Design of first-order differentiator utilising FIR and IIR sub-filters

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Abstract: In this paper, we propose a novel approach for constructing the first-order differentiator. It has been shown recently that an efficient Finite Impulse Response (FIR) wideband differentiator can be synthesised when applying the multirate and Frequency-Response Masking (FRM) techniques. The overall signal processing task is shared between a very low-order differentiator and a FIR half-band filter. The novelty of this paper is the usage of an approximately linear-phase IIR half-band filter based on the parallel connection of all-pass sub-filter and pure delay. The structure of resulting wideband differentiator consists of the approximately linear phase all-pass sub-filter, two low-order FIR polyphase sub-filters, and the delay element. The knowledge-based design has been used for the analysis and improvements of the wideband differentiator. The proposed system provides computational savings and exhibits a significantly smaller overall delay if compared with the corresponding solution based on FIR filters only.

Keywords: all-pass filter; approximately linear phase IIR filter; differentiator; half-band filter; multirate filtering.


Biographical notes:
Ljiljana D. Milić received the Dipl.-Eng., MSc and DSc degrees from the University of Belgrade in 1962, 1973 and 1978, respectively, all in Electrical Engineering. Her current research interest is in digital filters, multirate signal processing, and in communication applications of digital signal processing. She published more than 150 scientific papers and several book chapters all in the field of electrical engineering and digital signal processing. She is author of the book ‘Multirate Filtering for Digital Signal Processing: MATLAB Applications’. She is a senior member of IEEE and a member of The Academy of Engineering Sciences of Serbia.

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1 Introduction

Digital first-order differentiators are used to compute first-order derivatives of a continuous-time signal at discrete-time instances. The transfer function of a digital differentiator can be designed as a Finite Impulse Response (FIR) digital filter with the anti-symmetrical impulse response (Oppenheim and Schafer, 1989). To obtain the finite-order transfer function some don’t-care bands in the frequency domain should be specified. When the wideband differentiator is needed the don’t-care bands become narrow, and consequently, the transfer function order should be increased.

The application of multirate and Frequency-Response Masking (FRM) technique in the synthesis of efficient wideband differentiators has been proposed recently (Sheikh and Johansson, 2009; Sheikh and Johansson, 2011). The goal of the method proposed by Sheikh and Johansson (2009) and Sheikh and Johansson (2011) was to share the task of the wideband differentiator between two transfer functions: the differentiator designed for a half of the base-band, and the sharp half-band filter. The benefit of this approach is that the requirements for the differentiator transfer function are relaxed significantly. Solutions given in the work of Sheikh and Johansson (2009) and Sheikh and Johansson (2011) are based on FIR sub-filters and are suitable for wideband differentiators. However, we intend to investigate the usage of approximately linear-phase IIR half-band filter in order to decrease the overall delay of the system.

This paper is aimed to present a solution for the first-order differentiator which incorporates IIR and FIR sub-filters. We show that instead of using a linear-phase FIR half-band filter as shown in the work of Sheikh and Johansson (2009) and Sheikh and Johansson (2011), an approximately linear-phase IIR half-band filter can be used. We develop the structure of the overall differentiator composed of an approximately linear-phase all-pass filter, FIR polyphase sub-filters, and a pure delay element. We show that a wideband differentiator can be constructed by making use of an approximately linear-phase IIR sub-filter. Therefore, we introduce the IIR sub-filter into differentiator structure.

The design of high performance filters is heavily based on mathematical derivations, and the derivation can be simplified using computer algebra tools, such as in Pavlović et al.'s (2012) and Mladenović et al.'s (2012) study. Knowledge for filter synthesis can be accessible as script code so that user can modify, improve, optimise commands for filter synthesis, such as software described in the work of Lutovac et al. (2000) and Lutovac and Tošić (2010). Numeric only software tools can be usually used for testing and simulating real-time processing.

As a step towards the automatic understanding of electronic circuits, a logic grammar as a method of knowledge representation for structure and function of electronic circuits is presented by Tanaka (2009) and Tanaka (2011). A circuit is viewed as a sentence and its elements as words, but also it allows deriving the meaning of a given circuit as relationships between its syntactic structure and basic circuit functions.

This means that computer algebra systems can be used for embedding knowledge in a form of circuit description (as automatically generating digital filter structure based on known number of filter coefficient or filter order). In this way, more complex solutions can be designed which cannot be manually derived. Accordingly, an efficient tool is obtained for investigating properties of the DSP systems, and directly influence towards the system improvements as presented in the paper.

This paper is divided in six major sections. In Section 2, the principal structure of the wideband differentiator is presented. Section 3 presents through examples two approaches for designing the wideband differentiator: solution based on the low-order differentiator and IIR half-band filter (IIR/FIR), and solution based on the low-order differentiator and FIR half-band filter (FIR/FIR). In Section 4, we compare characteristics of the IIR/FIR wideband differentiator with an example in the work of Sheikh and Johansson (2009) and Sheikh and Johansson (2011) presenting an optimised FIR/FIR solution based on the FRM approach. In Section 5, we introduce the knowledge-based design for the analysis and improvements of the IIR/FIR wideband differentiator which followed Section 6 with conclusion.

2 Differentiator structure

2.1 Linear-phase FIR differentiator

The frequency response of an ideal causal linear-phase first-order differentiator \( H_D(j\omega) \) is given by Oppenheim and Schafer (1989)

\[
H_D(j\omega) = (j\omega)e^{-j\omega/2}, \quad -\pi < \omega < \pi
\]
Usually, the differentiator is approximated with a linear-phase FIR transfer function $H(z)$,

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n} \quad (2)$$

where the impulse response $h[n]$ exhibits the anti-symmetry property, $h[n] = -h[M-n]$. Therefore, the first-order differentiator is implemented as a linear-phase FIR filter: Type III (when $M$ is even) or Type IV (when $M$ is odd) (Oppenheim and Schafer, 1989).

The FIR filter $H(z)$ approximates an ideal differentiator in a limited band of frequencies. Thereby, some don’t-care bands should be tolerated. When the don’t-care bands become narrower, the transfer function order $M$ increases, that is for wideband differentiators, very high values of $M$ are needed.

A solution for wideband differentiators based on the two-rate signal processing and FRM technique has been proposed by Sheikh and Johansson (2009) and Sheikh and Johansson (2011). The basic two-rate filtering model is shown in Figure 1. According to the model, the sampling rate is increased by two at the input, the signal processing is performed at the increased rate, and the original sampling rate is reconstructed by the factor-of-two down-sampler. The signal processing is performed by the low-pass half-band filter $F(z)$ and the differentiator $G(z)$. With this approach, the don’t-care band for the differentiator $G(z)$ is a half of the base-band, i.e. $G(z)$ is to be designed to approximate the ideal differentiator for a half of the desired frequency band. The half-band filter $F(z)$ undertakes the task of interpolation and decimation filter.

**Figure 1** Two-rate filtering model

$$x[n] \xrightarrow{2} F(z) \xrightarrow{G(z)} y[n]$$

A single-rate model can be developed from the two-rate model in Figure 1 when presenting $F(z)$ and $G(z)$ in polyphase forms, and applying the cascade equivalences (Sheikh and Johansson, 2009, Sheikh and Johansson, 2011).

The half-band FIR filter $F(z)$ can be represented in the form

$$F(z) = F_0(z^2) + z^{-1}F_1(z^2) \quad (3)$$

where $F_0(z)$ and $F_1(z)$ are the polyphase components. For $F(z)$ being a half-band FIR filter, every second coefficient is zero valued, except the central coefficient whose value is 0.5. When the filter-order $N_F$ is an even number of the form $N_F = 4m + 2$ ($m$ is an integer), the polyphase component $F_1(z)$ is given by

$$F_1(z) = 0.5z^{-(N_F/2-1)/2} \quad (4)$$

and $F_0(z)$ is a polynomial in $z^{-1}$. The low-order differentiator $G(z)$ is a Type III FIR filter expressible in terms of the polyphase components $G_0(z)$ and $G_1(z)$

$$G(z) = G_0(z^2) + z^{-1}G_1(z^2) \quad (5)$$

By making use of equations (3)–(5), and employing the cascade equivalences (Milic, 2009), we obtain the transfer function of the single-stage equivalent denoted as $H_{eq1}(z)$, which is given by the following expression (Sheikh and Johansson, 2009; Sheikh and Johansson, 2011):

$$H_{eq1}(z) = 4F_0(z)G_0(z) + 2z^{-(N_F/2-1)/2}G_1(z) \quad (6)$$

In this way, the two-rate system as shown in Figure 1 is replaced with the single-rate system represented by equation (6), which operates at the lower sampling rate. It was shown in the work of Sheikh and Johansson (2009) and (Sheikh and Johansson, 2011) that considerable computational savings can be achieved if the FRM technique is used to implement polyphase sub-filter $F_0(z)$.

In the next subsection, we introduce the approximately linear-phase IIR half-band filter to implement the half-band filter $F(z)$ in Figure 1.

### 2.2 Application of approximately linear-phase IIR half-band filter

Let us introduce an IIR half-band filter with approximately linear phase for the role of interpolation/decimation filter in the two-rate system in Figure 1.

The transfer function of a low-pass approximately linear-phase IIR half-band filter is expressible as a sum of an all-pass function $A(z^2)$ and a pure delay (Schüssler and Steffen, 2001), i.e.

$$F(z) = \frac{A(z^2) + z^{-D}}{2} \quad (7)$$

In the two-rate model in Figure 1, we introduce $F(z)$ implemented as a parallel combination of $A(z^2)$ and the delay $z^{-D}$ as given in equation (7), and the low-order differentiator $G(z)$ in the polyphase form as given in equation (5). By employing the cascade equivalences, we obtain the transfer function of the single-rate equivalent for the two-rate system in Figure 1. The resulting transfer function of the single-rate equivalent $H_{eq2}(z)$ is given by the following expression:

$$H_{eq2}(z) = 2A(z)G_0(z) + 2z^{-(D+1)/2}G_1(z) \quad (8)$$

The block diagram depicted in Figure 2 represents the realisation structure which implements equation (8). In this way, an IIR/FIR solution is introduced for implementing the first-order differentiator.

**Figure 2** The block diagram implementing differentiator with IIR half-band filter (IIR/FIR solution)

$$x[n] \xrightarrow{2} A(z) \xrightarrow{G_0(z)} \xrightarrow{z^{-(D+1)/2}} y[n]$$
3 Design of IIR/FIR and FIR/FIR differentiators

The purpose of this section is twofold: First, we demonstrate the implementation of the first-order differentiator by making use of an approximately linear-phase IIR half-band filter and the low-order differentiator according to the structure shown in Figure 2, and equations (5), (7) and (8), (IIR/FIR solution). Second, we compare the IIR/FIR solution with the solution based on FIR filters (FIR/FIR) initially proposed by Sheikh and Johansson (2009) and Sheikh and Johansson (2011), and exposed in the previous section.

In this section, we consider the construction of a differentiator for the frequency range \([0, 0.9\pi]\).

3.1 Approximately linear-phase differentiator (IIR/FIR)

We design \(F(z)\) as the 10th-order approximately-linear phase IIR half-band filter with the passband cut-off frequency \(\omega_p = 0.45\pi\) using the algorithm given in the work of Schüssler and Steffen (2001). The full line in Figure 3 plots the gain response of the filter. Notice that because of the half-band filter symmetry condition, the stop-band cut-off is \(\omega_s = 0.55\pi\). The all-pass branch \(A_0(z^2)\) is the 10th-order function where the every second coefficient is zero valued, and the delay \(D\) amounts nine samples.

Figure 3  Gain responses of IIR and FIR half-band filters: IIR filter full line; FIR filter dashed line

Before designing the low-order differentiator, we should make choice between the Type III and Type IV FIR filter. Since an integer delay is desirable for the overall differentiator, we first examine the phase delay of the auxiliary filter \(A(z)+z^{-\frac{D+1}{2}}\). The results are shown in Figure 4.

Figure 4  Phase delay of \(A(z)+z^{-\frac{D+1}{2}}\)

According to Figure 4, the phase delay is approximately 4.5 samples. This implies that the filter order of \(G(z)\) should be of the form \(2(m+1)\), \(m\) is even, that is we need the Type III FIR filter for \(G(z)\).

Next, we design the low-order differentiator \(G(z)\) as the 6th-order Type III FIR filter using MATLAB function \texttt{firgr} (Signal Processing Toolbox, 2006) for the frequency range \([0, 0.45\pi]\). The MATLAB code is given bellow

\[
g = \text{firgr}(6,[0,0.45],[0,0.45*pi],'
\text{differentiator}');
\]

The magnitude response \(|G(e^{j\omega})|\) is shown in Figure 5.

Figure 5  Magnitude response of the 6th order differentiator

Finally, we compute the magnitude response of the overall system in Figure 2. The solid line in Figure 6 plots the magnitude response of the resulting differentiator.

Figure 6  Differentiator magnitude response: solid line plots the solution with IIR half-band filter and the 6th order differentiator; dashed line plots the solution with FIR half-band filter and the 6th order differentiator

The minimal number of multipliers for implementing the overall system is obtained when exploiting the coefficient symmetry and requires eight multiplication constants: five constants for \(A(z)\), and three constants for the two polyphase components of the 6th-order \(G(z)\): \(G_0(z)\) and \(G_1(z)\).

The overall delay of the system amounts \(N = 6\) samples: 4.5 samples from \(A(z)\), and 1.5 samples from \(G_0(z)\).
3.2 Linear-phase differentiator (FIR/FIR)

For the sake of comparison with the results of previous subsection, we design the corresponding FIR differentiator consisting of a linear-phase half-band FIR filter and a low-order differentiator for the frequency range \([0, 0.45\pi]\). In order to provide a very small passband ripple we chose the 58th-order FIR half-band filter. We design the 58th-order optimal half-band filter by means of the MATLAB function `firhalfband` (Signal Processing Toolbox, 2006).

\[ N_f = 58; \]
\[ h_{fir} = firhalfband(N_f,0.45); \]

The gain response is plotted in Figure 3, dashed line. Since the delay of the half-band filter is an odd integer \((N_f/2 = 29)\), we chose the 6th-order Type III FIR filter for \(G(z)\),

\[ g = firgr(6,[0,0.45],[0,0.45*pi], 'differentiator'); \]

We compute the magnitude response of the overall system according to equation (6). The resulting magnitude response is shown in Figure 6, dashed line.

The overall system requires 19 multiplication (16 for \(F_0(z)\) and \(F_1(z)\) and three constants for the two polyphase components of the 6th-order \(G(z)\); \(G_0(z)\) and \(G_1(z)\) constants for implementation if the symmetry/anti-symmetry coefficient property is exploited. In the case of an even order half-band filter, the polyphase component \(F_D(z)\) exhibits the coefficient symmetry, and the both polyphase components of \(G(z)\) exhibit anti-symmetry properties if the order of \(G(z)\) is even.

The total delay of the system amounts \(N = 16\) samples: 14.5 samples from \(F_0(z)\), and 1.5 samples from \(G_0(z)\).

Figure 7 displays the errors of the normalised magnitude response versus the ideal differentiator as computed by equation (9).

\[ Er(\omega) = |H_{eq}(\omega)|/|\omega - 1| \quad (9) \]

**Figure 7** The magnitude response error. Solid line: IIR half-band filter and low-order differentiator. Dashed line: FIR half-band filter and low-order differentiator

The solid line shown in Figure 7 is for the scheme consisting of IIR half-band filter and low-order differentiator, and the dashed line is for FIR half-band filter and low-order differentiator.

Table 1 summarises the number of multipliers and the total delay for both systems. Notice that the coefficient symmetry properties are exploited to minimise the number of multiplication constants.

<table>
<thead>
<tr>
<th>Differentiator structure</th>
<th>No. Multipl.</th>
<th>Total delay (samples)</th>
</tr>
</thead>
<tbody>
<tr>
<td>IIR hb filter and differentiator (IIR/FIR)</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>FIR hb filter and differentiator (FIR/FIR)</td>
<td>19</td>
<td>16</td>
</tr>
</tbody>
</table>

Figure 7 and Table 1 show that for a similar magnitude response error, the IIR/FIR differentiator provides savings in the number of multiplication constants, and exhibits smaller delay of the system in comparison with FIR/FIR solution.

4 Properties and efficiency comparisons

It is demonstrated in Section 3 that considerable computational savings in the differentiator can be achieved when filter \(F(z)\) is implemented as an IIR half-band filter instead of an FIR half-band exhibiting the similar passband ripple. Moreover, the use of IIR half-band filter significantly decreases delay of the overall differentiator.

The low passband ripple of \(F(z)\) is desirable in order to minimise the magnitude response error of the overall differentiator \(H_{eq}(z)\). The inherent property of an IIR half-band filter is a very small passband ripple. At the contrary, the requirement for a very small passband ripple increases rapidly the order of an FIR filter transfer function.

The computational efficiency of FIR/FIR differentiators can be improved remarkably when using FRM technique (Sheikh and Johansson, 2009; Sheikh and Johansson, 2011). In this section, by means of example, we compare IIR/FIR differentiator proposed in the paper with an optimal FRM-based FIR/FIR solution presented in Sheikh and Johansson’s (2009) and Sheikh and Johansson’s (2011).

We consider Example 1 from the above mentioned references: the wideband differentiator for the range \([0, 0.95\pi]\) with the normalised magnitude error of \(\pm0.01\) is achieved applying the FRM approach and minimax optimisation technique. The overall realisation consists of 19 multiplications and 30 additions, and the overall delay of the system amounts 34 samples.

In order to compare the above mentioned performances with the performances of an IIR/FIR differentiator, we use the implementation scheme in Figure 2 and design the wideband differentiator \(H_{eq}(z)\) for the range \([0, 0.95\pi]\).

First, we design the 6th-order Type III FIR filter for \(G(z)\),

\[ g = firgr(6,[0,0.45],[0,0.475*pi], 'differentiator'); \]

Second, we design the 12th-order approximately linear-phase IIR half-band filter \(F(z)\) with the passband edge frequency \(\omega_p = 0.475\pi\) (Schüssler and Steffen, 2001).

Figures 8–12 illustrate the properties of the resulting wideband differentiator \(H_{eq}(z)\).
Figure 8 plots the passband ripple of \( F(z) \) and the ripple of the normalised magnitude of \( G(z) \). It is expected that the small passband ripple of the 12th-order filter \( F(z) \) will cause the acceptable derogations of the overall differentiator magnitude response.

**Figure 8** The passband ripple of the 12th order half-band filter \( F(z) \) (solid line), and ripple of the normalized magnitude response of differentiator \( G(z) \) (dashed line).

The normalised magnitude error \( Er(\omega) \) for the overall wideband differentiator \( H_{\text{eq}}(z) \) has been computed. Figure 9 displays the results. One observes that \( Er(\omega) \) varies between the limits \(-0.01 \) and \(+0.01 \) in the prescribed frequency range \([0 < \omega < 0.95\pi]\) as has been achieved by the example FRM-based FIR/FIR differentiator (Sheikh and Johansson, 2009; Sheikh and Johansson, 2011). Moreover, Figure 7 shows that \( Er(\omega) \) remains within \( \pm 0.01 \) in the entire basis band \([0 < \omega < \pi]\).

**Figure 9** The magnitude response error of the wideband differentiator \( H_{\text{eq}}(z) \) achieved with the 12th order IIR half-band filter and 6th order differentiator.

The phase response error (deviation of a strictly linear phase response in radians) is shown in Figure 10. The phase error of the differentiator \( H_{\text{eq}}(z) \) (solid line) slightly differs from the equiripple phase error of the all-pass filter \( A(z) \) (dashed line), see Figure 2.

**Figure 10** The phase response error of the wideband differentiator \( H_{\text{eq}}(z) \) (solid line), and that of all-pass filter \( A(z) \) (dashed line).

The pole/zero plot for the 6th-order all-pass filter \( A(z) \) that is used to implement the wideband differentiator of this example is given in Figure 11. Evidently, the critical pole pair is placed on the real axis of the \( z \)-plane with the value \( p = -0.891074036616219 \). Plots in Figure 12 show how the module of the critical pole pair of \( A(z) \) approaches unity when increasing the differentiator bandwidth.

**Figure 11** The pole/zero plot of the 6th order all-pass filter \( A(z) \).

The efficiency of the examined example is summarised in Table 2. To achieve a maximal error of \( \pm 0.01 \) of the normalised magnitude response for the differentiator in the frequency range \([0, 0.95\pi]\), shown in Figure 9, IIR/FIR solution requires smaller number of multiplication constants and thus provides considerable computational savings when compared with the optimal FRM-based FIR/FIR solution. Additionally, the overall delay of the IIR/FIR differentiator is about five times smaller than that of FIR/FIR differentiator.

**Table 2** Number of multiplications and the total delay

<table>
<thead>
<tr>
<th>Differentiator structure</th>
<th>No. Multipl.</th>
<th>Total delay (samples)</th>
</tr>
</thead>
<tbody>
<tr>
<td>IIR hb filter and differentiator (IIR/FIR)</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>FIR/FRM hb filter and differentiator (FIR/FIR/FRM)</td>
<td>19</td>
<td>34</td>
</tr>
</tbody>
</table>

Besides the computational savings and considerable decrease in the overall delay of the system, the IIR/FIR solution suffers from the following deficiencies:

- Sharp half-band filter (as usually needed for the wideband differentiator) has at least one pole pair close to the unit circle. Consequently, the all-pass filter \( A(z) \) has a pole close to the unit circle as shown in Figures 11 and 12. It implies that a tolerable value of the critical pole pair should be taken into account when choosing the differentiator bandwidth.
- Due to the limited word length for implementation, the filter stability and possible derogations in the frequency response should be considered.
- Strictly linear phase response cannot be achieved, see Figure 10.
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Figure 12  Values of the critical pole of \( A(z) \) versus the bandwidth of the IIR/FIR wideband differentiator. \( N_\lambda = 5, 6, 7, 8 \) is the order of \( A(z) \)

Considering the analysis presented Sections 3 and 4, one concludes that the IIR/FIR approach offers a convenient solution in the cases when the computational efficiency and (or) low overall delay of wideband differentiator are the main requirement.

5 Knowledge based design

In previous sections, the design was performed using code obtained by Matlab and special function firgr (Signal Processing Toolbox, 2006). The code is actually based on generalised Remez method for the design of FIR digital filters. This code is optimal and that means there is no better solution for this type of filters. From Figures 7 and 9 it follows that the resulting error of the cascade of IIR and FIR filters is not optimal. On the other side, we do not know any other algorithm that can be used in the design presented in previous chapters.

This motivates us to use symbolic tools to prove the theory and find FIR filter coefficients so that the maximum of the error can be reduced.

In this section, using the example described in Section 3, we present how the knowledge from previous sections can be built into computer algebra system such as Mathematica.

First of all, the code in Mathematica is in m-files with embedded knowledge and so-called notebooks that can be used as templates for modification and simple adaptation to different applications. It is very important to specify the minimal number of parameters, so that the main code can be used with minimum of changes. For example, we are going to design filters that consist of defined structure, and at the beginning of the code we specify the symbolic values of all coefficients. Next, we can use the knowledge embedded into symbolic tools (Lutovac and Tošić, 2010) and generate schematic of digital filter for known parameters of all-pass filter \( \{a_0, a_1, a_2, a_3 \text{ and } a_4\} \) (Figure 13).

Figure 13  Automated generation of filter schematic (see online version for colours)

The description of the schematic is generated by specifying symbolic values of coefficients and the knowledge of the structure of IIR filters. The same description is used to derive transfer function

\[
\text{myTransferFunction} = \text{DiscreteSystemTransferFunction}[\text{schematicA}];
\]

\[
\text{Az} = \text{myTransferFunction[[1, 1]][[1]]};
\]

\[
\text{DiscreteSystemDisplayForm}[%]
\]

This generates the transfer function of \( A(z) \). From Figure 15 it is clear that this is the transfer function of the all-pass digital filter due to the symmetry of the coefficients in the numerator and denominator.

Figure 15  Derived transfer function from the filter schematic

The filter has symmetric coefficients and the symbolic values in the numerator are in the reversed order with respect to the denominator. This code generates schematic that is presented in Figure 14.

Figure 14  Schematic of all pass digital IIR filter \( A(z) \) (see online version for colours)
coefficients can be specified as replacement rules rather than to set numeric values to symbolic names of variables:

\[
\text{myValues} = \{ a0 \rightarrow 0.03468637614042004, \\
a1 \rightarrow -0.02563024802380627, \\
a2 \rightarrow 0.05323135688979297, \\
a3 \rightarrow -0.1176218749843463, \\
a4 \rightarrow 0.4950451781133837, \\
a5 \rightarrow -1, \\
d \rightarrow -9, \\
g01 \rightarrow 0.03215689511209947, \\
g02 \rightarrow 0.8058293375973158, \\
g11 \rightarrow -0.2005840708632795 \}\]

Instead of \( A(z) \), equation (7) can be verified as half-band filter for squared argument. This can be simply computed as shown in Figure 16.

**Figure 16** Derived transfer function from the filter schematic (see online version for colours)

\[
A_2 = \left( \frac{A z}{z - z^2} \right) + z^{-d} \\
A2n = A2 /. \text{myValues}; \\
\text{DiscreteSystemFrequencyResponse}[A2n, \{0, 0.45\}];
\]

At this point, we have verified that the design and numeric values of the half-band IIR filter coefficients are correct. The required efforts are to retype numeric values from another program or text in a form of the replacement rules, to specify symbolic values of the filter coefficients and to evaluate the knowledge of digital IIR filter schematic description.

In a similar way we can generate schematics of all other filter parts as it is shown by block diagram depicted in Figure 2. The resulting filter schematic is presented in Figure 17.

**Figure 17** Schematic of differentiator with IIR half-band filter (see online version for colours)

The transfer function of the filter illustrated in Figure 17 can be derived automatically in terms of symbolic values of filter coefficients. Some of coefficients can be replaced by numeric values (say for the IIR filter \( A(z) \)). Other coefficient can be kept in a symbolic form (in this case for \( g01, g02, g11 \)). A set of frequencies can be specified, and the squared error can be derived at those frequencies in terms of unknown coefficients \( g01, g02, g11 \). Numeric optimisation can be started (command that finds minimum of a function) and as a result new numeric values are computed

\[
\text{sol2} = \{ g01 \rightarrow 0.03672371231903042, \\
g02 \rightarrow 0.8145429766900261, \\
g11 \rightarrow -0.21030576732406597 \}\]

The magnitude response is shown in Figure 18, and the error is illustrated in Figure 19. The error is lower than that obtained by using Matlab code.

**Figure 18** Differentiator magnitude response of the filter designed using computer algebra system (see online version for colours)

**Figure 19** The error of the filter designed using computer algebra system (solid line) and numeric Matlab code (dashed line)

By a slight adaptation of frequency position at which we minimise the squared error, the maximum of the error is reduced from 0.00282 to 0.00254 with respect to the initial numeric solution designed using Matlab as shown in Figure 7.

Note that manual derivation of the error function with symbolic parameters of some of the coefficients is almost
impossible task, and even symbolically derived result using computer algebra system cannot be presented on several pages.

6 Conclusion and future work

In this paper, we demonstrate that an approximately linear phase half-band IIR filter in combination with very low order FIR differentiator can be used for constructing the wideband differentiator. In comparison with previously published solution based on FIR half-band filter and low-order FIR differentiator, the structure proposed in the paper exhibits lower overall delay of the system and requires smaller number of multiplication constants for implementation. The cost to be paid is some tolerance in the phase response linearity. The practical limits for the application of an IIR/FIR differentiator are determined by the fact that the critical pole pair of the IIR sub-filter become very close to the unit circle when the bandwidth of the differentiator approaches to the half of the sampling rate.

It is shown in the paper that knowledge-based system design can be applied for verification, analysis and improvements of the system performances.

The goal of future investigations is to develop a new algorithm for optimising error of the IIR/FIR differentiator in a minimax sense. Owing to the existence of local minima, the successful optimisation is not an easy task. It is intended to apply the knowledge-based system design to develop sensitivity functions in the closed forms in order to provide better insight into the optimisation process.

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References


