The mean square bandwidth for stream \( p \):
  \[
  W_p^2 = \frac{\int_{-\infty}^{+\infty} (2\pi f)^2 |H_p(f)|^2 df}{\int_{-\infty}^{+\infty} |H_p(f)|^2 df}
  \]  

(7)

III. TIMING SENSITIVITY

For the various detectors investigated in this paper to produce the estimates of the transmitted symbols, these estimates write:

\[
\hat{I}_{p,\rho}(n) = I_{p,\rho}(n) + w_{p,\rho}(n)
\]  

(8)

where \( I_{p,\rho}(n) = \sum_{p'=0}^{N-1} \sum_{m=-\infty}^{+\infty} x_{p',\rho}(m) I_p'(n - m) \) and \( w_{p,\rho}(n) \) represents the contribution of the additive noise. The interference coefficients \( x_{p',\rho}(m) \) may be obtained by sampling a given set of continuous-time interference waveforms \( x_{p',\rho}(t) \) at the symbol rate, with a timing error \( \epsilon \):

\[
x_{p',\rho}(m) = x_{p',\rho}(mNT + \epsilon)
\]  

(9)
In the $z$ domain, the estimation errors are:
\[ e_n(z) = (X(z) - E_{\mu}(z))I(z) + W(z) \tag{10} \]
where matrix $X(z)$ has element $(p, p')$ given by the $z$-transform of the sequence $x_{pp',\epsilon}(m)$. The error variance $\sigma^2_{ep}(\epsilon)$ on each output includes two contributions:
\[ \sigma^2_{ep}(\epsilon) = x_p(\epsilon)\sigma^2_I + y_p\sigma^2_n \tag{11} \]

The first term represents the residual interference between symbol streams, the ISI inside each symbol stream and the self-interference $(x_{pp',\epsilon}(0) - 1)$. The second term gives the contribution of the additive noise: its variance is independent of the timing error. The interference factor may be decomposed into:
\[ x_p(\epsilon) = x_{p1}(\epsilon) + x_{p2}(\epsilon) + 1 \tag{12} \]
with the following definitions:
\[ x_{p1}(\epsilon) = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega})X^H(e^{j\Omega}) ~d\Omega \right)_{pp} \]
\[ = \sum_{m=-\infty}^{N-1} \sum_{p'=0}^{\infty} \left| x_{pp',\epsilon}(m) \right|^2 \tag{13} \]
\[ x_{p2}(\epsilon) = - \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) + X^H(e^{j\Omega}) ~d\Omega \right)_{pp} \]
\[ = -2x_{pp1}(0) \tag{14} \]

For the receivers investigated in the sequel, it can be shown that the $x_p(\epsilon)$ functions are even, with a global minimum at 0. For small timing errors, the following approximation is valid:
\[ x_p(\epsilon) \approx x_{p1}(\epsilon) + \frac{\epsilon^2}{2} x_{p2}(0). \tag{15} \]

The signal to noise plus interference ratio (SNIR) at the output of the $p^{th}$ symbol stream is given by:
\[ \rho_p(\epsilon) \approx \frac{\sigma^2_I}{\sigma^2_I x_{p1}(\epsilon) + y_p \sigma^2_n} \tag{16} \]

With the last approximation, the 3 dB loss offset is inversely proportional to the initial signal to noise ratio and to the second derivative:
\[ \epsilon_{p,\Delta B} \approx \sqrt{\frac{2}{\rho_p(0)x_{p2}(0)}} \tag{17} \]

In the sequel, we propose to use this value $\bar{x}_p(0)$ as a measurement of the timing sensitivity. We just need to remember that the real sensitivity also depends on the initial signal to noise ratio at perfect synchronisation $\rho_p(0)$.

Let us show as in [2] that the coefficients $x_{p1}(\epsilon)$ can be expressed as the sum of a few sinusoids. From (9), and letting $\Omega = 2\pi fNT$, the discrete time spectra $X_n(e^{2\pi jfNT})$ are linked to the continuous time spectra $X_n(f)$ by:
\[ X_n(e^{2\pi jfNT}) = \frac{1}{NT} \int_{-\infty}^{\infty} X_n(f - \frac{n}{NT}) e^{2\pi j(f - \frac{n}{NT})} ~df \tag{18} \]

By substitution of (18) into (13), we get:
\[ x_{p1}(\epsilon) = \left\{ \sum_{\Delta n=1}^{M} 2\Re \left[ e^{2\pi j\epsilon} \frac{\Delta n}{NT} X_n((\Delta n)) \right] \right\}_{pp} \tag{19} \]

where the series of matrices $R_{xx}(\Delta n)$ is defined as:
\[ R_{xx}(\Delta n) = \frac{1}{NT} \int_{-\infty}^{\infty} X_n(f)X_n^H(f - \frac{\Delta n}{NT})df \tag{20} \]
and $M = \lfloor N(1 + \alpha) \rfloor$. It has been explicitly assumed that the spectra $X_{pp'}(f)$, including the transmit filter, are zero for $|f| > \frac{1}{2T}$.

The first contribution to the timing sensitivity is finally obtained as a sum:
\[ \bar{x}_{p1}(0) = -\frac{2\pi}{T} \sum_{\Delta n=1}^{M} \int_{-\infty}^{\infty} 2(\Delta n)^2 R_{xx}(\Delta n)_{pp} df \tag{21} \]

The second contribution to the timing sensitivity can also be expressed as a function of the continuous time spectra $X_n(f)$:
\[ \bar{x}_{p2}(0) = \frac{2}{(2\pi f)^2} \int_{-\infty}^{\infty} \left| X_n(f) \right|_{pp} df \tag{22} \]

IV. THE MATCHED FILTER RECEIVER

A sufficient statistic of the received signal is obtained by the use of a matched filter bank at the receiver. With this kind of receiver, the continuous time interference waveforms write:
\[ x_{pp'}(t) = h_p(t - \tau) \otimes h_{p'}(t) \tag{23} \]

This operation can be done by first filtering the signal with $h(-t)$ in the analog domain and sampling the result at a rate $1/T$:
\[ q_\epsilon(t) = r_\epsilon(t) \otimes h(-t) = \sum_{n=0}^{\infty} p(n)g(t - nT) + \nu(t) \tag{24} \]

where $g(t) = \int_{-\infty}^{\infty_0} h(\tau)h(\tau - t) d\tau$ is the channel autocorrelation function and $\nu(t) = n(t) \otimes h(-t)$ is the filtered noise with covariance $\Gamma_\nu(t) = \sigma_n^2 g(t)$. The sampling operation then provides:
\[ q_\epsilon(n) = q_\epsilon(nT + \epsilon) \tag{25} \]

where $\epsilon$ is the timing offset caused by a non-perfect synchronisation. The corresponding polyphase components $q_{\epsilon,p}(n) = q_\epsilon(nN + \rho)$ are expressed, with $z$ transforms, as:
\[ Q(z) = G_{\sum}(z)P(z) + \sum_{n=0}^{\infty} \Sigma(z). \tag{26} \]

The polyphase matrix $G(z)$ is toeplitz with element $(i, j)$ given by the $z$ transform of $g[nN + (i - j)T + \epsilon]$. We have the property:
\[ G(z) = \sum_{n=0}^{\infty} \Sigma(z) \tag{27} \]

The noise covariance spectrum is $\Sigma(z) = \sigma_n^2 \sum_{n=0}^{\infty} \Sigma(z)$. The symbol estimates are computed by passing the channel matched filter
output through a bank of analysis filters matched to the synthesis bank:
\[ \mathbf{\hat{L}}(z) = \mathbf{S}^H \left( 1/z^* \right) \mathbf{G}_B(z) = \mathbf{L}_j(z) + \mathbf{W}_j(z) \quad (28) \]

with \( \mathbf{L}_j(z) = \left( \mathbf{S}^H \left( 1/z^\ast \right) \mathbf{G}_j(z) \mathbf{S}(z) \right) \mathbf{H}(z) \) and \( \mathbf{W}_j(z) = \mathbf{S}^H \left( 1/z^\ast \right) \mathbf{V}_j(z) \). Using property (4), the error covariance spectrum writes:
\[ \mathbf{S}_{e_k}(z) = \mathbf{S}^H \left( 1/z^\ast \right) \left[ \sigma^2 \left( \mathbf{G}_k(z) - \mathbf{F}_N \right) \left( \mathbf{G}_j(z) - \mathbf{F}_N \right)^H \right] + \sigma_n^2 \mathbf{G}_k(z) \mathbf{S}(z) \quad (29) \]

An important conclusion may be drawn from the last expression: at the output of a paraunitary FB transmission scheme with matched filtering receiver, the arithmetic mean of the error variances does not depend on the filter bank. In particular, the size of the FB has no impact on this arithmetic mean, which is equal to the unique error variance obtained with a single carrier system. This property remains valid for an arbitrary timing error \( \epsilon \). To demonstrate this property, we first write the expression of the average error variance:
\[ \frac{1}{N} \sum_{p=0}^{N-1} \sigma_{e_p}(\epsilon) = \frac{1}{2\pi N} \int_{-\pi}^{\pi} \text{tr} \left[ \mathbf{S}_{e_k}(e^{j\Omega}) \right] d\Omega \quad (30) \]
Substitution of (29) into (30) gives the proof, along with the following properties recalled here for convenience:

1) Two matrices \( \mathbf{A} \) and \( \mathbf{B} \) are similar (i.e. they have the same eigenvalues) if there exists an invertible matrix \( \mathbf{C} \) such that \( \mathbf{A} = \mathbf{C}^{-1} \mathbf{B} \mathbf{C} \).

2) The trace of a matrix is equal to the eigenvalue sum.

The first property can be applied to (29) thanks to the paraunitary property (4). As a consequence, we may expect that the average timing sensitivity \( \frac{1}{\pi} \sum_{p=0}^{N-1} \delta_p(0) \) should also be independent of the FB. To check this, let us derive a more adequate form for the general results shown in (21) and (22).

The \( R_{ij,j} \) covariance matrix appears to be:
\[ R_{ij,j}(\epsilon) = \frac{\sigma^2}{2\pi} \int_{-\pi}^{\pi} \mathbf{S}^H(e^{j\Omega}) \mathbf{G}_j(e^{j\Omega}) \mathbf{G}_i(e^{j\Omega}) \mathbf{S}(e^{j\Omega}) d\Omega \quad (31) \]
The channel polyphase matrix \( \mathbf{G}_{pq}(e^{j\Omega}) \) is derived from the analog channel transmittance \( G(f) \) as:
\[ \left[ \mathbf{G}_{pq}(e^{j\Omega}) \right]_{pq} = \frac{1}{NT} \sum_{n=-\infty}^{+\infty} G(f - \frac{n}{NT}) e^{2\pi j (p-q) T} (f - \frac{n}{NT}) \quad (32) \]
It follows:
\[ \left[ \mathbf{G}_j(e^{j\Omega}) \mathbf{G}_j(e^{j\Omega}) \right]_{pq} = \frac{1}{(NT)^2} \sum_{k=1}^{N} \sum_{n=-\infty}^{+\infty} G(f - \frac{n}{NT}) G(f - \frac{n}{NT} - \frac{\Delta n}{NT}) e^{2\pi j \epsilon (p-k)(f - \frac{n}{NT} - \frac{\Delta n}{NT})} \quad (33) \]
Using the following property:
\[ \sum_{k=1}^{N} e^{2\pi j \epsilon (k-q) (-\frac{\Delta n}{NT})} = \begin{cases} N & \text{if } \Delta n = \pm N, l \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \quad (34) \]
it appears that the non-zero terms in the last equation are limited to \( \Delta n = 0, \pm N \). For parity reasons, computation of equation (31) simplifies into:
\[ R_{ij,j}(0) = 2 \cos \left( \frac{2\pi \epsilon}{T} \right) - 1 \quad (35) \]
where \( R_{ij,j}(e^{j\Omega}) \) is defined as the polyphase matrix of \( \frac{1}{T} \left( g(t) \otimes g(t) \cos(2\pi f T) \right) \), which is non zero only if the excess bandwidth \( \alpha \) is non zero. By comparison with the general expression (19), we observe that the use of a paraunitary FB reduces the size of the sum to only one term.

The second derivative of the continuous time interference waveforms can be expressed as:
\[ \frac{d^2 x_{pp,a}(t)}{dt^2} \bigg|_{t=0} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \mathbf{S}^H(e^{j\Omega}) \mathbf{R}_{ij,j}(e^{j\Omega}) \mathbf{S}(e^{j\Omega}) \right]_{pp} d\Omega \]
\[ = \int_{-\infty}^{\infty} (2\pi f)^2 |H_p(f)|^2 df = -W_p^2 \frac{F_p}{\sigma_f^2} \quad (36) \]
where \( R_{ij,j}(e^{j\Omega}) \) is the polyphase matrix of \( \tilde{g}(t) \).

Using these results, we can now compute the second derivative of the error covariance matrix with respect to the timing offset, and derive the timing sensitivity for each symbol stream \( \tilde{x}_p(0) \):
\[ \tilde{x}_p(0) = \text{diag} \left( \frac{1}{\pi} \int_{-\pi}^{\pi} \mathbf{S}^H(e^{j\Omega}) \left[ \frac{2\pi}{T} \right]^2 \mathbf{R}_{ij,j}(e^{j\Omega}) \right. \]
\[ - \mathbf{R}_{ij,j}(e^{j\Omega}) \mathbf{S}(e^{j\Omega}) d\Omega \quad (37) \]
From the last expression, we conclude again that the average timing sensitivity does not depend on the selected FB, as already expected.

V. EXAMPLES WITH AWGN CHANNEL
A. A single PAM system
Let us analyse in more details the general result presented above on a simple example: a single baseband PAM modulation \((N=1, S(z)=1)\) and a frequency-flat channel \(c(t)=\delta(t-\tau)\). In this scenario, the channel correlation corresponds to a full raised cosine filter. The corresponding spectrum \(G(f)\) is recalled here for convenience:

\[
G(f) = \begin{cases} 
\frac{T}{2} & |f| < f_1 \\
0 & f_1 \leq |f| \leq f_2 \\
\frac{1}{2} - \frac{1}{2} \cos\left(\frac{\pi f}{f_2}\right) & |f| > f_2
\end{cases}
\]

(38)

with \(f_1 = \frac{1-\alpha}{T}\) and \(f_2 = \frac{1+\alpha}{T}\), \(\alpha\) being the roll-off factor. The received symbol energy is \(E_s = \sigma_r^2 \int_{-\infty}^{\infty} G(f) \, df = \sigma_r^2\).

At perfect synchronism, the interference disappears and we get \(\rho(0) = \sigma_r^2/\sigma_n^2\). Equation (19) becomes:

\[
x_p(\epsilon) = \left[ \frac{1}{T} \int_{-\infty}^{\infty} |G(f)|^2 \, df \right]^{1/2} = 2 \cos^2\left(\frac{2\pi \epsilon}{T}\right) \left[ \frac{1}{T} \int_{-\infty}^{\infty} G(f) G^*(f - \frac{1}{T}) \, df \right]^{1/2}
\]

(39)

where (38) has been used to obtain the last expression. The first contribution to the timing sensitivity is thus:

\[
\bar{x}_p(0) = -\left(\frac{2\pi}{T}\right)^2 \frac{\alpha}{4}
\]

(40)

It appears that the concavity of the nyquist waveform at the origin increases with the rolloff. Equation (22) gives:

\[
\bar{x}_{p2}(0) = \left(\frac{2\pi}{T}\right)^2 \left(\frac{1}{2} - \frac{4}{\pi^2} \alpha^2 + \frac{1}{6}\right) = 2W^2
\]

(41)

using (38). Finally, the total timing sensitivity \(\bar{x}_p(0) = \bar{x}_{p1}(0) + \bar{x}_{p2}(0)\) decreases with the rolloff: this reflects the decrease of the nyquist waveform slopes around the zero-crossing points at higher \(\alpha\).

Figure 1 gives the profile of the SNIR \(\rho(\epsilon)\) as a function of the relative timing error \(\epsilon/T\). The different sets of curves correspond to various levels of additive noise: \(\rho(0) = 10\) to 50 dB with a 10 dB step. Following (17), the timing sensitivity is highly dependent on this parameter. The different line styles correspond to distinct rolloff values: \(\alpha = 0.1, 0.3\) and 0.5. A larger sensitivity at low \(\alpha\) can be observed on the curves.

B. The Discrete Multitone (DMT) system
Let us now investigate the effect of a timing error on a DMT transmission scheme, with the simple AWGN channel. Intuitively, we may expect a higher timing sensitivity for the higher tones, as their spectrum is located at higher frequencies. This is true for the term \(\bar{x}_{p1}(0) = 2W^2\). But concerning the first term \(\bar{x}_{p2}(0)\), the effect is inversed: the tones with higher spectral components will be more influenced by the rolloff region, and the resulting timing sensitivity is decreased.

Figure 2 gives the timing sensitivity profile vs. the tone index \(p\) for a DMT 512 transmission scheme. As expected, we observe an increase for the second term \(\bar{x}_{p2}(0)\) (upper curves). The total sensitivity (lower curves) first increases with the tone index, and finally decreases for the higher tones. This effect appears faster at high rolloff values. The curves corresponding to sine and cosine filters are almost superimposed, with an exact matching for \(p = N/2\).

For any rolloff factor \(\alpha\), it can be verified that the average \(\bar{x}_p(0)\) corresponds to the single PAM value computed by (40) + (41).
VI. MMSE RECEIVERS

In the case of a frequency-selective channel, the symbol estimates obtained at the output of the matched filter bank are severely corrupted by interference. A popular solution consists in the use of a MIMO equalizer that intends to compute more reliable estimates of the symbols on the basis of a minimum mean square error criterion (MMSE). The general expression of these estimates is:

\[ \hat{\mathbf{x}}_{\text{mmse},e}(z) = \mathbf{C}(z) \hat{\mathbf{x}}_{\text{mf},e}(z) - \left( \mathbf{B}(z) - \mathbf{E}_N \right) \mathbf{I}(z) \]  

where \( \mathbf{C}(z) \) is the feedforward MIMO filter and \( \mathbf{B}(z) \) is an optional decision feedback MIMO filter with a causality constraint:

\[ \mathbf{B}(z) = \sum_{n=0}^{\infty} \mathbf{B}_n z^{-n} \]  

where matrix \( \mathbf{B}_0 \) is lower triangular with '1's on the main diagonal. The matched filter outputs \( \hat{\mathbf{x}}_{\text{mf},e}(z) \) are given in (28). The equalizer is computed at modem startup by assuming perfect synchronism.

The MMSE solution of this problem is known to be:

\[ \mathbf{C}(z) = \mathbf{B}(z) \mathbf{S}^{-1}(z) \]  

where we introduce the key matrix \( \mathbf{S}(z) \) as follows:

\[ \mathbf{S}(z) = \begin{bmatrix} \sigma_i^2 & \mathbf{E}_N \mathbf{S}^H(1/z^*) \mathbf{G}_0(z) \mathbf{S}(z) \\ \mathbf{S}^H(1/z^*) \mathbf{S}_0(z) \mathbf{S}(z) \end{bmatrix} \]

\[ = \mathbf{S}^H(1/z^*) \mathbf{S}_0(z) \mathbf{S}(z) \]  

where \( \mathbf{S}_0(z) = \sigma_i^2 \mathbf{E}_N + \mathbf{G}_0(z) \) and the paraunitary property was used.

For a linear MIMO equalizer, \( \mathbf{B}_{\text{lin}}(z) = \mathbf{E}_N \). For a decisionfeedback MIMO equalizer, an optimal feedback matrix \( \mathbf{B}(z) \) has to be found in (44) to minimize the resulting error variance. This problem is solved by letting \( \mathbf{B}_{\text{def}}(z) = \mathbf{D}(z) \), the spectral factor of the key matrix \( \mathbf{S}(z) \) [8]:

\[ \mathbf{S}(z) = \mathbf{D}^H(1/z^*) \mathbf{L} \mathbf{D}(z) \]  

where \( \mathbf{L} \) is diagonal and \( \mathbf{D}(z) \) is causal, monic and stable.

For a paraunitary \( \mathbf{B} \), it can be shown that the resulting error covariance spectrum writes:

\[ \mathbf{S}_{\epsilon}(z) = \mathbf{B}(z) \mathbf{S}^H(1/z^*) \]

\[ + \frac{\sigma_i^2}{\mathbf{S}_0(z)} \left( \mathbf{G}^{-1}(z) \mathbf{S}_0^{-1}(z) - \mathbf{E}_N \right) \left( \mathbf{S}^H(1/z^*) \mathbf{S}_0^{-1}(z) - \mathbf{E}_N \right) \]

\[ + \sigma_i^2 \frac{\mathbf{S}_0^{-1}(z) \mathbf{G}^{-1}(z) \mathbf{S}_0^{-1}(z)}{\mathbf{S}_0(z)} \]  

This expression simplifies further in the absence of timing error.

In the linear scenario, the same conclusion may be drawn as with the matched filtering receiver: at the output of a paraunitary FB transmission scheme with linear MMSE receiver, the arithmetic mean of the symbol error variances does not depend on the filter bank. Substitution of (47) into (30) gives the proof. This is valid even in the presence of an arbitrary timing error \( \epsilon \).

The arithmetic mean of the symbol error variances is not preserved in the case of a DF equalizer, because of the (non-paraunitary) feedback matrix in (47). In [8], it was shown, however, that the geometrical mean of the symbol error variances is independent of the selected FB, at least at perfect synchronism:

\[ \sqrt[n]{\prod_{p=0}^{N-1} \sigma_{\epsilon p}^2(0)} = \sqrt{n} \frac{\sigma_n^2}{\sqrt{\det(\mathbf{L})}} \]  

This property does not remain valid in the presence of a timing error.

We finally provide a way to compute the timing sensitivity of the presented MMSE receivers:

\[ \bar{\epsilon}_V(t) = \left[ \frac{1}{\pi} \int_{-\pi}^{\pi} \mathbf{B}^H(\epsilon^{j\Omega}) \mathbf{S}^H(\epsilon^{j\Omega}) \right]^{-1} \left[ \frac{2\pi}{T} \left( \mathbf{S}_0^{-1}(\epsilon^{j\Omega}) \mathbf{B}_{\text{ff}}(\epsilon^{j\Omega}) \right) \mathbf{S}_0^{-1}(\epsilon^{j\Omega}) \right. 
\]

\[ \left. - \mathbf{S}_0^{-1}(\epsilon^{j\Omega}) \mathbf{B}_{\text{ff}}(\epsilon^{j\Omega}) \mathbf{S}_0^{-1}(\epsilon^{j\Omega}) \right] \mathbf{S}(\epsilon^{j\Omega}) \mathbf{B}^H(\epsilon^{j\Omega}) d\Omega \]  

From (49), it appears that the average timing sensitivity is independent of the selected FB in the linear case.

VII. CONCLUSIONS

This paper proposed an explicit method to compute the timing sensitivity of a general FB based transmission system working on a frequency-selective channel. It was shown that the average timing sensitivity is independent of the selected FB in the case of the matched filter and linear MMSE receivers, but not in the case of the decision feedback MMSE receiver. The AWGN channel scenario with a square root raised cosine transmit filter was further analyzed and the exact timing sensitivity was given as a function of the rolloff factor for the single PAM and DMT systems.

REFERENCES


