Channel Estimation Using Gaussian Approximation in a Factor Graph for QAM Modulation

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Abstract—Joint channel estimation and decoding using belief propagation on factor graphs requires the quantization of probability densities since continuous parameters are involved. We propose to replace these densities by standard messages where the channel estimate is accurately modeled as a Gaussian mixture. Upward messages include symbol extrinsic information and downward messages carry a mean and a variance for the Gaussian modeled channel estimate. Such unquantized message propagation leads to a complexity reduction and a performance improvement. For QAM modulated symbols, the proposed belief propagation almost achieves the performance of Expectation-Maximization under good initialization and surpasses it under bad initialization.

I. INTRODUCTION

Propagating messages in a suitable factor graph is a systematic tool for deriving iterative algorithms. Among various receiver issues solved using the belief propagation algorithm (BP), also called sum-product algorithm [2], we can cite decoding, channel estimation, synchronization and detection [3]. [4] presents a BP handling continuous variables, in which canonical distributions are used for quantizing probability distributions, in order to propagate discrete probability distributions. However, the degree of quantization has a strong impact on estimation accuracy and performance. The expectation-maximization algorithm (EM) [5] [6], directly handling continuous variables through a posteriori probabilities (APPs) of transmitted symbols, achieves a better performance-complexity trade-off. Even adapting the quantization in each iteration of BP, as proposed in [7] and [8], does not fill the complexity gap between EM and BP.

Instead of relying on quantization, we propose here to model probability distributions as mixtures of Gaussian distributions. It allows for estimation improvement and complexity reduction simultaneously. High order QAM modulation is addressed here, so as to show that the Gaussian approximation principle also applies with a multiplicity of modulation symbols.

The paper is structured as follows. Section II explains how the transmission system is modeled using a factor graph and how BP is applied. Section III presents the approximation of the distribution of channel estimates in BP by a mixture of Gaussian distributions. In Section IV, APPs are computed from the approximated distribution. Channel estimation with continuous upward messages in the factor graph is presented in Section V. The paper ends with simulation results in Section VI.

In the sequel, messages that are not based on quantized densities will be referred to as continuous messages.

II. SYSTEM MODEL

We consider the transmission of $N$ encoded M-QAM symbols $x_k$ over a quasi-static single-input single-output (SISO) channel. Figure 1 depicts the encoding, interleaving and mapping of the information binary sequence $b_k$. The obtained M-QAM symbols $x_k$ experience a single-tap channel, i.e., they are multiplied by a complex Gaussian channel coefficient $h \sim \mathcal{C}\mathcal{N}(0,1)$ before addition of a complex Gaussian noise $n_k \sim \mathcal{C}\mathcal{N}(0,2\sigma_n^2)$. The receiver performs joint channel estimation and decoding based on the channel outputs $y_k$ in order to obtain the estimated information sequence $\hat{b}_k$. The system model is described by

$$y_k = h x_k + n_k, \quad 0 \leq k \leq N - 1.$$  (1)

The corresponding factor graph is built as in [4]. Figure 2 details the upward messages, where $g$ is a quantized estimate of $h$ and $p(g)$ represents the quantized distribution. In node CODE, a forward-backward algorithm computes the extrinsic information for each coded bit. Taken deinterleaving into account, the extrinsic information is propagated to nodes $x_k$, where the extrinsic information $\xi_k$ for each M-QAM symbol $x_k$ is computed. From each node $f_k$ to node $p(g)$, a discrete distribution $\mu_{f_k \rightarrow p(g)}$ of the quantized estimate of $h$ is computed and propagated based on a marginalization of the likelihood $p(y_k|x_k,g)$ with respect to the transmitted symbol $x_k$. All $\mu_{f_k \rightarrow p(g)}$ distributions are multiplied to build a single common discrete distribution $p(g)$. Figure 3 details the downward messages. The common distribution $p(g)$ is propagated down to all nodes $f_k$. The APP of each M-QAM symbol $x_k$ is computed based on this discrete distribution, marginalizing the likelihood $p(y_k|x_k,g)$ with respect to $g$. The APP of each coded bit is then computed from the APP
of the M-QAM symbol it belongs to. The whole process of propagating upward and downward messages is then iterated.

III. DISTRIBUTION OF CHANNEL ESTIMATE

In the iterative receiver, initial estimate is obtained from known pilots and subsequent estimates from data symbols. Thus, the distribution of channel estimate will differ depending on the iteration.

A. Estimation based on pilots

$L_p$ pilots $x_{\text{pilot}}$ are included in the transmitted sequence. From the $L_p$ messages provided by $f_{\text{pilot},k}$ nodes corresponding to pilots, we get the discrete distribution of $g$ [2]:

$$p(g) = \prod_{k=0}^{L_p-1} \mu_{f_{\text{pilot},k} - p(g)} \propto \sum_{i=0}^{q-1} \delta(g - g_i) \exp \left( -\frac{1}{2\sigma_n^2} \sum_{k=0}^{L_p-1} |y_{\text{pilot},k} - x_{\text{pilot},k}g_i|^2 \right),$$

where

$$y_{\text{pilot},k} = h x_{\text{pilot},k} + n_{\text{pilot},k},$$

(3)

$\{g_i\}$ is a quantization codebook of size $q$ for channel estimate’s probability density function (pdf) and $\delta(\cdot)$ denotes the Dirac delta function. For $L_p \gg 1$, we have

$$\sum_{k=0}^{L_p-1} |y_{\text{pilot},k} - x_{\text{pilot},k}g_i|^2 \approx L_p\epsilon_{\text{pilot}} |g_i - h|^2 + 2L_p\sigma_n^2,$$

(4)

where $\epsilon_{\text{pilot}} = |x_{\text{pilot},k}|^2$ represents the pilot energy. Surprisingly, this rough approximation for small $L_p$ also yields a good performance. Substituting (4) into (2), we obtain

$$p(g) \propto \sum_{i=0}^{q-1} \delta(g - g_i) \exp \left( -\frac{L_p\epsilon_{\text{pilot}}}{2\sigma_n^2} |g_i - h|^2 \right).$$

(5)

Hence, $p(g)$ can be approximated as a Gaussian distribution

$$\mathcal{CN}(h, \frac{2\sigma_n^2}{L_p\epsilon_{\text{pilot}}}).$$

B. Estimation based on data

Let $S = \{s_m, 0 \leq m \leq M - 1\}$ represent the constellation of M-QAM and $\xi_{k,m}$ be the extrinsic information for $x_k = s_m$. The message from $f_k$ to $p(g)$ can be expressed as [2]:

$$\mu_{f_k - p(g)} \propto \sum_{i=0}^{q-1} \delta(g - g_i) \exp \left( -\frac{|y_k - s_m g_i|^2}{2\sigma_n^2} \right) \xi_{k,m}.$$ 

(6)

Using (1), we have

$$\mu_{f_k - p(g)} \propto \sum_{i=0}^{q-1} \delta(g - g_i) \sum_{m=0}^{M-1} \exp \left( -\frac{|s_m g_i - h x_k - n_k|^2}{2\sigma_n^2} \right) \xi_{k,m}.$$ 

(7)

In (9), considering the items $s_k^* x_k h g_i$ and $s_k^* x_k^* h^* g_i$, the summation can be rewritten into five sub-summations where $s_k^* = x_k, s_k^* = -x_k, s_k^* = j x_k, s_k^* = -j x_k$ and $s_k^*$ equals other values. We get (10), where $U_1^j$ (resp. $U_2^j, U_3^j$ and $U_4^j$) is the number of items with $s_k^* = x_k$ (resp. $s_k^* = -x_k, s_k^* = j x_k$ and $s_k^* = -j x_k$) and $V$ represents the number of

$$p(g) = \prod_{k=0}^{N-1} \mu_{f_k - p(g)} \propto \sum_{i=0}^{q-1} \delta(g - g_i) \sum_{j=1}^{M^N} \left\{ \exp \left( -\frac{1}{2\sigma_n^2} \sum_{k=0}^{N-1} |s_k^j g_i - h x_k - n_k|^2 \right) \prod_{k=0}^{N-1} \xi_{k} \right\}.$$ 

(8)
all other items. $E_{av}$ is the average energy of M-QAM symbols. Substituting (10) into (8), we have (11).

1) With high SNR, the decoder almost provides perfect extrinsic information. Thus, for a single sequence $j$ with $U^j_1 = N$, all $\xi^j_{u_i} \rightarrow 1$ and other terms are null:

$$p (g) \propto \sum_{u_i=0}^{q-1} \exp \left\{ -\frac{N E_{av}}{2 \sigma_n^2} |g_i - h|\right\} \delta (g - g_i).$$  (12)

2) With low SNR, all extrinsic probabilities are close to 1/M. Thus, in (11), there are $M^N$ items. However, we can consider the following factors:

$$\exp \left\{ \frac{N E_{av}}{4 \sigma_n^2} \left( 1 - X^2 \right) |h|^2 \right\} \triangleq \epsilon_{12}$$  (13)

and

$$\exp \left\{ \frac{N E_{av}}{4 \sigma_n^2} \left( 1 - U^3_2 \right) |h|^2 \right\} \triangleq \epsilon_{34}.  \quad \text{(14)}$$

With $N \gg 1$, in the items where $X^2 + U^3_2 < 2$, $\epsilon_{12} \epsilon_{34} \rightarrow 0$. Hence, only sequences with $X^2 + U^3_2 \geq 2$ correspond to non-negligible terms in (11). In addition, there are also another two items in (11):

$$\exp \left\{ \frac{N E_{av}}{4 \sigma_n^2} |g_i + X^j_2 h|^2 \right\} \exp \left\{ \frac{N E_{av}}{4 \sigma_n^2} |g_i + jX^3_4 h|^2 \right\}. \quad \text{(15)}$$

Since the variance $4\sigma_n^2/N E_{av}$ of the two Gaussian distributions in (15) are very small, the product can be neglected when they have different mean values. Thus, only the items with $X^2 = 0$ or $U^3_2 = 0$ are non-negligible in (11). Furthermore, among these non-negligible items, only four items with $U^j_1 = N$, $U^3_2 = N$, $U^3_3 = N$ and $U^3_4 = N$ are dominant taking into account of (13) and (14). The item corresponding to $U^j_3 = N$ is:

$$\sum_{k=0}^{N-1} |s^j_k g_i - h x_k - n_k|^2 \approx \frac{N E_{av}}{2} \left[ |g_i + X^j_2 h|^2 + (1 - X^2) |h|^2 + |g_i + jX^3_4 h|^2 + (1 - X^3_4) |h|^2 \right] + 2N \sigma_n^2,$$

where $X^j_2 \triangleq 2 \left( U^j_2 - U^j_1 \right) / N$ and $X^j_3 \triangleq 2 \left( U^j_3 - U^j_4 \right) / N$. (10)

$$p (g) \propto \sum_{i=0}^{q-1} \delta (g - g_i) \sum_{j=1}^{M^N} \prod_{u_i=0}^{u_i-1} \prod_{u_2=0}^{U^2_2-1} \prod_{u_3=0}^{U^3_3-1} \prod_{u_4=0}^{U^4_4-1} \prod_{v=0}^{V-1} \sum_{U^j_1=0}^{U^j_1-1} \sum_{U^j_2=0}^{U^j_2-1} \sum_{U^j_3=0}^{U^j_3-1} \sum_{U^j_4=0}^{U^j_4-1} \sum_{V=0}^{V-1} \propto \exp \left\{ -\frac{N E_{av}}{2 \sigma_n^2} |g_i - \alpha_u h|^2 \right\},$$  (16)

where $u = \{1, 2, 3, 4\}$. So, (11) can be simplified into:

$$p (g) \propto \sum_{i=0}^{q-1} \exp \left\{ \frac{N E_{av}}{2 \sigma_n^2} |g_i - \alpha_u h|^2 \right\} \times \prod_{k=0}^{N-1} \xi^j_k (u).$$  (17)

where $\xi^j_k (u)$ is the extrinsic probability of $\alpha_u x_k$. Especially, when all extrinsic probabilities are close to 1/M, the four items in (17) have the same factors.

So, combining the approximations with high and low SNR, the pdf of channel estimate with M-QAM in factor graph can be approximated as a mixture of four Gaussian distributions:

$$p (g) \propto \sum_{i=0}^{q-1} \sum_{u=1}^{4} \beta_u \exp \left\{ -\frac{N E_{av}}{2 \sigma_n^2} |g_i - h|^2 \right\},$$  (18)

where $\beta_u, u = 1, \ldots, 4$, represent the products of extrinsic probabilities and $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 1$. Our simulations confirm this conclusion, as shown in Fig. 4.

IV. APP EVALUATION FROM DOWNWARD MESSAGES

With the conclusions in Section III, the discrete channel distribution can be approximated as a sum of four Gaussian distributions. However, our simulations show that there is always one dominant Gaussian distribution (with mean value $h$) among the four ones even with low SNR. Hence, when calculating APP, we only consider the dominant one ($\beta_1 = 1$) and the discrete distribution message $\mu (p(g) - f_k)$ can be reduced to one pair $(h, \sigma_n^2)$ characterizing $p (g)$. The parameter $h$ is computed as the mean value of the discrete distribution in (2) (resp. (8)) and $\sigma_n^2$ is equal to $\sigma_n^2 / \langle L(E_{av}) \rangle$ (resp. $\sigma_n^2 / (N E_{av})$). Thus, we can calculate each downward message APP$_k$ in a continuous way, instead of computing it for each codebook...
value \( g_i \), and then marginalizing with respect to \( g \). It reduces the computation complexity.

The APP_{k,m} that transmitted symbol \( x_k \) equals \( s_m \) can be expressed as:

\[
\text{APP}_{k,m} \propto \int_g p(y_k|x_k = s_m, g) p(g) \, dg.
\]  

(19)

With some calculation, we have

\[
\text{APP}_{k,m} \propto \frac{1}{8\pi (\sigma^2_h s_m^2 + \sigma^2_n)} \exp \left\{ \frac{1}{2} \frac{|y_k - s_m \hat{h}|^2}{\sigma^2_h s_m^2 + \sigma^2_n} \right\}. 
\]  

(20)

In Fig. 5, BP using APP computation based on the Gaussian approximation (BP-QT-DGA) is compared with BP using APP computation based on the discrete distribution (BP-QT). We observe that BP-QT-DGA performs as well as BP-QT. Nevertheless, thanks to the computation in (20), a single APP computation instead of \( q \) computations is performed for each symbol \( x_k \) with the Gaussian approximation. Since the derivation of each message \( \mu_j, f_j, p_j(g) \) still involves \( q \) computations, one per codebook value, the global complexity reduction brought by the Gaussian approximation in the downward messages is approximately 50\% for large \( q \).

V. ESTIMATION FROM UPWARD MESSAGES

We observe in Fig. 5 that quantization in the upward message leads to an error floor. Thus, we propose to increase the accuracy of \( \hat{h} \) using a continuous upward message.

A. Estimation based on messages from pilots

By replacing the quantization codebook \( \{g_i\} \) in (2) with a continuous value \( g \) and considering normalization, we get the following estimated mean value of \( g \):

\[
\hat{h} = \frac{\int_g \exp \left\{ -\frac{1}{2\sigma^2_n} \sum_{k=0}^{L_p-1} |y_{\text{pilot,}\, k} - x_{\text{pilot},\, g}|^2 \right\} \, dg}{\int_g \exp \left\{ -\frac{1}{2\sigma^2_n} \sum_{k=0}^{L_p-1} |y_{\text{pilot,}\, k} - x_{\text{pilot},\, g}|^2 \right\} \, dg}. 
\]  

(21)

The integration of numerator and denominator in (21) results in the following simple formula:

\[
\hat{h} = \frac{x_{\text{pilot}}}{L_p} \frac{\sum_{k=0}^{L_p-1} y_{\text{pilot,}\, k}}{g_{\text{pilot}}}. 
\]  

(22)

B. Estimation based on messages from data

Replacing the discrete computation in (6) by an integral, we get the mean value of \( g \):  \[
\hat{h} = \frac{\int_g \prod_{k=0}^{N-1} \sum_{m=0}^{M-1} \exp \left\{ -\frac{|y_{k} - s_m g|^2}{2\sigma^2_n} \right\} \xi_{k,m} \, dg}{\int_g \prod_{k=0}^{N-1} \sum_{m=0}^{M-1} \exp \left\{ -\frac{|y_{k} - s_m g|^2}{2\sigma^2_n} \right\} \xi_{k,m} \, dg}. 
\]  

(23)

Finally, the integration can be performed using the Gaussian approximation:

\[
\hat{h} = \left\{ \sum_{j=1}^{M^N} \Delta_j \prod_{k=0}^{N-1} \sum_{m=0}^{M-1} \exp \left\{ \frac{1}{2\sigma^2_n} \sum_{k=0}^{N-1} |s_m|^2 \right\} \xi_{k,m} \right\}^{-1} 
\]  

\[
\Delta_j = \exp \left\{ \frac{1}{2\sigma^2_n} \sum_{k=0}^{N-1} |s_k|^2 \right\}. 
\]  

(26)
in which
\[
\prod_{k=0}^{N-1} \sum_{m=0}^{M-1} \exp \left( -\frac{|y_{km} - s_{km}|^2}{2\sigma^2_m} \right) \xi_{k,m} = \sum_{j=1}^{M^N} \exp \left( -\frac{1}{2\sigma^2} \sum_{k=0}^{N-1} |y_k - s_j^k|^2 \right) \Delta_j,
\]
(24)
where \(s_j^k\) represents the value of the \(k\)th symbol in sequence \(j\) and
\[
\Delta_j = \prod_{k} \xi_j^k.
\]
(25)
After some calculations, we obtain (26). In order to reduce the computation in (26), we only consider the item with the largest \(\Delta_j\) in both numerator and denominator:
\[
\hat{h} \approx \frac{\sum_{k=0}^{N-1} s_{j_{\text{max}}}^k y_k}{\sum_{k=0}^{N-1} |s_{j_{\text{max}}}^k|^2},
\]
(27)
where \(j_{\text{max}} = \arg \max_j \Delta_j\).

VI. NUMERICAL RESULTS

We now compare BP with continuous downward and upward messages (BP-DUGA) with EM. From Fig. 6, we observe that, with 10 pilots, i.e., good initialization, the proposed BP-DUGA does not exhibit an error floor and achieves bit error rate (BER) performance close to EM. Furthermore, from Fig. 7, with 1 pilot for initial estimate, we observe that the proposed BP-DUGA performs better than EM: for BER performance, we obtain a gain of about 1.5 dB with BP-DUGA compared to EM. Using continuous downward and upward messages brings a complexity reduction compared to the quantization method. As a result, BP-DUGA complexity becomes equivalent to EM complexity.

VII. CONCLUSION

Thanks to an approximation of the distribution of the channel estimate as a mixture of Gaussian distributions, we improved the performance of BP and reduced its complexity by propagating continuous messages in the factor graph. The proposed BP-DUGA almost achieves EM performance under good initialization and outperforms it under a bad initialization. BP-DUGA and EM have equivalent complexity. This paper is focusing on a single-path channel with QAM modulation. Nevertheless, the extension of the Gaussian approximation principle to a multipath channel is natural.

REFERENCES