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Abstract. This paper give sufficient conditions for the stability of linear singular continuous and descriptor time delay systems of the form $E\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau)$ and $E x(k) = A_0 x(k) + A_1 x(k - 1)$, respectively. These new, delay-independent conditions are derived using approach based on Lyapunov’s direct method. Approach that has been applied is based on crucial idea presented in paper of Owens, Debeljkovic (1985). Numerical examples have been working out to show the applicability of results derived.

Key words: Singular continuous time delay systems, Descriptor discrete time delay systems, Lyapunov stability

1 Introduction

It should be noticed that in some systems we must consider their character of dynamic and static state at the same time. Generalized state space systems (also referred to as degenerate, singular, descriptor, generalized, differential - algebraic systems or semi - state) are those the dynamics of which are governed by a mixture of algebraic and differential (difference) equations. Recently many scholars have paid much attention to singular and descriptor systems and have obtained many good consequences. The complex nature of singular and discrete systems causes many difficulties in the analytical and numerical treatment of such systems, particularly when there is a need for their control.
The problem of investigation of time delay systems has been exploited over many years. Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc.

The existence of pure time lag, regardless if it is present in the control or/and the state, may cause undesirable system transient response, or even instability.

Consequently, the problem of stability analysis for this class of systems has been one of the main interests for many researchers. In general, the introduction of time delay factors makes the analysis much more complicated.

We must emphasize that there are a lot of systems that have the phenomena of time delay and singular simultaneously, we call such systems as singular differential (difference) systems with time delay. These systems have many special characters. If we want to describe them more exactly, to design them more accurately and to control them more effectively, we must paid tremendous endeavor to investigate them, but that is obviously very difficult work.

In recent references authors had discussed such systems and got some consequences. But in the study of such systems, there are still many problems to be considered. When the general time delay systems are considered, in the existing stability criteria, mainly two ways of approach have been adopted. Namely, one direction is to contrive the stability condition which does not include the information on the delay, and the other is the method which takes it into account. The former case is often called the delay - independent criteria and generally provides simple algebraic conditions.

In that sense the question of their stability deserves great attention.

In the short overview, that follows, we shall be familiar only with results achieved in the area of Lyapunov stability of linear, both continuous and discrete descriptor time delay systems (LCSTDS) and (LDDTDS), respectively. In that sense we will not discuss contributions presented in papers concerned with problem of robust stability, stabilization of this class of systems with parameter uncertainty, see the list of references, as well as with other questions in connection with stability of (LCSTDS) or (LDDTDS) being necessarily, transformed by Lyapunov - Krasovski functional, to the state space model in the form of differential - integral equations, Fridman (2001, 2002).

To the best of our knowledge only one paper has been published on the matter of Lyapunov stability of (LDDTDS), so we shall discuss its contribution latter and more carefully. To be familiar with other problems successfully solved, for this class of systems, check the attached list of references.

To the best of our knowledge, some attempts in stability investigation of (LCSTDS) was due to Sarić (2001, 2002a, 2002b) where sufficient conditions for convergence of appropriate fundamental matrix were established.

Recently, in the paper of Xu et al. (2002) the problem of robust stability and stabilization for uncertain (LCSTDS) was addressed and necessary and sufficient conditions were obtained in terms of strict LMI. Moreover in the
same paper, using suitable canonical description of (LCSTDS) a rather simple criteria for asymptotic stability testing was also proposed. Paper of Xu et al. (2004) is addressed to the problems of robust stabilization and robust $H_\infty$ control of uncertain (LDDTDS) and parameter uncertainties. The (LDDTDS) under consideration is not necessarily regular. Moreover presented Lemma enables one to check systems asymptotic stability using LMI techniques. But still there are a lot of problems in solving complex system of inequalities.

In our paper we present quite another approach to this problem. Namely, our result is expressed directly in terms of matrices $E, A_0$ and $A_1$ naturally occurring in the system model and avoid the need to introduce any canonical form into the statement of the Theorem or to solve complex matrix system of inequalities.

The geometric theory of consistency leads to the natural class of positive definite quadratic forms on the subspace containing all solutions. This fact makes possible the construction of Lyapunov stability theory even for the (LCSTDS) or (LDDTDS) in that sense that asymptotic stability is equivalent to the existence of symmetric, positive definite solutions to a weak form of Lyapunov matrix equation incorporating condition which refer to time delay term.

In the descriptor discrete case, the concept of of smoothness has little meaning but the idea of consistent initial conditions being these initial conditions $x_0$ that generate solution sequence $(x(k) : k \geq 0)$ has a physical meaning.

The certain aim of this paper is to present a new results concerning asymptotic stability of a particular class of linear descriptor discrete time delay systems. In order to have the best insight in these problems, the first part of paper is devoted to new contributions in the field of linear continuous singular systems and the second part is concerned with linear discrete descriptor time delay systems.

Our aim is to derive a quite new results concerning asymptotic stability of a particular class of linear singular continuous and discrete descriptor time delay systems.
2 Notations

\[ \mathbb{R} \] Real vector space
\[ \mathbb{C} \] Complex vector space
\[ I \] Unit matrix
\[ F = (f_{ij}) \in \mathbb{R}^{n \times n}, \] real matrix
\[ F^T \] Transpose of matrix \( F \)
\[ F > 0 \] Positive definite matrix
\[ F \geq 0 \] Positive semi definite matrix
\[ \mathbb{R}(F) \] Range of matrix \( F \)
\[ (F) \] Null space (kernel) of matrix \( F \)
\[ \lambda(F) \] Eigenvalue of matrix \( F \)
\[ \sigma(F) \] Singular value of matrix \( F \)
\[ \|F\| \] Euclidean matrix norm of \( F \)
\[ F^D \] Drazin inverse of matrix \( F \)
\[ \Rightarrow \] Follows
\[ \mapsto \] Such that

3 Linear continuous time delay systems

Generally, the linear singular continuous systems with time delay can be written as:

\[
E(t)\dot{x}(t) = f(t, x(t), x(t-\tau), u(t)), \quad t \geq 0 \tag{3.1}
\]

\[
x(t) = \varphi(t), \quad -\tau \leq t \leq 0 \tag{3.3}
\]

where \( x(t) \in \mathbb{R}^n \) is a state vector, \( u(t) \in \mathbb{R}^l \) is a control vector, \( E(t) \in \mathbb{R}^{n \times n} \) is a singular matrix, \( \varphi \in C = C([-\tau, 0][\mathbb{R}^n]) \) is an admissible initial state functional, \( C = C([-\tau, 0][\mathbb{R}^n]) \) is the Banach space of continuous functions mapping the interval \([-\tau, 0]\) into \( \mathbb{R}^n \) with topology of uniform convergence.

3.1 Some Preliminaries

Consider a linear continuous singular system with state delay, described by

\[
E\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau), \tag{3.2}
\]

with known compatible vector valued function of initial conditions

\[
x(t) = \varphi(t), \quad -\tau \leq t \leq 0, \tag{3.3}
\]
Stability in the sense of Lyapunov of GSSTDS: A Geometric Approach

where $A_0$ and $A_1$ are constant matrices of appropriate dimensions. Moreover we shall assume that $\text{rank}E = r < n$.

**Definition 3.1** The matrix pair $(E, A_0)$ is said to be regular if $\det(sE - A_0)$ is not identically zero, Xu et al. (2002).

**Definition 3.2** The matrix pair $(E, A_0)$ is said to be impulse free if $\deg(\det(sE - A_0)) = \text{rang}E$, Xu et al. (2002).

The linear continuous singular time delay system (3.2-3.3) may have an impulsive solution, however, the regularity and the absence of impulses of the matrix pair $(E, A_0)$ ensure the existence and uniqueness of an impulse free solution to the system under consideration, which is defined in the following Lemma.

**Lemma 3.1** Suppose that the matrix pair $(E, A_0)$ is regular and impulsive free and unique on $[0, \infty)$, Xu et al (2002).

Necessity for system stability investigation makes need for establishing a proper stability definition. So one can has:

**Definition 3.3** a) Linear continuous singular time delay system, (3.2-3.3) is said to be regular and impulsive free if the matrix pair $(E, A_0)$ is regular and impulsive free.

b) Linear continuous singular time delay system, (3.2-3.3), is said to be stable if for any $\varepsilon > 0$ there exist a scalar $\delta(\varepsilon) > 0$ such that, for any compatible initial conditions $\varphi(t)$, satisfying condition: $\sup_{-\tau \leq t \leq 0} \| \varphi(t) \| \leq \delta(\varepsilon)$, the solution $x(t)$ of system (3.2-3.3) satisfies $\| x(t) \| \leq \varepsilon$, $\forall t \geq 0$.

Moreover if $\lim_{t \to \infty} \| x(t) \| \to 0$, system is said to be asymptotically stable, Xu et al (2002).

3.2 Some previous result

Let us consider the case when the subspace of consistent initial conditions for singular time delay and singular nondelay system coincide.

3.2.1 Pandolfi’s approach

Our result is stated as follows:

**Theorem 3.1** Suppose that the system matrix $A_0$ is nonsingular, e.i. $\det A_0 \neq 0$.

Then we can consider system (3.2) with known compatible vector valued function of initial conditions and we shall assume that $\text{rank} E_0 = r < n$. 

Matrix $E_0$ is defined in the following way $E_0 = A_0^{-1}E$.
The system (3.2-3.3) is asymptotically stable, independent of delay, if
\[
\|A_1\| < \sigma_{\min} \left( Q^{\frac{1}{2}} \right) \sigma_{\max}^{-1} \left( Q^{-\frac{1}{2}} E_0^T P \right),
\] (3.4)
and if there exist:
(i) $(n \times n)$ matrix $P$, being the solution of Lyapunov matrix:
\[
E_0^T P + PE_0 = -2I_\Omega,
\] (3.5)
with the following properties:
\[a)\]
\[
P = P^T
\] (3.6)
\[b)\]
\[
Pq(t) = 0, \quad q(t) \in \Lambda
\] (3.7)
\[c)\]
\[
q^T(t)Pq(t) > 0, \quad q(t) \neq 0, \quad q(t) \in \Omega
\] (3.8)
where:
\[\Omega = \left(I - EE^D\right),
\] (3.9)
\[\Lambda = \left(EE^D\right),
\] (3.10)
with matrix $I_\Omega$ representing generalized operator on $\mathbb{R}^n$ and identity matrix on subspace $\Omega$ and zero operator on subspace $\Lambda$ and matrix $Q$ being any positive definite matrix.
Moreover matrix $P$ is symmetric and positive definite on the subspace of consistent initial conditions.
Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ have the same meaning as in the previous section.

**Proof.** For the sake of brevity, the proof is omitted here and can be found in Debeljković et al. (2005.c, 2006.a).

### 3.3 Main result
#### 3.3.1 Owens-Debeljković approach

**Theorem 3.2** Suppose that the matrix pair $(E, A_0)$ is regular with system matrix $A_0$ being nonsingular, i.e. $\det A_0 \neq 0$.
The system (3.2-3.3) is asymptotically stable, independent of delay, if there
exist a positive definite matrix \( P \), being the solution of Lyapunov matrix equation

\[
A_0^T P E + E^T P A_0 = -2(S + Q),
\]

(3.11)

with matrices \( Q = Q^T > 0 \) and \( S = S^T \), such that:

\[
x^T(t)(S + Q)x(t) > 0, \quad \forall x(t) \in W_k \setminus \{0\},
\]

(3.12)

is positive definite quadratic form on \( W_k \setminus \{0\} \), \( W_k \) being the subspace of consistent initial conditions, and if the following condition is satisfied:

\[
\| A_1 \| < \sigma_{\min} \left( Q^{\frac{1}{2}} \right) \sigma_{\max}^{-1} \left( Q^{-\frac{1}{2}} E^T P \right),
\]

(3.13)

Here \( \sigma_{\max}(\cdot) \) and \( \sigma_{\min}(\cdot) \) are maximum and minimum singular values of matrix(\( \cdot \)), respectively.

**Proof 3.1** Let us consider functional

\[
V(x(t)) = x^T(t) E^T P E x(t) + \int_{t-\tau}^{t} x^T(\gamma) Q x(\gamma) d\gamma,
\]

(3.14)

Note that and Lemma A1 and Theorem A1 indicates that

\[
V(x(t)) = x^T(t) E^T P E x(t),
\]

(3.15)

is positive quadratic form on \( W_k \), and it is obvious that all smooth solutions \( x(t) \) evolve in \( W_k \), so \( V(x(t)) \) can be used as a Lyapunov function for the system under consideration, Owens, Debeljković (1985).

It will be shown that the same argument can be used to declare the same property of another quadratic form present in (3.14).

Clearly, using the equation of motion of (3.2-3.3), we have

\[
\dot{V}(x(t)) = x^T(t) \left( A_0^T P E + E^T P A_0 + Q \right) x(t)
\]

\[+ 2x^T(t) (E^T P A_1) x(t - \tau) - x^T(t-\tau) Q x(t-\tau),
\]

(3.16)

and after some manipulations, yields to

\[
\dot{V}(x(t)) = x^T(t) \left( A_0^T P E + E^T P A_0 + 2Q + 2S \right) x(t)
\]

\[+ 2x^T(t) (E^T P A_1) x(t - \tau) - x^T(t-\tau) Q x(t-\tau) - x^T(t) Q x(t) - x^T(t) S x(t) - x^T(t-\tau) Q x(t-\tau)
\]

(3.17)

From (3.16) and the fact that the choice of matrix \( S \), can be done, such that

\[
x^T(t) S x(t) \geq 0, \quad \forall x(t) \in W_k \setminus \{0\},
\]

(3.18)
one can have the following result
\[ \dot{V}(x(t)) \leq 2x^T(t) (E^T P A_1) x(t - \tau) - x^T(t) Q x(t) - x^T(t - \tau) Q x(t - \tau) \] (3.19)
and based on well known inequality.
\[ 2x^T(t) E^T P A_1 x(t - \tau) = 2x^T(t) \left( E^T P A_1 Q^{-\frac{1}{2}} Q^\frac{1}{2} \right) x(t - \tau) \leq x^T(t) E^T P A_1 Q^{-1} A_1^T P E \cdot x(t) + x^T(t - \tau) Q x(t - \tau) \] (3.20)
and by substituting into (3.19), it yields
\[ \dot{V}(x(t)) \leq -x^T(t) Q x(t) + x^T(t) E^T P A_1 Q^{-1} A_1^T P E x(t), \] (3.21)
or
\[ \dot{V}(x(t)) \leq -x^T(t) Q^\frac{1}{2} \Gamma^* Q^\frac{1}{2} x(t), \] (3.22)
with matrix \( \Gamma^* \) defined by
\[ \Gamma^* = \left( I - Q^{-\frac{1}{2}} E^T P A_1 Q^{-\frac{1}{2}} A_1^T P E Q^{-\frac{1}{2}} \right). \] (3.23)
\( \dot{V}(x(t)) \) is negative definite if
\[ 1 - \lambda_{\max} \left( Q^{-\frac{1}{2}} E^T P A_1 Q^{-\frac{1}{2}} A_1^T P E Q^{-\frac{1}{2}} \right) > 0, \] (3.24)
which is satisfied if
\[ 1 - \sigma_{\max}^2 \left( Q^{-\frac{1}{2}} E^T P A_1 Q^{-\frac{1}{2}} A_1^T P E Q^{-\frac{1}{2}} \right) > 0. \] (3.25)
Using the properties of the singular matrix values, Amir-Moez (1956), the condition (3.25) holds if
\[ 1 - \sigma^{\frac{2}{2}}_{\max} \left( Q^{-\frac{1}{2}} E^T P \right) \sigma^{\frac{2}{2}}_{\max} \left( A_1 Q^{-\frac{1}{2}} \right) > 0, \] (3.26)
which is satisfied if
\[ 1 - \frac{\|A_1\|^2 \sigma^{\frac{2}{2}}_{\max} \left( Q^{-\frac{1}{2}} E^T P \right)}{\sigma^{\frac{2}{2}}_{\min} \left( \Omega^{-\frac{1}{2}} \right)} > 0, \] (3.27)
what completes Proof, Debeljković et al. (2007). Q.E.D.

**Remark 3.1** Equations (3.11-3.12) are, in modify form, taken from Owens, Debeljković (1985).
Remark 3.2 If the system under consideration is just ordinary time delay, e.g. \( E = I \), we have result identical to that presented in Tissir, Hmamed (1996).

Remark 3.3 Let us discuss first the case when the time delay is absent. Then the singular (weak) Lyapunov matrix equation (3.11) is natural generalization of classical Lyapunov theory. In particular

a) If \( E \) is nonsingular matrix, then the system is asymptotically stable if and only if \( A = E^{-1}A_0 \) Hurwitz matrix.

Equation (3.11) can be written in the form

\[
A^T E^T P E + E^T P E A = -Q
\]

with matrix \( Q \) being symmetric and positive definite, in whole state space, since then \( W_{k^*} = W_{k^*} = \mathbb{R} (E^{k^*}) = \mathbb{R}^n \).

In this circumstances \( E^T P E \) is a Lyapunov function for the system.

b) The matrix \( A_0 \) by necessity is nonsingular and hence the system has the form

\[
E_0 \dot{x}(t) = x(t), \quad x(0) = x_0.
\]

Then for this system (3.2-3.3) to be stable must hold also, and has familiar Lyapunov structure

\[
E_0^T P + PE_0 = -Q,
\]

where \( Q \) is symmetric matrix but only required to be positive definite on \( W_{k^*} \).

Remark 3.4 There is no need for system, given (3.2-3.3), to posses properties given in 3.2, since this is obviously guaranteed by demand that all smooth solutions \( x(t) \) evolve in \( W_{k^*} \).

In what follows we give an example to show the effectiveness of proposed method.

Example 3.1 Consider the linear continuous singular time delay system with matrices as follows:

\[
E = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad A_0 = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & -1 & -1
\end{pmatrix}, \quad A_1 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Based on the above presented procedure, one can easily find the following data, Debeljiković et. al.(1996.b):

\[
E = (\lambda E + A_0)^{-1} \quad \Rightarrow \quad \mathbb{R} (E^{k^*}) = \mathbb{R}^n
\]

\[
E = A_0^{-1} E = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 1 & 0
\end{pmatrix}, \quad \Rightarrow
\]

\[
E = A_0^{-1} E = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 1 & 0
\end{pmatrix}, \quad \Rightarrow
\]
\[ \hat{E}^D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \]

\[ (I - \hat{E}^D)(I - \hat{E}^D)x_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}x_0 = 0, \quad \Rightarrow \]

\[ (I - \hat{E}^D) = W_{k^*} = \{x : x_1 \in, \ x_2 \in, \ x_2 = -x_3 \} . \]

\[ \det A_0 \neq 0, \ \exists \lambda \mapsto \det (\lambda E - A_0) \neq 0, \ \text{rang} \ E = 2, \ \text{deg det} (sE - A_0) = 2 . \]

One can adopt \[ Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = Q^T > 0, \ S = S^T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{pmatrix}, \]

\[ (S + Q) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \Rightarrow \]

\[ x^T(t)Sx(t) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \]

\[ = (2x_1x_2 + 2x_1x_3 - 2x_2x_3 - x_3^2)x_2 = x_2 - x_3 \]

\[ = x_2^2 - x_3^2 > 0, \ \forall x(t) \in W_{k^*} \setminus \{0\} . \]

\[ x^T(t)Qx(t) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \]

\[ = (x_1^2 + x_2^2 + x_3^2)x_2 = x_1^2 + 2x_2^2 > 0, \ \forall x(t) \in W_{k^*} \setminus \{0\} . \]

Moreover we have \[ x^T(t)Qx(t) = x_1^2 + x_2^2 + x_3^2 > 0, \ \forall x(t) \in W_{k^*} \setminus \{0\} , \ \text{det} \ Q = 1 \neq 0 . \]

and

\[ x^T(t)(S + Q)x(t) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \]

\[ = (x_1^2 + x_2^2 + 2(x_2 + x_3) + 2x_3^2)x_2 = x_1^2 + 3x_2^2 > 0, \]

\[ x(t) \in W_{k^*} \setminus \{0\} . \]
Also, one can compute
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
p_{11} & p_{12} & p_{13} \\
p_{12} & p_{22} & p_{23} \\
p_{13} & p_{23} & p_{33}
\end{pmatrix}
\begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
p_{11} & p_{12} & p_{13} \\
p_{12} & p_{22} & p_{23} \\
p_{13} & p_{23} & p_{33}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
= -2 \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & -1 \\
1 & -1 & 0
\end{pmatrix}
\]

\[P = \begin{pmatrix}
1 & 0 & 2 \\
0 & 3 & -2 \\
2 & -2 & 6
\end{pmatrix}, \Rightarrow \Delta_3(p_{33}) > 0, \Rightarrow p_{33} > \frac{16}{3} \Rightarrow P = P^T > 0.
\]

Generally we can adopt
\[P = \begin{pmatrix}
1 & 0 & 2 \\
0 & 3 & -2 \\
2 & -2 & 6
\end{pmatrix} = P^T > 0.
\]

Finally, we have to check condition (3.13)
\[\|A_1\| = 0.10, \quad \sigma\{Q\} = \{1, 1, 1\},
\]

\[Q^{\frac{1}{2}} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad Q^{-\frac{1}{2}} E^T P = \begin{pmatrix}
-1 & -\frac{1}{2} & 1 \\
-\frac{1}{2} & -3 & 2 \\
0 & 0 & 0
\end{pmatrix}
\]

\[\sigma_{\min}(Q^{\frac{1}{2}}) = 1, \quad \sigma_{\max}(Q^{-\frac{1}{2}} E^T P) = 3.82
\]

\[0.10 = \|A_1\| < \frac{\sigma_{\min}(Q^{\frac{1}{2}})}{\sigma_{\max}(Q^{-\frac{1}{2}} E^T P)} < 0.26,
\]

so, the system under consideration is asymptotically stable.

For the sake of further investigation let us, for the previous case, adopt situation when \(E = I\).

Then one can calculate
\[\sigma\{A_0\} = \{\lambda_1, \lambda_2, \lambda_3\} = \{-1, -1, -1\}, \quad \lambda_{\max} = -1.
\]

\[\|A_0\| = \sigma_{\max}(A_0) = \sqrt{\lambda_{\max}(A_0^T A_0)}.
\]
\[ A_0 + A_0^T = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & -1 & -2 \end{pmatrix}, \]
\[ \lambda_i (A_0 + A_0^T) = \{-1, -2, -3\}, \]
\[ \mu(A_0) = \frac{1}{2} \lambda_{\text{max}}(A_0 + A_0^T) = -\frac{1}{2}. \]

Following Mori et al. (1981):
\[ \mu(A_0) + \|A_1\| < 0, \quad \iff \quad -0.5 + 0.10 = -0.4 < 0. \]

asymptotic stability of the non-delay system under consideration is confirmed. Moreover we have
\[ x^T(t) E^T P E x(t) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (x_1^2 + 3x_2^2)x_2 > 0, \quad \forall x(t) \in W_k \setminus \{0\} \]

so \( V(x(t)) \) can be used as a Lyapunov function for the system (3.2-3.3). A simple calculation, also shows:
\[ f(s) = (s + 1)(s + 1 - 0.1e^{-s}) = 0 \]
\begin{enumerate}
  \item \((s + 1) = 0 \quad \Rightarrow \quad s = -1, \)
  \item \((s + 1 - 0.1e^{-s}) = 0 \quad \Rightarrow \quad t = s + 1, \quad t - 0.1e^{-t} = 0, \quad te^t = e/10 \quad \Rightarrow \quad t = \text{lambert}(e/10) \)
  \quad \Rightarrow \quad s = t - 1 = \text{lambert}(e/10) - 1, \\
  \quad t = \sigma + j\omega = w_ne^{j\varphi}, \quad te^t = w_ne^{j\varphi}e^{\sigma+j\omega} = w_ne^{\varphi}e^{j(\omega+\varphi)} = e/10e^{2k\pi} \]
  \quad \sqrt{\sigma^2 + \omega^2}e^{\varphi} = e/10, \quad \varphi + \omega = 2k\pi, \quad k = 0, \pm 1, \pm 2 \ldots \\
  \quad k = 0 \quad \Rightarrow \quad t = 0.2185 \quad \Rightarrow \quad s = t - 1 = -0.7815 \\
  \quad k = \pm 1 \quad \Rightarrow \quad t = -2.9173 \pm j4.0932 \quad \Rightarrow \quad s = t - 1 = -3.9173 \pm j4.0932 \\
  \quad k = \pm 2 \quad \Rightarrow \quad t = -3.7267 \pm j10.6592 \quad \Rightarrow \quad s = t - 1 = -4.7267 \pm j10.6592 \\
\end{enumerate}

and so on.

4 Linear discrete descriptor systems

4.1 Some preliminaries

(LDDTDS) is described by
\[ E x(k+1) = A_0 x(k) + A_1 x(k-1), \quad (4.1) \]
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where \( x(k) \in \mathbb{R}^n \) is a state vector.
The matrix \( E \in \mathbb{R}^{n \times n} \) is a necessarily singular matrix, with property \( \text{rank} \ E = r < n \) and with matrices \( A_0 \) and \( A_1 \) of appropriate dimensions.
For a (LDDTDS), (4.1), we present the following definitions taken from, Xu et al. (2004).

**Definition 4.1** The (LDDTDS) is said to be regular if \( \det \ (z^2 E - zA_0 - A_1) \), is not identically zero.

**Definition 4.2** The (LDDTDS) is said to be causal if is regular and
\[
\deg (z^n \det (zE - A_0) - z^{-1}A_1) = n + \text{rang} \ E.
\]

**Definition 4.3** The (LDDTDS) is said to be stable if it is regular and \( \rho (E, A_0, A_1) \subset D(0, 1) \), where
\[
\rho (E, A_0, A_1) = \{ z \mid \det (z^2 E - zA_0 - A_1) = 0 \}.
\]

**Definition 4.4** The (LDDTDS) is said to be admissible if it is regular, causal and stable.

**Lemma 4.1** ? The (LDDTDS) is admissible if there exist a matrix \( Q > 0 \) and an invertible symmetric matrix \( P \) such that
i) \( E^T P E \geq 0 \)
ii) \( A_0^T P A_0 - E^T P E + A_1^T P A_1 (Q - A_1^T P A_1)^{-1} A_1^T P A_0 + Q < 0 \)
iii) \( Q - A_1^T P A_1 > 0 \),
Xu et al. (2004).

**Proof.** See, Xu et al. (2004).

### 4.2 Main results

**Theorem 4.1** Suppose that (LDDTDS) is regular and causal with system matrix \( A_0 \) being nonsingular, e.i. \( \det A_0 \neq 0 \).
Moreover, suppose matrix \( (Q - A_1^T P A_1) \) is regular.
The system (4.1) is asymptotically stable, independent of delay, if
\[
\| A_1 \| < \frac{\sigma_{\text{min}} \left( (Q - A_1^T P A_1)^{\frac{1}{2}} \right)}{\sigma_{\text{max}} \left( Q^{-\frac{1}{2}} A_0^T P \right)},
\]
(4.2)
and if there exist a symmetric positive definite matrix \( P \), being the solution of discrete Lyapunov matrix equation
\[
A_0^T P A_0 - E^T P E = -2 (S + Q),
\]
(4.3)
with matrices $Q = Q^T > 0$ and $S = S^T$, such that
\[ x^T (k) (S + Q) x (k) > 0, \quad \forall x (k) \in W_0^d \setminus \{0\}, \] (4.4)
is positive definite quadratic form on $W_0^d \setminus \{0\}$, $W_0^d$ being the subspace of consistent initial conditions for both time delay and non-time delay discrete descriptor system.

**Proof 4.1** Let us consider functional
\[ V (x (k)) = x^T (k) E^T P E x (k) + x^T (k - 1) Q x (k - 1). \] (4.5)
with matrices $P = P^T > 0$ and $Q = Q^T > 0$.

**Remark 4.1** Equations (4.3 - 4.4) are, in modify form, taken from Owens, Debeljković (1985).

Note that Lemma B1 and Theorem B1 indicates that
\[ V (x (k)) = x^T (k) E^T P E x (k), \] (4.6)
is positive quadratic form on $W_0^d$, and it is obvious that all solutions $x (k)$ evolve in $W_0^d$, so $V (x (k))$ can be used as a Lyapunov function for the system under consideration, Owens, Debeljković (1985).

It will be shown that the same argument can be used to declare the same property of another quadratic form present in (4.5).

Clearly, using the equation of motion of (4.1), we have
\[ \Delta V (x (k)) = V (x (k + 1)) - V (x (k)) = \]
\[ = x^T (k) (A_0^T P A_0 - E^T P E + Q) x (k) + 2x^T (k) (A_0^T P A_1) x (k - 1) - x^T (k - 1) Q x (k - 1) \]
\[ = x^T (k) (A_0^T P A_0 - E^T P E + 2Q + 2S) x (k) - x^T (k) Q x (k) - 2x^T (k) S x (k) + 2x^T (k) (A_0^T P A_1) x (k - 1) - x^T (k - 1) (Q - A_1^T P A_1) x (k - 1) \] (4.7)

From (4.3) and the inequality:
\[ 2x^T (k) A_0^T P A_1 x (k - 1) = \]
\[ = 2x^T (k) (A_0^T P A_1 (Q - A_1^T P A_1)^{-\frac{1}{2}} (Q - A_1^T P A_1)^{\frac{1}{2}}) x (k - 1) \]
\[ \leq x^T (k) A_0^T P A_1 (Q - A_1^T P A_1)^{-\frac{1}{2}} (Q - A_1^T P A_1)^{-\frac{1}{2}} A_0^T P A_1 x (k) \]
\[ + x^T (k - 1) (Q - A_1^T P A_1)^{\frac{1}{2}} (Q - A_1^T P A_1)^{\frac{1}{2}} x (k - 1) \] (4.8)

one can obtain
\[ \Delta V (x (k)) = -x^T (k) Q x (k) - x^T (k) S x (k) + x^T (k) \times \]
\[ \times (A_0^T P A_1 (Q - A_1^T P A_1)^{-\frac{1}{2}} (Q - A_1^T P A_1)^{-\frac{1}{2}} A_0^T P A_0) x (k) \]
\[ \leq -x^T (k) S x (k) \]
\[ -x^T (k) Q^{\frac{1}{2}} \left( I - Q^{-\frac{1}{2}} A_0^T P A_1 \times (Q - A_1^T P A_1)^{-1} A_1^T P A_0 Q^{-\frac{1}{2}} \right) Q^{\frac{1}{2}} x (k) \] (4.9)
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From the fact that the choice of matrix $S$, can be done, such that
\[ x^T(k) S x(k) \geq 0, \quad \forall x(k) \in W_d \setminus \{0\}, \tag{4.10} \]
and after some manipulations, (4.9) yields to
\[ \Delta V(x(k)) \leq -x^T(k) Q^{\frac{1}{2}} \Pi Q^{\frac{1}{2}} x(k), \tag{4.11} \]
with matrix $\Pi$ defined by
\[ \Pi = \left( I - Q^{-\frac{1}{2}} A_0^T P A_1 Q^{-\frac{1}{2}} Q^{-\frac{1}{2}} A_1^T P A_0 Q^{-\frac{1}{2}} \right). \tag{4.12} \]
and following the procedure presented in Tissir, Hmamed (1996), $V(x(k))$ is negative definite if
\[ 1 - \lambda_{\text{max}} \left( Q^{-\frac{1}{2}} A_0^T P A_1 (Q - A_1^T P A_1)^{-\frac{1}{2}} \times (Q - A_1^T P A_1)^{-\frac{1}{2}} A_1^T P A_0 Q^{-\frac{1}{2}} \right) > 0, \tag{4.13} \]
which is satisfied if
\[ 1 - \sigma_{\text{max}}^2 \left( Q^{-\frac{1}{2}} A_0^T P A_1 (Q - A_1^T P A_1) Q^{-\frac{1}{2}} \right) > 0. \tag{4.14} \]
Using the properties of the singular matrix values, Amir-Moez (1956), the condition (4.14) holds if
\[ 1 - \sigma_{\text{max}}^2 \left( Q^{-\frac{1}{2}} A_0^T P \right) \times \sigma_{\text{max}} \left( A_1 (Q - A_1^T P A_1) Q^{-\frac{1}{2}} \right) > 0, \tag{4.15} \]
which is satisfied if
\[ 1 - \|A_1\|^2 \sigma_{\text{max}}^2 \left( Q^{-\frac{1}{2}} A_0^T P \right) \sigma_{\text{min}}^2 \left( (Q - A_1^T P A_1)^{-\frac{1}{2}} \right) > 0, \tag{4.16} \]
what completes proof. Q.E.D.

In the sequel we give an example to show the effectiveness of proposed method.

**Example 4.1** Consider the linear discrete descriptor time delay system with matrices as follows:

\[
E = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0.10 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.10 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
Based on the above presented procedure, one can easily find the following data, Debeljković et. al. (1996.b):

\[ \dot{E} = (zE + A_0)^{-1}E = \dot{E} = A_0^{-1}E = \begin{pmatrix} -20 & 0 & 0 \\ 20 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \Rightarrow \]

\[ \Rightarrow \dot{E}^D = \begin{pmatrix} -0.05 & 0 & 0 \\ 0.05 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ (I - \dot{E}\dot{E}^D) = (I - \dot{E}\dot{E}^D)x_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}x_0 = 0, \Rightarrow \]

\[ (I - \dot{E}\dot{E}^D) = \{ \mathbf{x} \in \mathbb{R}^n : x_{10} + x_{20} = 0 \land x_{30} = 0 \}. \]

\[ \det A_0 \neq 0, \quad \text{rang } E = 1 \]

\[ \det (z^2E - zA_0 - A_1) = z(2z^2 - 0.01z - 1)(z - 0.10) \neq 0 \]

\[ \deg (z^n \det (zE - A_0 - z^{-1}A_1)) = n + \text{rang } E \Rightarrow \]

\[ n = 3, \quad \Rightarrow \deg (z^4) = 4, \quad n + \text{rank } E = 3 + 1 = 4 \]

One can adopt

\[ Q = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = Q^T > 0, \quad S = S^T = \begin{pmatrix} 0 & -2 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \]

\[ (S + Q) = \begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow \]

\[ x^T(k)Sx(k) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{pmatrix} 0 & -2 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (-2x_1x_2 - 2x_1x_2 - 3x_2^2 - 2x_3^2)x_{x=0} \]

\[ = x_1^2 > 0, \quad \forall x(k) \in \mathbb{R}_+^3 \setminus \{0\} \]

\[ x^T(k)Qx(k) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (x_1 + x_2)^2 + x_2^2 + x_3^2x_{x=0} \]

\[ = x_2^2 > 0, \quad \forall x(k) \in \mathbb{R}_+^3 \setminus \{0\} \]
Moreover we have
\[ x^T(k) Q x(k) = (x_1 + x_2)^2 + x_2^2 + x_3^2 > 0, \quad \forall x(k) \in \mathbb{R}^n, \]
\[ \det Q = 1 \neq 0, \]
and
\[ x^T(k) (S + Q) x(k) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (x_1^2 + 2x_2^2 - x_2^2 - x_3^2) \bigg|_{x_1 = -x_2} = x_1^2 + x_2^2 > 0, \quad \forall x(k) \in W^d_k \setminus \{0\} \]
Also, one can compute
\[ \begin{pmatrix} 0.10 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{pmatrix} \begin{pmatrix} 0.10 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \]
\[ = -2 (S + Q) = -2 \begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow \]
\[ P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{pmatrix} = \]
\[ = \begin{pmatrix} -3.99p_{11} + 0.2p_{12} + p_{22} & 0.1p_{12} + p_{22} & 0.1p_{13} + p_{23} \\ 0.1p_{12} + p_{22} & 0.1p_{13} + p_{23} & p_{23} \\ 0.1p_{13} + p_{23} & p_{23} & p_{33} \end{pmatrix} \]
with solution
\[ P = \begin{pmatrix} 0.501 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix} = P^T > 0. \]
Moreover we have
\[ x^T(k) E^T P E x(k) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{pmatrix} 0.501 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (0.501x_1^2 + 2x_2^2 + 2x_3^2) \bigg|_{x_1 = -x_2} = \]
\[ = (2.501x_1^2) \bigg|_{x_1 = -x_2} > 0, \quad \forall x(k) \in W^d_k \setminus \{0\} \]
so $V(x(k))$ can be used as a Lyapunov function for the system (4.1). Finally, we have to check condition (4.3)

$$\|A_1\| = 0.10 \quad \sigma \{Q\} = \{0.382, 1.00, 2.618\}$$

$$\Omega = Q - A_1^T P A_1 = \begin{pmatrix} 0.995 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_{\min} \left( (Q - A_1^T P A_1)^{\frac{1}{2}} \right) = 0.618, \quad \sigma_{\max} = \left( Q^{-\frac{1}{2}} A_0^T P \right) = 1.42$$

$$0,10 = \|A_1\| < \frac{\sigma_{\min} \left( (Q - A_1^T P A_1)^{\frac{1}{2}} \right)}{\sigma_{\max} \left( Q^{-\frac{1}{2}} A_0^T P \right)} = 0.4361,$$

so, the system under consideration is asymptotically stable. Moreover it should be noticed that there are eight different solutions for $Q^\frac{1}{2}$, $Q^{-\frac{1}{2}}$ and of course for expression $Q^{-\frac{1}{2}} A_0^T P$. But solutions for $\sigma_{\min} \left( Q^\frac{1}{2} \right)$ and $\sigma_{\max} = \left( Q^{-\frac{1}{2}} A_0^T P \right)$ are unique in all cases. In particular case, when $E = I$, we have result identical to that presented in Debeljković et al. (2005.c).

Now we investigate the general case, namely when one can let that basic system matrix is singular, e.g. $\det A_0 = 0$.

Note that this is impossible case for linear continuous singular system see, 3.1 and 3.2.

**Theorem 4.2** Suppose that (LDDTDS) is regular and causal. Moreover, suppose matrix $(Q_\lambda - A_1^T P_\lambda A_1)$ is regular, with $Q_\lambda = Q_\lambda^T > 0$.

The system (4.1) is asymptotically stable, independent of delay, if:

$$\|A_1\| < \frac{\sigma_{\min} \left( (Q - A_1^T P A_1)^{\frac{1}{2}} \right)}{\sigma_{\max} \left( Q^{-\frac{1}{2}} A_0^T P \right)}.$$  \hspace{1cm} (4.17)

and if there exist real positive scalar $\lambda^* > 0$ such that for all $\lambda$ within the range $0 < |\lambda| < \lambda^*$ there exist symmetric positive definite matrix $P_\lambda$, being the solution of discrete Lyapunov matrix equation:

$$(A_0 - \lambda E)^T P_\lambda (A_0 - \lambda E) - E^T P_\lambda E = -2 (S_\lambda + Q_\lambda),$$  \hspace{1cm} (4.18)

with matrix $S_\lambda = S_\lambda^T$, such that:

$$x^T(k) (S_\lambda + Q_\lambda) x(k) > 0, \quad \forall x(k) \in W^d, \{0\},$$  \hspace{1cm} (4.19)

is positive definite quadratic form on $W^d, \{0\}$, $W^d$ being the subspace of consistent initial conditions for both time delay and non-time delay discrete
Using the same procedure, as in the previous case, one can get:

Clearly, using the equation of motion of 4.1, we have property of another quadratic form, present in 4.20. It will be shown that the same argument can be used to declare the same under consideration, Owens, Debeljković (1985).

Proof 4.2 Let us consider functional

\[ V(x(k)) = x^T(k)E^TP_\lambda Ex(k) + x^T(k-1)Q_\lambda x(k-1). \]

(4.20)

with matrices \( P_\lambda = P_\lambda^T > 0 \) and \( Q_\lambda = Q_\lambda^T > 0 \).

Remark 4.2 Equations (4.18-4.19) are, in modified form, taken from Owens, Debeljković (1985).

Note that Lemma B1 and Theorem B1 indicates that:

\[ V(x(k)) = x^T(k)E^TP_\lambda Ex(k), \]

(4.21)

is positive quadratic form on \( W^d \), and it is obvious that all solutions \( x(k) \) evolve in \( W^d \), so \( V(x(k)) \) can be used as a Lyapunov function for the system under consideration, Owens, Debeljković (1985).

It will be shown that the same argument can be used to declare the same property of another quadratic form, present in 4.20.

Clearly, using the equation of motion of 4.1, we have

\[
\Delta V(x(k)) = V(x(k+1)) - V(x(k)) = \\
= x^T(k) \left( (A_0 - \lambda E)^T P_\lambda (A_0 - \lambda E) \right) x(k) \\
- x^T(k) \left( Q_\lambda - E^T P_\lambda E \right) x(k) + \\
2x^T(k) \left( (A_0 - \lambda E)^T P_\lambda A_1 \right) x(k-1) \\
- x^T(k-1) \left( Q_\lambda - A_1^T P_\lambda A_1 \right) x(k-1) \\
= x^T(k) \left( (A_0 - \lambda E)^T P_\lambda (A_0 - \lambda E) \right) x(k) \\
+ x^T(k) \left( 2Q_\lambda - E^T P_\lambda E + 2S_\lambda \right) x(k) - \\
- x^T(k) Q_\lambda x(k) - 2x^T(k) S_\lambda x(k) \\
+ 2x^T(k) \left( (A_0 - \lambda E)^T P_\lambda A_1 \right) x(k-1) \\
- x^T(k-1) \left( Q_\lambda - A_1^T P_\lambda A_1 \right) x(k-1)
\]

(4.22)

Using the same procedure, as in the previous case, one can get:

\[
2x^T(k) (A_0 - \lambda E)^T P_\lambda A_1 x(k-1) = \\
= 2x^T(k) \left( (A_0 - \lambda E)^T P_\lambda A_1 \left( Q_\lambda - A_1^T P_\lambda A_1 \right)^{-\frac{1}{2}} \left( Q_\lambda - A_1^T P_\lambda A_1 \right)^{\frac{1}{2}} \right) x(k-1) \\
\leq x^T(k) (A_0 - \lambda E)^T P_\lambda A_1 \left( Q_\lambda - A_1^T P_\lambda A_1 \right)^{-\frac{1}{2}} \left( Q_\lambda - A_1^T P_\lambda A_1 \right)^{\frac{1}{2}} A_1^T P_\lambda A_0 x(k) \\
+ x^T(k-1) \left( Q_\lambda - A_1^T P_\lambda A_1 \right)^{\frac{1}{2}} \left( Q_\lambda - A_1^T P_\lambda A_1 \right)^{-\frac{1}{2}} A_1^T P_\lambda A_0 x(k)
\]

(4.23)

so that:
\[
\Delta V(x(k)) = -x^T(k)Q\lambda x(k) - x^T(k)S_\lambda x(k)
\]
\[
+ x^T(k) \left( (A_0 - \lambda E)^T P_\lambda A_1 (Q_\lambda - A_1^T P_\lambda A_1)^{-\frac{1}{2}} \times (Q_\lambda - A_1^T P_\lambda A_1)^{-\frac{1}{2}} A_1^T P_\lambda A_0 \right) x(k)
\]
\[
\leq -x^T(k)S_\lambda x(k) - x^T(k)Q_\lambda^\frac{1}{2} \times
\]
\[
\times \left( I - Q_\lambda^{-\frac{1}{2}} (A_0 - \lambda E)^T P_\lambda A_1 \times (Q_\lambda - A_1^T P_\lambda A_1)^{-\frac{1}{2}} A_1^T P_\lambda (A_0 - \lambda E) Q_\lambda^{-\frac{1}{2}} \right) Q_\lambda^\frac{1}{2} x(k)
\]
\[
(4.24)
\]
From the fact that the choice of matrix \(S_\lambda\), can be done, such that:
\[
x^T(k)S_\lambda x(k) \geq 0, \quad \forall x(k) \in W^d_0 \setminus \{0\}, \quad (4.25)
\]
and after some manipulations, 4.24 yields to:
\[
\Delta V(x(k)) \leq -x^T(k)Q_\lambda^\frac{1}{2} \Upsilon_\lambda Q_\lambda^\frac{1}{2} x(k), \quad (4.26)
\]
with matrix \(\Upsilon_\lambda\) defined by:
\[
\Upsilon_\lambda = \left( I - Q_\lambda^{-\frac{1}{2}} (A_0 - \lambda E)^T P_\lambda A_1 Q_\lambda^{-\frac{1}{2}} Q_\lambda^{-\frac{1}{2}} A_1^T P_\lambda (A_0 - \lambda E) Q_\lambda^{-\frac{1}{2}} \right) \quad (4.27)
\]
and following the procedure presented in Tissir, Hnaied (1996), \(V(x(k))\) is negative definite if
\[
1 - \lambda_{\text{max}} \left( Q_\lambda^{-\frac{1}{2}} (A_0 - \lambda E)^T P_\lambda A_1 Q_\lambda^{-\frac{1}{2}} Q_\lambda^{-\frac{1}{2}} A_1^T P_\lambda (A_0 - \lambda E) Q_\lambda^{-\frac{1}{2}} \times (Q_\lambda - A_1^T P_\lambda A_1)^{-\frac{1}{2}} A_1^T P_\lambda (A_0 - \lambda E) Q_\lambda^{-\frac{1}{2}} \right) > 0 \quad (4.28)
\]
which is satisfied if
\[
1 - \sigma_{\text{max}}^2 \left( Q_\lambda^{-\frac{1}{2}} (A_0 - \lambda E)^T P_\lambda (Q_\lambda - A_1^T P_\lambda A_1) Q_\lambda^{-\frac{1}{2}} \right) > 0 \quad (4.29)
\]
Using the properties of the singular matrix values, Amir-Moez (1956), the condition 4.29 holds if:
\[
1 - \sigma_{\text{max}}^2 \left( Q_\lambda^{-\frac{1}{2}} (A_0 - \lambda E)^T P_\lambda \times A_1^T (Q_\lambda - A_1^T P_\lambda A_1) Q_\lambda^{-\frac{1}{2}} \right) > 0, \quad (4.30)
\]
which is satisfied if:
\[
1 - \frac{\|A_1\|^2 \sigma_{\text{max}}^2 \left( Q_\lambda^{-\frac{1}{2}} (A_0 - \lambda E)^T P_\lambda \right)}{\sigma_{\text{min}}^2 \left( (Q_\lambda - A_1^T P_\lambda A_1)^{-\frac{1}{2}} \right)} > 0, \quad (4.31)
\]
what completes proof. \(Q.E.D.\)
In particular case, when $E = I$ and $\det A_0 \neq 0$, we have result identical to that presented in Debeljković et al. (2005c).

In particular case, when $\det A_0 \neq 0$, we have result identical to that presented in Debeljković et al. (2006b).

**Example 4.2** Consider the linear discrete descriptor time delay system with matrices as follows:

\[
E = \begin{pmatrix}
-1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad A_0 = \begin{pmatrix}
0 & 0 & 0 \\
-1 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad A_1 = \begin{pmatrix}
\frac{1}{10} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

In comparison with the ???, it should be underlined that here we have fact that matrix $A_0$ singular, e.g. $\det A_0 = 0$.

Based on the above presented procedure, one can easily find the following data, Debeljković et al. (2006b):

\[
\dot{E} = (zE + A_0)_{z=1}^{-1} \cdot E = \dot{E}_{z=1} = I \cdot E = \begin{pmatrix}
-1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \Rightarrow \dot{E}^D = \begin{pmatrix}
-1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
\mathbb{N} \left( I - \dot{E} \dot{E}^D \right) = \left( I - \dot{E} \dot{E}^D \right) x_0 = \begin{pmatrix}
0 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} x_0 = 0, \quad \Rightarrow
\]

\[
\mathbb{N} \left( I - \dot{E} \dot{E}^D \right) = \{ x \in \mathbb{R}^n : x_{10} + x_{20} = 0 \land x_{30} = 0 \}.
\]

\[
\det A_0 = 0, \quad \rang E = 1,
\]

**Def. 1**

\[
\det (z^2 E - z A_0 - A_1) = -z (z^2 + 0.10) \neq 0,
\]

\[
\det (z E - A_0 - z^{-1} A_1) = - (z - \frac{0.10}{z})
\]

**Def. 2**

\[
\deg \left( z^n \det (z E - A_0 - z^{-1} A_1) \right) = n + \rang E \Rightarrow
\]

\[n = 3, \quad \Rightarrow \deg (z^4) = 4, \quad n + \rang E = 3 + 1 = 4 \]
One can adopt:

\[ Q_\lambda = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = Q_\lambda^T > 0, \]

\[ S_\lambda = S_\lambda^T = \begin{pmatrix} 0 & -2 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad S_\lambda + Q_\lambda = \begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

\[ x^T (k) S_\lambda x (k) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} 0 & -2 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\]

\[ = ( -2x_1x_2 - 2x_1x_2 - 3x_2^2 - 2x_3^2 )_{x_1=0, x_2=0} = x_1^2 > 0, \quad \forall x (k) \in W^d_k \setminus \{0\} \]

\[ x^T (k) Q_\lambda x (k) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\]

\[ = ( (x_1 + x_2)^2 + x_2^2 + x_3^2 )_{x_1=0, x_2=0} = x_2^2 > 0, \quad \forall x (k) \in W^d_k \setminus \{0\} \]

Moreover we have:

\[ x^T (k) Q_\lambda x (k) = (x_1 + x_2)^2 + x_2^2 + x_3^2 > 0, \quad \forall x (k) \in \mathbb{R}^n \]

\[ \det Q = 1 \neq 0, \]

and:

\[ x^T (k) (S_\lambda + Q_\lambda) x (k) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\]

\[ = (x_1^2 + 2x_2^2 - x_2^2 - x_3^2)_{x_1=0, x_2=0} = x_1^2 + x_2^2 > 0, \quad \forall x (k) \in W^d_k \setminus \{0\} \]

Also, one can compute:

\[ (A_0 - \lambda E)^T_{\lambda=\lambda} P_{\lambda} (A_0 - \lambda E)_{\lambda=\lambda} =
\]

\[ = \begin{pmatrix} 4p_{11} - 12p_{12} + 9p_{22} & 3p_{22} - 2p_{12} & 3p_{23} - 2p_{13} \\ 3p_{22} - 2p_{12} & p_{22} & p_{23} \\ 3p_{23} - 2p_{13} & p_{23} & p_{33} \end{pmatrix} \]
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\[
E^T P_\lambda E = \begin{pmatrix}
p_{11} - 2p_{12} + p_{22} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

so finally, we have:

\[
\begin{pmatrix}
4p_{11} - 14p_{12} + 8p_{22} & 3p_{22} - 2p_{12} & 3p_{23} - 2p_{13} \\
3p_{22} - 2p_{12} & p_{22} & p_{23} \\
3p_{23} - 2p_{13} & p_{23} & p_{13}
\end{pmatrix} =
\begin{pmatrix}
-2 & 2 & 0 \\
2 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}
\]

with solution:

\[
P_\lambda = \begin{pmatrix}
\frac{10}{3} & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix} = P_\lambda^T > 0.
\]

Moreover we have:

\[
x^T(k) E^T P_\lambda E x(k) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} \frac{10}{3} & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix} \begin{pmatrix} x_1 \\
x_2 \\
x_3
\end{pmatrix}
\]

\[
= \left(100x_1^2 + 2x_2^2 + 2x_3^2\right)_{x_1 = -x_2 = x_3 = 0} = \left(3.11x_1^2\right)_{x_1 = -x_2 > 0, \forall x \in \mathbb{R}_d}
\]

so \(V(x(k))\) can be used as a Lyapunov function for the system 4.1.

Finally, we have to check condition ??

\[
\|A_1\| = 0.10 \quad \sigma \{Q_\lambda\} = \{0.382, 1.00, 2.618\}
\]

\[
\Omega_\lambda = Q_\lambda - A_1^T P_\lambda A_1 = \begin{pmatrix}
0.997 & 1 & 0 \\
1 & 2 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
\sigma_{\min} \left(\left(Q_\lambda - A_1^T P_\lambda A_1\right)^{\frac{1}{2}}\right) = 0.616, \quad \sigma_{\max} = \left(Q_\lambda^{\frac{1}{2}} A_1^T P_\lambda\right) = 1.422
\]

\[
0, 10 = \|A_1\| < \frac{\sigma_{\min} \left(\left(Q_\lambda - A_1^T P_\lambda A_1\right)^{\frac{1}{2}}\right)}{\sigma_{\max} \left(Q_\lambda^{\frac{1}{2}} A_1^T P_\lambda\right)} = 0.481,
\]

so, the system under consideration is asymptotically stable.

The same conclusion can be derived directly from the locations of roots of characteristic equation.
5 Conclusion

A quite new sufficient delay–independent criteria for asymptotic stability of (LDDTDS) is presented. In some sense this result may be treated as the further extension of results derived in Debeljković et. al (2006.b).

In that sense it seems as the extension of weak Lyapunov equation (4.3) and 4.18 to discrete descriptor time delay systems.

One should say that this approach gives an insight in the structure of singular continuous and discrete descriptor systems under consideration, so it may be called: a geometric approach to the Lyapunov stability of this particular class of systems.

In comparison with some other papers on this matter, there is no need for linear transformations of basic system, as well there is no need of solving the systems of high order linear matrix inequalities.

Two numerical examples are presented to show the applicability of results derived

6 Appendix A

The fundamental geometric tool in the characterization of the subspace of consistent initial conditions, for linear singular system without delay, is the subspace sequence

\[ W_0 = \mathbb{R}^n \]

\[ : \]

\[ W_{j+1} = A_0^{-1}(EW_j), \quad j \geq 0, \]

where \( A_0^{-1}(\cdot) \) denotes inverse image of \((\cdot)\) under the operator \( A_0 \).

**Lemma 6.1** The subsequence \( \{W_0, W_1, W_2, \ldots\} \) is nested in the sense that:

\[ W_0 \supset W_1 \supset W_2 \supset W_3 \supset \ldots \]

Moreover:

\[ \mathbb{N}(A) \subset W_j, \quad \forall j \geq 0, \]

and there exist an integer \( k \geq 0 \), such that:

\[ W_{k+j} = W_k. \]

Then it is obvious that:

\[ W_{k+j} = W_k, \quad \forall j \geq 1. \]
If \( k^* \) is the smallest such integer with this property, then:

\[
W_k \cap \mathbb{N}(E) = \{0\}, \quad k \geq k^*,
\]

(6.7)

provide that \((\lambda E - A_0)\) is invertible for some \( \lambda \in \mathbb{R} \).

**Theorem 6.1** Under the conditions of 6.1, \( x_0 \) is a consistent initial condition for the system under consideration if and only if \( x_0 \in W_{k^*} \).

Moreover \( x_0 \) generates a unique solution \( x(t) \in W_{k^*}, \ t \geq 0 \), that is real analytic on \( \{t : t \geq 0\} \).

**Proof 6.1** See Owens, Debeljković (1985).

## 7 Appendix B

The fundamental geometric tool in the characterization of the subspace of consistent initial conditions, for linear discrete descriptor system without delay 4.1, is the subspace sequence

\[
W^d_0 = \mathbb{R}^n,
\]

(7.1)

\[
W^d_{j+1} = A^{-1}_0 (E W^d_j), \quad j \geq 0,
\]

(7.2)

where \( A^{-1}_0 (\cdot) \) denotes inverse image of (\( \cdot \)) under the operator \( A_0 \).

**Lemma 7.1** 7.1 is identical to the 6.1.

One should, only, change \( W_0 = W^d_0, \ W_1 = W^d_1, \ldots, \ W_k = W^d_k \).

Final result follows as:

\[
W^d_k \cap \mathbb{N}(E) = \{0\}, \quad k \geq k^*,
\]

(7.3)

provide that \((z E - A_0)\) is invertible for some \( z \in \mathbb{C} \).

**Proof 7.1** See Owens, Debeljković (1985).

**Theorem 7.1** Under the conditions of 7.1, \( x_0 \) is a consistent initial condition for the system under consideration, e.g. linear discrete singular system without delay if and only if \( x_0 \in W^d_{k^*} \).

Moreover \( x_0 \) generates a discrete solution sequence \( (x(k) : k \geq 0) \) such that \( x(k) \in W^d_k, \ \forall k \geq 0 \).

**Proof 7.2** See Owens, Debeljković (1985).
Appendix C

Definition 8.1 Let \( E \) be a square matrix, if there exists a matrix \( E^D \) satisfying:
1. \( E E^D = E^D E \)
2. \( E^D E E^D = E^D \) (8.1)
3. \( E^{\varphi+1} E^D = E^\varphi \)

we call the Drazin inverse matrix of matrix \( E \), simply \( D \) - inverse matrix. \( \varphi \) is the index of the matrix \( E \), it is the smallest nonnegative integer which makes:
\[ \text{rank } E^{\varphi+1} = \text{rank } E^\varphi, \] (8.2)

be true.

Lemma 8.1 ?? For any square matrix \( E \), its Drazin inverse matrix is existent and unique.

If the Jordan normalized form of \( E \) is
\[ E = T \begin{pmatrix} J & 0 \\ 0 & N \end{pmatrix} T^{-1} \] (8.3)
then:
\[ E^D = T \begin{pmatrix} J^{-1} & 0 \\ 0 & 0 \end{pmatrix} T^{-1} \] (8.4)

Here \( N \) is a nilpotent matrix, and \( T \) are invertible matrices.

9 References


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