RECURSIVE REALIZATION OF THE ROBUST STFT

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ABSTRACT

An efficient procedure is proposed for the realization of robust short-time Fourier transform (STFT) based on the recursive realization of the standard STFT. In the initial stage it uses the robust STFT calculated by iterative or sorting procedures and, after that, the recursive relation is applied. Basic difference from the standard STFT realization is in limiting samples in the next instants, since they can be corrupted by an impulse noise. It results in a numerically efficient form of the STFT, keeping robustness to the impulse noise influence.

1. INTRODUCTION

The robust STFT has been proposed recently in order to produce an accurate estimate of a non-noisy signal's STFT from the noisy observations. It behaves well for the signal embedded in an impulse noise where the standard STFT's performance is poor. In the Gaussian noise environment, the robust STFT is only slightly worse than its standard counterpart. Three forms of the robust STFT are defined so far. The robust M-STFT that can be realized by using an appropriate iterative procedure is proposed in [1]. The marginal-median and the L-filter forms are introduced in [2] and [3]. They can be implemented by using sorting algorithms. All these forms are computationally demanding, since the corresponding procedures should be applied for each point in the time-frequency (TF) plane. A recursive procedure for realization of the robust STFT is proposed in this paper. In the initial instant, the robust STFT is calculated by using some of the previously mentioned procedures. The robust STFT in the next instant is calculated from its value in the previous one, and the signal estimate in the next instant. The recursive relationship developed in the standard STFT case [4] cannot be applied directly here, since the new sample can be corrupted by the impulse noise. In order to avoid this case, we limited value of the new sample. In this way, we obtained a robust STFT form with very similar performance to the previously mentioned transforms and with significant calculation savings.

2. REVIEW OF THE ROBUST STFT

The standard STFT is a very important tool for the TF analysis since it has very simple form and higher-order TF distributions can be realized by using the STFT in the initial stage [5]-[11]. It can be defined as:

$$STFT(n, k) = \arg \min_{\mu} \sum_{m=0}^{N-1} F(x(n + m) \exp(-j2\pi km/N) - \mu),$$  \hspace{1cm} (1)

where \(x(n) = f(n) + \nu(n)\) is the observation containing useful signal \(f(n)\) and white noise \(\nu(n)\), with the loss function \(F(e) = |e|^2\):

$$STFT(n, k) = \frac{1}{N} \sum_{m=0}^{N-1} x(n + m) \exp(-j2\pi km/N)$$

$$= \text{mean}\{x(n + m) \exp(-j2\pi km/N), m \in [0, N]\}.$$  \hspace{1cm} (2)

Like many other TF transforms, the STFT is sensitive to the impulse noise influence. Several methods for solving minimization problems like (1) for signals embedded in the impulse noise are proposed in robust statistics. In many practical situations, introducing the loss function \(F(e) = |e|^2\) gives satisfactory results. It has motivated the introduction of this loss function for the problem given by (1). Due to the complex-valued nature of the loss function argument, \(x(n + m) \exp(-j2\pi km/N) - \mu\), the solution can be represented in the implicit form [1]:

$$STFT_M(n, k) = \frac{1}{N} \sum_{m=0}^{N-1} \frac{1}{\rho |x(n + m) \exp(-j2\pi km/N) - STFT(n, k)|}$$

$$\times \sum_{m=0}^{N-1} x(n + m) \exp(-j2\pi km/N)$$

$$= \text{median}\{|\text{Re}\{x(n + m) \exp(-j2\pi km/N)\}, m \in [0, N]\}\} + j \text{median}\{|\text{Im}\{x(n + m) \exp(-j2\pi km/N)\}, m \in [0, N]\}\}.$$  \hspace{1cm} (3)

The \(STFT_M(n, k)\) can be determined by an iterative procedure [1]. In order to avoid iterative procedures the loss function \(F(e) = |\text{Re}\{e\}| + |\text{Im}\{e\}|\) is introduced in [2]. It produces the marginal-median form of the robust STFT:

$$STFT_M(n, k) = \text{median}\{|\text{Re}\{x(n + m) \exp(-j2\pi km/N)\}, m \in [0, N]\}\} + j \text{median}\{|\text{Im}\{x(n + m) \exp(-j2\pi km/N)\}, m \in [0, N]\}\}.$$  \hspace{1cm} (4)

For a mixture of the impulse and Gaussian noise robust statistics proposes usage of the L-filters. The L-filter form of the robust TF distributions is introduced in [3]. Since other two robust STFT forms can produce spectral distortion due to their nonlinearity, the L-filter forms that can reduce this...
effect, are even more important here. The L-filter form of the robust STFT is defined as:

\[ STFT_l(n, k) = \sum_{i=0}^{N-1} a_l r_i(n, k) + j i_i(n, k), \]

where \( r_i(n, k) \) and \( i_i(n, k) \) are elements from the sets \( R(n, k) = \{ \Re \{ x(n+m) \exp(-j2\pi km/N) \}, m \in [0, N] \} \) and \( I(n, k) = \{ \Im \{ x(n+m) \exp(-j2\pi km/N) \}, m \in [0, N] \} \), respectively, sorted into a non-decreasing order.

\[ r_i(n, k) \leq r_{i+1}(n, k), \quad i_i(n, k) \leq i_{i+1}(n, k). \]

The L-filters are usually generated in such a way to produce the unbiased estimate of the input signal [12]. This condition holds for:

\[ \sum_{i=0}^{N-1} a_l = 1 \quad \text{and} \quad a_l = a_{N-1-l} \quad l \in [0, N]. \]

The \( \alpha \)-trimmed version of the L-filters will be used in this paper:

\[ a_l = \begin{cases} \frac{1}{N-2|\alpha|} & l \in [(N-2)|\alpha|, (2-N)|\alpha|+N-1] \\ 0 & \text{elsewhere,} \end{cases} \]

where \( \alpha \in [0, 0.5] \). Note that the \( \alpha \)-trimmed form for \( \alpha = 0 \) reduces to the standard STFT, while for \( \alpha = 0.5 \) it reduces to the robust STFT in the marginal-median filter form. All three presented robust STFT forms are very demanding for calculation since the iterative or sorting procedures should be performed for each point in the TF plane. This is the reason for proposing the recursive realization of the robust STFT in the next section.

### 3. Recursive Realization

It is well known that the standard STFT can be calculated recursively in the following way [4]:

\[ STFT(n+1, k) = \frac{1}{N} x(n+N) - \frac{1}{N} x(n) + STFT(n, k) e^{2\pi i k/N}. \]

For its realization it is necessary to know the STFT from the previous instant as well as the new sample \( x(n+N) \) (sample \( x(n) \) is already known since it is required for calculation of the STFT in the previous instant). The robust STFT is a very accurate estimate of the standard STFT for the impulse noise environment. Naturally, its realization can be done by using the recursive relation (9). Since the signal can be corrupted by the impulse noise, direct application of (9) can produce inaccurate results. Therefore, instead of the signal samples \( x(n+N) \) and \( x(n) \), their estimates, determined by the inverse DFT, will be used:

\[ \hat{f}(n+m) = N \sum_{i=0}^{N-1} STFT_{\Delta}(n, k) e^{2\pi i km/N}, \]

where \( STFT_{\Delta}(n, k) \) represents any of the robust forms (3), (4) or (5), and \( \Delta \) can be any of \( M, L \) or \( R \). Based on this relationship we obtained estimation of the noise-free signal \( f(n+m) \) for \( m \in [0, N] \). Estimation of \( f(n+N) \) can be performed based on the comparison of value \( x(n+N) \) with signal estimates \( \hat{f}(n+m) \) from the previous window, \( m \in [0, N] \). If the sample \( x(n+N) \) is larger than the maximal signal within the previous window it can be assumed as unreliable for determination of the STFT at the new instant. Then, the previous sample \( \hat{f}(n+N-1) \) can be used as the estimate of \( \hat{f}(n+N) \), \( \hat{f}(n+N) = \hat{f}(n+N-1) \), under the assumption that spectral content cannot be changed significantly within that interval. Based on the mentioned facts, the algorithm for recursive realization of the robust STFT reads:

Step 1: Calculation of the robust STFT \( STFT_{\Delta}(n, k) \) in the initial instant, by using one of the procedures (3), (4) or (5). Set \( R = 0, P = 0 \).

Step 2: Calculation of the inverse DFT based on the robust STFT:

\[ \hat{f}(n+m) = N \sum_{k=0}^{N-1} STFT_{\Delta}(n, k) e^{2\pi i km/N}, \]

where \( \hat{f}(n+m) \) is the maximal value of \( \hat{f}(m) \) within the interval:

\[ f_{\max} = \max\{|\hat{f}(n+m)|, m \in [0, N]\}. \]

Step 3: Calculation of the maximal value of \( \hat{f}(m) \) within the interval:

\[ f_{\max} = \max\{|\hat{f}(n+m)|, m \in [0, N]\}. \]

Step 4: If the following condition holds:

\[ |x(n+N)| \leq (1+\eta) f_{\max} \]

then \( f(n+N) = x(n+N) \) and \( P = 0 \); otherwise \( \hat{f}(n+N) = \hat{f}(n+N-1) \) and \( P = P + 1 \).

Step 5: Recursive calculation of the robust STFT in the next instant:

\[ STFT_{\Delta}(n+1, k) = \frac{1}{N} \hat{f}(n+N) - \frac{1}{N} \hat{f}(n) + STFT_{\Delta}(n, k) e^{2\pi i N} \]

and \( R = R + 1, n = n + 1 \).

Step 6: If \( R \leq R_{\max} \) and \( P \leq P_{\max} \), go to Step 3, otherwise go to Step 1 and recalculate the robust STFT by using the iterative or sorting procedures.

Comments on the algorithm. It can happen that the true new sample \( f(n+N) \) is larger than the maximal sample of the noise-free signal within the previous window \( f_{\max} \). In order to avoid this, we set larger value than \( f_{\max} \) in (13), \( \eta \geq 0 \). In our numerical calculation \( \eta = 0.25 \) is used. In order to avoid accumulation of the discretization error in the registers, recursive realization of the standard STFT needs recalculation by using the FFT techniques after relatively large number of samples. It should be done in the recursive realization of the robust STFT as well. However, since the recursive robust STFT is not equal to any of the directly calculated forms, we set number of samples for recursive calculation of the robust STFT relatively small, \( R_{\max} = 32 \). Furthermore, it can happen that there is no signal within a wide interval, and that \( f_{\max} \) is close to zero. Then, a new
The second row are for the marginal median STFT, while

\[ E \text{ussian noise with unit variance } \mathcal{N}(\mu_1(n) | \sigma_1(n)) = \delta(i - j). \]

This noise is used as a model of the impulse environment in [1]. In all examples, the STFT with \( N = 256 \) is considered.

The STFT forms for these signals are depicted in Figures 1-3, respectively. The left columns represent the case of small noise \( \alpha = 0.1 \), while the right ones depict higher impulse noise \( \alpha = 0.8 \). The first rows present the standard STFTs, the second rows are for the marginal-median STFT, while

the \( \alpha \)-trimmed form is depicted in the third rows. The recursive STFT (with the \( \alpha \)-trimmed form in the initial stage) is shown in the fourth rows. It can be seen that the robust forms behave similarly and that now the signal components can easily be recognized even in a high impulse noise environment. We have performed statistical study for noise amounts within a range \( \alpha \in [0, 2] \). As a measure of the STFT forms quality the mean squared difference between the STFT forms of the noisy signal and the standard STFT of the non-noisy signal is used. This measure is shown in Fig. 4 for the standard and robust forms of the STFT. It can be easily concluded that the robust forms behave better than the standard one, when the amount of noise increases. However, both recursive robust STFTs, calculated with L-filter and marginal-median forms as initial representations, behave almost the same like as initial forms evaluated for each point in the TF plane.

**Figure 1:** Signal \( f_1(t) \): Left column - small noise \( \alpha = 0.1 \); Right column - impulse noise \( \alpha = 0.8 \); First row - Standard STFT; Second row - Marginal median form; Third row - \( \alpha \)-trimmed form for \( \alpha = 3/8 \); Fourth row - Recursive STFT calculated with \( \alpha \)-trimmed form in the initial stage.

**Figure 2:** Signal \( f_2(t) \): Left column - small noise \( \alpha = 0.1 \); Right column - impulse noise \( \alpha = 0.8 \); First row - Standard STFT; Second row - Marginal median form; Third row - \( \alpha \)-trimmed form for \( \alpha = 3/8 \); Fourth row - Recursive STFT calculated with \( \alpha \)-trimmed form in the initial stage.

4. **NUMERICAL EXAMPLES**

We have considered three signals:

\[
f_1(t) = \exp(j48\pi t^2) \tag{15}
\]

\[
f_2(t) = \exp(-24\pi t^2 - j96\pi t) \exp(-16(t + 0.6)^2) + \exp(-j48\pi t^2 - j96\pi t) \exp(-16(t - 0.6)^2) \tag{16}
\]

\[
f_3(t) = 2\exp(-j48\pi t^2/2) \exp(-8(t + 0.6)^2) + \exp(j48\pi t) \tag{17}
\]

within \( t \in [-1/2, 1/2] \), with sampling rate \( \Delta t = 1/256 \). They are corrupted by noise \( \nu(n) = (a_1(n))^{\alpha} + j(a_2(n))^{\alpha} \), where \( a(n), i = 1, 2 \) are mutually independent white Gaussian noises with unit variance \( E[\nu(n)\nu^*(n)] = \delta(\cdot - j) \).

5. **CONCLUSION**

The recursive realization of the robust STFT is proposed. The marginal-median or L-filter forms are used in the initial stage, while in the consecutive instants the recursive relation is applied. The recursive realization produces the same accuracy as the calculation for each point in the TF plane but with significant calculation savings. The proposed transform is tested on several numerical examples.

**ACKNOWLEDGMENTS** - Work of A. Ohsumi is supported by Ministry of Education, Culture, Sports, Science
and Technology under Grant (C) I3650069. The work of LJ. Stanković and I. Djurović is supported by the Volkswagen Stiftung, Federal Republic of Germany.

6. REFERENCES


