Abstract

In this paper we study the geometrical properties of the set of reachable states of a single input discrete–time LTI system with positive controls. This set is a cone and it can be expressed as the direct sum of a linear subspace and a proper cone. In order to give a complete geometrical characterization of the reachable set, we provide a formula to evaluate the dimension of the largest reachable subspace and conditions for polyhedrality of the proper cone.

1 Introduction

In this paper we study the geometrical properties of the set of reachable states \( x_k \) of a single input discrete–time LTI system of the form:

\[
x_{k+1} = F x_k + g u_k \quad k = 0, 1, \ldots
\]

with \( F \in \mathbb{R}^{n \times n} \), \( g \in \mathbb{R}^n \) when the input function \( u_k \) is nonnegative for all times \( k \). This situation is frequently encountered, for example, in medical, ecological, chemical and economical applications where the controls have a unidirectional influence [2]. Moreover, this may also occur in electro-mechanical applications (see the examples discussed in [10]).

It is worth noting that nonnegativity of the input implies that the reachable set is a convex cone. In fact, the set of states reachable in \( k \) steps can be written as

\[
\mathcal{R}_k(F, g) = \left\{ x : x = \sum_{i=0}^{k-1} F^{k-i-1} g u(i), \ u(i) \geq 0 \right\} = \text{cone} \left( g, F g, \ldots, F^{k-1} g \right)
\]

In what follows, we will consider the geometrical properties of the reachable set \( \mathcal{R}(F, g) \) of a reachable pair \( (F, g) \) defined as

\[
\mathcal{R}(F, g) = \text{cl} \left\{ \bigcup_{k=1}^{\infty} \mathcal{R}_k(F, g) \right\} = \text{cl} \left\{ \text{cone} \left( g, F g, F^2 g, \ldots \right) \right\} = \mathcal{R}_F(F, g)
\]

where the sum of two cones, as proved in [8], Theorem 3.8, coincides with the set of all finite nonnegative combinations of vectors belonging to the two cones. Since the reachability set \( \mathcal{R}(F, g) \subseteq \mathbb{R}^n \) is a cone, then it can be obviously written as

\[
\mathcal{R}(F, g) = \mathcal{S}(F, g) \oplus \mathcal{K}(F, g)
\]

where \( \mathcal{S}(F, g) \) is the maximal linear subspace contained in \( \mathcal{R}(F, g) \) and \( \mathcal{K}(F, g) \) is a proper cone contained in the subspace \( \mathcal{S}(F, g) \) complementary to \( \mathcal{S}(F, g) \) in \( \mathbb{R}^n \).

The problem of characterizing the geometrical properties of the reachable set \( \mathcal{R}(F, g) \) of linear system has been studied by Evans and Murthy and Son in [6, 11] for discrete–time systems and by Brammer, Saperstone and Yorke, and Ohta et al. in [4, 10, 7] for continuous–time systems. Evans and Murthy, and Brammer derived conditions for complete controllability, i.e. \( \mathcal{R}(F, g) = \mathcal{S}(F, g) = \mathbb{R}^n \) for discrete and continuous–time respectively. Ohta et al. provided a simple formula to evaluate the dimension of the largest reachable subspace, i.e. the dimension of \( \mathcal{S}(F, g) \) for single–input continuous–time systems. Saperstone and Yorke, and Son consider complete controllability in the case of bounded inputs for continuous and discrete–time systems, respectively.

In this paper we deal with single–input discrete–time systems and provide a complete geometrical characterization of the reachable set \( \mathcal{R}(F, g) \), i.e. of both \( \mathcal{S}(F, g) \) and \( \mathcal{K}(F, g) \). More precisely, we give the dimension of the largest reachable subspace \( \mathcal{S}(F, g) \) and provide conditions for polyhedrality of \( \mathcal{K}(F, g) \). Some preliminary results have appeared in [5].

Proofs of theorems are omitted hereafter for the sake of brevity.

2 Definitions

A set \( \mathcal{K} \subseteq \mathbb{R}^m \) is said to be a cone provided that \( \alpha \mathcal{K} \subseteq \mathcal{K} \) for all \( \alpha \geq 0 \). If a cone \( \mathcal{K} \subseteq \mathbb{R}^m \) contains an open ball of \( \mathbb{R}^m \) then it is said to be solid and if \( \mathcal{K} \cap
\{-K\} = \{0\} it is said to be pointed. A cone which is closed, convex, solid and pointed is said a proper cone. A cone \(K\) is said to be polyhedral if it is expressible as the intersection of a finite family of closed half-spaces. The notation cone\((v_1, \ldots, v_M)\) indicates the convex cone consisting of all nonnegative linear combinations of vectors \(v_1, \ldots, v_M,\) with \(M\) possibly infinite.

Given a square matrix \(F, p_F(\lambda)\) is its characteristic polynomial, \(\sigma_F\) denotes the set of its eigenvalues and \(\deg \lambda_i\), with \(\lambda_i \in \sigma_F\), is the size of the largest block containing \(\lambda_i\) in the Jordan canonical form of \(F\). If the matrix \(F\) has at least one nonnegative real eigenvalue, then \(\omega_F\) equals the maximal nonnegative real eigenvalue of \(F\); otherwise \(\omega_F = 0\). Using the above definitions, the set \(\sigma_F\) can be partitioned in the following disjoint subsets:

\[
\begin{align*}
\sigma_F^{(1)} &= \{\lambda_i \in \sigma_F : |\lambda_i| > \omega_F\} \\
\sigma_F^{(2)} &= \{\lambda_i \in \sigma_F : |\lambda_i| = \omega_F \text{ and } \deg \lambda_i > \deg \omega_F\} \\
\sigma_F^{(3)} &= \{\lambda_i \in \sigma_F : |\lambda_i| = \omega_F \text{ and } \deg \lambda_i \leq \deg \omega_F\} \\
\sigma_F^{(4)} &= \{\lambda_i \in \sigma_F : |\lambda_i| < \omega_F\}
\end{align*}
\]

so that \(\sigma_F := \sigma_F^{(0)} = \sigma_F^{(1)} \cup \sigma_F^{(2)} \cup \sigma_F^{(3)} \cup \sigma_F^{(4)}\). Moreover, given a set of eigenvalues \(\sigma_F^{(k)}\), we define

\[
\rho(\sigma_F^{(k)}) = \max_{\lambda_i \in \sigma_F^{(k)}} \{|\lambda_i|\}
\]

and every eigenvalue \(\lambda_i \in \sigma_F^{(k)}\) such that \(|\lambda_i| = \rho(\sigma_F^{(k)})\) will be called a dominant eigenvalue of \(\sigma_F^{(k)}\).

If \(F\) is nonderogatory, then w.l.o.g. we can assume the matrix to be in following pseudo-Jordan form

\[
F = \begin{pmatrix}
J \left(\sigma_F^{(1)}\right) & 0 & 0 & 0 \\
0 & J' \left(\sigma_F^{(2)}\right) & 0 & 0 \\
0 & 0 & J'' \left(\sigma_F^{(3)}\right) & 0 \\
0 & 0 & 0 & J^{(4)} \left(\sigma_F^{(4)}\right)
\end{pmatrix}
\]

where

\[
J \left(\sigma_F^{(k)}\right) = \text{diag}_{\lambda_i \in \sigma_F^{(k)}} (J_{\deg \lambda_i} (\lambda_i)) \quad k = 1, 3, 4
\]

\[
J' \left(\sigma_F^{(2)}\right) = \text{diag}_{\lambda_i \in \sigma_F^{(2)}} (J_{\deg \lambda_i - \deg \omega_F} (\lambda_i))
\]

\[
J'' \left(\sigma_F^{(3)}\right) = \text{diag}_{\lambda_i \in \sigma_F^{(3)}} (J_{\deg \omega_F} (\lambda_i))
\]

and \(J_k(\lambda)\) is a \(k \times k\) upper triangular matrix of the form

\[
J_k(\lambda) = \begin{pmatrix}
\lambda & 1 & 0 & \cdots \\
\lambda & 1 & \cdots & \ddots \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & \lambda
\end{pmatrix}
\]

3 Main Results

As stated in the Introduction, we will first present a result which provides the dimension of the largest reachable subspace \(S(F,g)\) and how the cone \(K(F,g)\) can be generated.

**Theorem 1** Let the pair \((F,g)\) be reachable. Then the dimension of the largest reachable subspace \(S(F,g)\) is

\[
\mu = \sum_{\lambda_i \in \sigma_F^{(1)}} \deg \lambda_i + \sum_{\lambda_i \in \sigma_F^{(2)}} (\deg \lambda_i - \deg \omega_F)
\]

and that of \(A\) is

\[
\chi = n - \mu = \sum_{\lambda_i \in \sigma_F^{(2)}} \deg \omega_F + \sum_{\lambda_i \in \sigma_F^{(3)} \cup \sigma_F^{(4)}} \deg \lambda_i
\]

where summation over the empty set is considered to be zero.

**Example 1** For the sake of illustration, consider a matrix \(F\) having a spectrum as in figure 1 and a vector \(g\) such that the pair \((F,g)\) is reachable.

In this case, \(\omega_F = \lambda_3\) so that

\[
\begin{align*}
\sigma_F^{(1)} &= \{\lambda_1, \lambda_1^*, \lambda_2\} \\
\sigma_F^{(2)} &= \{\lambda_3, \lambda_3^*\} \\
\sigma_F^{(3)} &= \{\lambda_5, \lambda_5^*\} \\
\sigma_F^{(4)} &= \{\lambda_6, \lambda_6^*, \lambda_7\}
\end{align*}
\]

In view of the above theorem, the dimension of the largest reachable subspace \(S(F,g)\) is \(\mu = 5\).
It is worth stating the following corollaries which directly follow from the above theorem and characterize the two special cases of $\mathcal{R}(F,g) = \mathcal{S}(F,g) = \mathbb{R}^n$ and $\mathcal{R}(F,g) = \mathcal{K}(F,g)$.

**Corollary 2** [6] Let the pair $(F,g)$ be reachable. Then, $\mathcal{R}(F,g) = \mathbb{R}^n$, that is $\mu = n$, if and only if the matrix $F$ has no real nonnegative eigenvalues. In this case $F = A'$ and $g = b'$.

**Corollary 3** Let the pair $(F,g)$ be reachable. Then, $\mathcal{R}(F,g)$ is a proper cone, that is $\mu = 0$, if and only if

$$\sigma_F^{(1)} \cup \sigma_F^{(2)} = \emptyset$$

or equivalently if and only if $\rho(\sigma_F) \in \sigma_F$ and $\deg \rho(\sigma_F) \geq \deg \lambda_i$ for each $\lambda_i$ such that $|\lambda_i| = \rho(\sigma_F)$.

In this case $F = A$ and $g = b$.

Secondly, we present hereafter a Lemma which provides conditions for polyhedrality of $\mathcal{K}(F,g)$. Define

$$K_i(A,b) := \text{cone}(b, Ab, A^2b, \ldots, A^{i-1}b) = \mathcal{R}_i(F,g) \bigcap \mathcal{S}(F,g)^c$$

and

$$\hat{K}(A,b) := \bigcup_{i=1}^{\infty} K_i(A,b) = \text{cone}(b, Ab, A^2b, \ldots)$$

so that we have

$$\mathcal{K}(F,g) = \text{cl} \hat{K}(A,b) := K(A,b) \quad (4)$$

Moreover, by definition,

$$K_1(A,b) \subseteq K_2(A,b) \subseteq K_3(A,b) \subseteq \ldots$$

and if $K_N(A,b) = K_{N+1}(A,b)$ then $K(F,g) = \mathcal{K}(A,b)$ for all $N \in \mathbb{N}$. In other words, if the matrix $A$ is nilpotent and $A^N = 0$. Hence

$$\mathcal{K}(F,g) = \hat{K}(A,b) = K_N(A,b) = \text{cone}(b, Ab, A^2b, \ldots)$$

is polyhedral. Consequently, w.l.o.g. in the sequel we will assume $\omega_F > 0$.

**Lemma 1** Let the pair $(F,g)$ be reachable and $\omega_F > 0$. Then, $\mathcal{K}(F,g)$ is a polyhedral proper cone if and only if there exists a finite positive integer $r$ such that the following limits

$$\lim_{k \to \infty} \frac{A^{rk+b}b}{\|A^{rk+b}b\|} = v_{\infty}^{(b)} \neq 0 \quad h = 0,\ldots,r-1$$

exist with $v_{\infty}^{(i)} \neq v_{\infty}^{(j)}$ for $i \neq j$, and there exists a finite value $N$ such that

$$K_{N+1}(A^r, A^rb) + \text{cone}(v_{\infty}^{(b)}) = \mathcal{K}_N(A^r, A^rb) + \text{cone}(v_{\infty}^{(b)}) \quad (5)$$

for every $h = 0,\ldots,r-1$.

Figure 2: The cones $\hat{K}(A^2, b) + v_{\infty}^{(0)}$ and $\hat{K}(A^2, Ab) + v_{\infty}^{(1)}$ (left) and the cone $K(A,b)$ (right).

**Example 2** In order to illustrate the previous theorem, consider the matrices

$$F = \text{diag}(-2,1,-1,0.8), \quad g = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

In this case we have

$$A = \text{diag}(1,-1,-0.8), \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and, for $r = 2$ the following limits exist

$$\lim_{k \to \infty} \frac{A^{2k}b}{\|A^{2k}b\|} = v_{\infty}^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\lim_{k \to \infty} \frac{A^{2k+1}b}{\|A^{2k+1}b\|} = v_{\infty}^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

and for $N = 1$ equalities (5) holds for $h = 0,1$ as figure 2 clearly shows. Moreover, the figure also makes clear that, as shown in the proof of Theorem 1, any extremal vectors of the form $A' b$ of $\hat{K}(A^2, b) + v_{\infty}^{(0)}$ and of $\hat{K}(A^2, Ab) + v_{\infty}^{(1)}$ is also an extremal vector of $\mathcal{K}(A,b)$.

In what follows we provide the main result of the paper, that is a spectral characterization of polyhedrality of the cone $\mathcal{K}(F,g)$.

**Theorem 4** Let the pair $(F,g)$ be reachable and $\omega_F > 0$. Then $\mathcal{K}(F,g)$ is a polyhedral proper cone if and only if one of the following sets of conditions holds:

a1. $\deg \omega_F \leq 2$;

a2. the eigenvalues in $\sigma_F^{(2)} \cup \sigma_F^{(3)}$ are among the $r$-th roots of $\omega_F$ for some positive integer $r$;

a3. taking the minimal value of $r$, no nonzero eigenvalue in $\sigma_F^{(4)}$ has an argument which is an integer multiple of $2 \pi/r$.

or
b1. \( \deg \omega_F = 1 \)

b2. the dominant eigenvalues of \( \sigma_F^{(4)} \) are simple;

b3. the eigenvalues in \( \sigma_F^{(2)} \cup \sigma_F^{(3)} \) are among the \( r \)-th roots of \( \omega_F \), for some positive integer \( r \) and the dominant eigenvalues of \( \sigma_F^{(4)} \) are among the \( s \)-th roots of \( \rho(\sigma_F^{(4)})^s \), for some positive integer \( s \);

b4. taking the minimal value of \( r \) and \( s \), then no nonzero non dominant eigenvalue of \( \sigma_F^{(4)} \) has an argument which is an integer multiple of \( 2\pi/r \), where \( \tilde{r} \) is the least common multiple between \( r \) and \( s \).

Example 3 In order to illustrate the above theorem, consider the following pair

\[
F = \text{diag}(1, \lambda_2, \lambda_3), \quad g = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\]

with \( \lambda_2, \lambda_3 \) real and such that \( |\lambda_3| < |\lambda_2| < 1 \). Hence, \( A = F, b = g, \omega_F = 1 \) and \( \deg \omega_F = 1 \). Furthermore, condition a1 holds and condition a2 holds with \( r = 1 \). Lastly, conditions b1, b2 hold and condition b3 holds with \( r = 1 \) and \( s \leq 2 \).

When \( \lambda_2 = -0.9 \) and \( \lambda_3 = -0.6 \), then also condition a3 holds. Moreover, as expected, condition b4 fails since the dominant eigenvalue of \( \sigma_F^{(4)} \) is \( \lambda_2 = -0.9 \) so that \( s = 2, \tilde{r} = 2 \) and \( \lambda_3 = -0.6 \) has a phase equal to \( 2\pi \). Hence, the cone \( \mathcal{K}(A, b) \) is polyhedral as shown on the left hand side of figure 3.

When \( \lambda_2 = 0.9 \) and \( \lambda_3 = -0.8 \), then condition a3 fails since the eigenvalue \( \lambda_2 = 0.9 \) has a phase equal to \( 2\pi \). By contrast, condition b3 holds with \( s = 1 \) so that \( \tilde{r} = 1 \) and consequently condition b4 holds since \( \lambda_3 = -0.8 \) has a phase which is not an integer multiple of \( 2\pi \). Hence, the cone \( \mathcal{K}(A, b) \) is polyhedral as shown in the middle picture of figure 3.

Finally, when \( \lambda_2 = -0.9 \) and \( \lambda_3 = 0.8 \), then condition a3 fails since the eigenvalue \( \lambda_3 = 0.8 \) has a phase equal to \( 2\pi \). Moreover, also condition b4 fails since the dominant eigenvalue of \( \sigma_F^{(4)} \) is \( \lambda_2 = -0.9 \) so that \( s = 2, \tilde{r} = 2 \) and \( \lambda_3 = 0.8 \) has a phase which is an integer multiple of \( \pi \). Hence, the cone \( \mathcal{K}(A, b) \) is not polyhedral as shown in the right hand side of figure 3.

Note that, when \( \chi = 2 \) the cone \( \mathcal{K}(F, g) \) is always polyhedral since obviously any cone in \( \mathbb{R}^2 \) is polyhedral. In fact, in this case, the conditions of the theorem are always met as one can easily check.

From the proof of the above Theorem, immediately follows the next corollaries which provide a geometrical and the corresponding spectral characterization of systems for which the cone \( \mathcal{K}(F, g) \) is reachable in a finite number of steps. This property is clearly equivalent to requiring polyhedrality of \( \mathcal{K}(A, b) \), or that the condition \( \mathcal{K}(F, g) = \mathcal{K}(A, b) \) holds.

Corollary 5 Let the pair \( (F, g) \) be reachable and \( \omega_F > 0 \). Then, \( \mathcal{K}(A, b) \) is a polyhedral proper cone if and only if there exists a finite value \( N \) such that

\[
\mathcal{K}_{N+1}(A, b) = \mathcal{K}_N(A, b)
\]

Corollary 6 Let the pair \( (F, g) \) be reachable and \( \omega_F > 0 \). Then, \( \mathcal{K}(A, b) \) is a polyhedral proper cone if and only if the following conditions hold:

1. \( \deg \omega_F = 1 \)
2. the eigenvalues in \( \sigma_F^{(2)} \cup \sigma_F^{(3)} \) are among the \( r \)-th roots of \( \omega_F \), for some positive integer \( r \);
3. taking the minimal value of \( r \), no nonzero eigenvalue of \( \sigma_F^{(4)} \) has an argument which is an integer multiple of \( 2\pi/r \).

Finally, we conclude the paper, with the special case of \( \mathcal{K}(F, g) \) simplicial.
Theorem 7 Let the pair $(F, g)$ be reachable and $\chi > 0$. Then $K(F, g)$ is a polyhedral proper cone with $\chi$ extremal vectors (simplicial) if and only if the polynomial

$$p(\lambda) := \prod_{\lambda_i \in \sigma_A^{(3)} \cup \sigma_A^{(4)}} (\lambda - \lambda_i)$$

has all nonpositive coefficients.

References


