Stochastic Multi-symplectic Wavelet Collocation Method
for 3D Stochastic Maxwell Equations
with Multiplicative Noise

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Abstract

In this paper, we propose a stochastic multi-symplectic wavelet collocation method to numerically solve the three-dimensional (3D) stochastic Maxwell equations with multiplicative noise. We show that the proposed method preserves the discrete stochastic multi-symplectic conservation law. Meanwhile, we find that the equations have an energy conservation law in the sense of almost surely (a.s.) and the method can preserve discrete version of energy conservation law a.s. Theoretical analysis shows that the method has the first mean-square convergence order in temporal direction. Numerical experiments, focusing on the 3D stochastic Maxwell equations with multiplicative noise, match the theoretical results well and have the stability in long-time computations.

Keywords: Stochastic multi-symplectic structure, Stochastic Maxwell equations, Wavelet collocation method, Multi-symplectic conservation law, Energy conservation law, Mean-square convergence order

1. Introduction

The stochastic Maxwell equations describe many physical phenomena and play an important role in aeronautics, electronics, and biology, see \cite{5, 8, 11} and references therein. Up to now there have been a certain number of articles devoted to these equations but many problems, including theoretical and numerical aspects, still need to be solved. It started with the work of K. Liaskos \cite{9}, the mild, strong and classical

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well posedness for the Cauchy problem of the stochastic Maxwell equations are given. In most of the articles dealing with stochastic Maxwell equations, the technique is to rewrite the stochastic Maxwell equations into Helmholtz equation and investigate the time-harmonic case. For instance, in [5], the author investigates the time harmonic stochastic Maxwell equations driven by a color noise by applying discontinuous Galerkin method and obtains the error estimates. In [1], the authors analyze the uncertainty of the time-harmonic Maxwell equations for the simulation of a coplanar waveguide with uncertain material parameters based on stochastic collocation with stroud and sparse grid points. Interest in the problem of 3D stochastic Maxwell equations with multiplicative noise (see Section 2) arises in several research areas. Its renewed interest is motivated by the fundamental role, which it plays in the near-field radiative heat transfer and approximate controllability problems [3][5]. Only very recently, the study of directly solving stochastic Maxwell equations numerically has started. Based on the stochastic version of variational principle, [6] gives a method to get stochastic multi-symplectic structure for 3D stochastic Maxwell equations with additive noise and proposes a stochastic multi-symplectic method that preserves the discrete stochastic multi-symplectic conservation law and stochastic energy dissipative properties.

The aim of the present work is to analyze the energy evolution law and construct an energy-conserving method for 3D stochastic Maxwell equations with multiplicative noise. But similar to the deterministic problem, some difficulties arise when applying some traditional numerical methods, e.g., finite difference method (FDM), finite element method (FEM), spectral method. First of all, they are difficult in programming because of the huge scale of the algebraic equations and substantial computational cost. For example, for a 3D problem, they are required to solve at least one $10^6$ scale algebraic equation at every time step provided that the considered spatial domain is divided into $100 \times 100 \times 100$ cells. This is insolvable by a personal computer because of the limitation of memory and the performance of CPU up to now. Particularly, due to the existence of multiplicative noise, the computational cost is more expensive than 3D deterministic case. A second difficulty which arises, is the fact that the solutions to the stochastic Maxwell equations are non-smoothness. Since the stochastic process is nowhere differentiable and is of infinite variation on each subinterval. Motivated by the papers [4][16] and references therein, we develop an effective and robust numerical method for the space discretization of 3D stochastic Maxwell equations with multiplicative noise.

Wavelet-based numerical methods have been widely investigated both from the theoretical and computational points of view in many papers. [2] firstly proposes the wavelet collocation method which is based on the autocorrelation function of Daubechies scaling functions. The advantage of this method is that the partial differential equations are reduced to a system of algebraic equations with a sparse space differentiation matrix, leading to a numerical algorithm of reduced computational cost compared to the FDMs and the FEMs. In addition, this method has higher order of accuracy, and can capture the singularity of system, for the further analysis of this method, see [2][12] and references therein. In [4], the authors apply the biorthogonal interpolating wavelets to the Maxwell equations through the time-domain wavelet collocation method. The resulting scheme maintains high accuracy and has typically one ninth of CPU time and one fifth of memory of those of the convectional finite-difference time-domain. Re-
cently, Zhu et al. [16] proposes a multi-symplectic wavelet collocation method for 3D deterministic Maxwell equations. Theoretical analysis shows that the proposed method is multi-symplectic, unconditionally stable and energy preserving under periodical boundary conditions. In additional, symplectic and multi-symplectic numerical methods for stochastic Hamiltonian ordinary differential equations and partial differential equations have been developed in [6, 7, 10, 13, 14] and references therein. To the best of our knowledge, there has been no work in the literature which studies the stochastic multi-symplectic wavelet collocation method for the stochastic Maxwell equations with multiplicative noise in $\mathbb{R}^d$ ($d = 2, 3$). To this end, we rewrite the 3D stochastic Maxwell equations with multiplicative noise in the sense of Sratonovich into the form of the stochastic Hamiltonian partial differential equation and introduce the stochastic multi-symplectic structure. Meanwhile, we propose a stochastic multi-symplectic wavelet collocation method to solve the 3D stochastic Maxwell equations with multiplicative noise and present some theoretical analysis and numerical results.

The rest of this paper is organized as follows. In Section 2, we begin with some preliminary results about stochastic Maxwell equations. In Section 3, we present and analyze the stochastic multi-symplectic structure for stochastic Maxwell equations with multiplicative noise. In Section 4, a stochastic multi-symplectic wavelet collocation method for 3D stochastic Maxwell equations is given. In Section 5, numerical experiments are performed to testify the effectiveness and accuracy of the method. Concluding remarks are given in Section 6.

2. Preliminary results

2.1. Stochastic Maxwell equations with multiplicative noise

We consider the stochastic Maxwell equations with multiplicative noise in the following form (see [5]):

\[
\frac{\partial}{\partial t} \mathbf{E} = \nabla \times \mathbf{H} - \lambda \mathbf{H} \circ \dot{\chi}, \quad t \in [0, T], \ (x,y,z) \in \Theta,
\]

\[
\frac{\partial}{\partial t} \mathbf{H} = -\nabla \times \mathbf{E} + \lambda \mathbf{E} \circ \dot{\chi}, \quad t \in [0, T], \ (x,y,z) \in \Theta
\]

with an initial condition

\[
\mathbf{E}(0,x,y,z) = (E_{10}, E_{20}, E_{30}),
\]

\[
\mathbf{H}(0,x,y,z) = (H_{10}, H_{20}, H_{30}),
\]

and perfectly electric conducting (PEC) boundary condition

\[
\mathbf{E} \times \mathbf{n} = 0, \quad \text{on} \quad (0,T) \times \partial \Theta,
\]

where $T > 0$, $\Theta$ is a bounded and simply connected domain of $\mathbb{R}^3$ with smooth boundary $\partial \Theta$ and $\mathbf{n}$ is the unit outward normal of $\partial \Theta$. $\dot{\chi}$ is a space-time white noise and $\lambda > 0$ stands for the noise amplitude. $\circ$ stands for a Stratonovich product in the right hand side of (2.1).
It is convenient at this point to give a precise mathematical definition of $\dot{\varphi}$ and the following setting and assumptions.

In the sequel, we assume that we have given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a normal filtration $(\mathcal{F}_t)_{t \geq 0}$. We also assume that $W$ is a standard $\mathcal{Q}$-Wiener process on $L^2(R;R)$ associated with $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$, so that for any orthonormal basis $(e_m)_{m \in \mathbb{N}}$ of $L^2(R;R)$, there is a sequence $(\beta_m)_{m \in \mathbb{N}}$ of real independent Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ such that

$$W(t,x,\omega) = \sum_{m=0}^{\infty} \sqrt{\eta_m} \beta_m(t,\omega) e_m(x), \quad t \geq 0, \ x \in R, \ \omega \in \Omega.$$  \hfill (2.3)

Here, $\mathcal{Q}$ is nonnegative, symmetric with finite trace, i.e.,

$$Tr(\mathcal{Q}) < \infty, \ \mathcal{Q}e_m = \eta_m e_m, \ \eta_m \geq 0, \ m \in \mathbb{N},$$

where $\eta_m$ are the eigenvalues of $\mathcal{Q}$ with corresponding eigenvectors $e_m, m \in \mathbb{N}$. Then, we set $\dot{\varphi} = \frac{dW}{dt}$.

### 2.2. Well-posedness results

In this subsection, we recall the results on existence and uniqueness to the stochastic Maxwell equations with multiplicative noise.

In order to have a correct interpretation of (2.1), we define here some tools used in next sections. Let $U$ be a real separable and infinite dimensional Hilbert space. We consider $H = L^2(\Theta)^3 \times L^2(\Theta)^3$, and the predictable $\sigma$-field $\mathcal{P}_T$ in the space $\Omega_T = [0,T] \times \Omega$. We also consider the measurable spaces $(U, \mathcal{B}(U)), (L^2, \mathcal{B}(L^2)), (\Omega_T, \mathcal{P}_T)$ (as usual $\mathcal{B}$ is the Borrel $\sigma$-field) and $(L^2_0, \mathcal{B}(L^2_0))$, where by $L^2_0$ we denote the space of all Hilbert-Schmidt operators in $L^2(U_0,H)$ with $U_0 = Q^{1/2}(U)$, and $Q \in \mathcal{L}(U)$ is a nonnegative, trace operator.

We define the vector operators

$$X = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \ A = \begin{pmatrix} 0 & \text{curl} \\ -\text{curl} & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}.$$

Using the above notations, the system (2.1) can be rewritten as an abstract Cauchy problem in $\mathcal{H}$

$$dX = AX dt + BX \circ dW(t), \quad X(0) = X_0.$$  \hfill (2.4)

We will use the equivalent Itô equation. Defining the function

$$\Psi(x) = \sum_{m=0}^{\infty} (\sqrt{\eta_m}e_m(x))^2, \ x \in R,$$  \hfill (2.5)

which is independent of the basis $(e_m)_{m \in \mathbb{N}}$, this equivalent Itô equation may be written as

$$dX = (AX + \frac{1}{2}B^2X\Psi)dt + BXdW, \quad X(0) = X_0.$$  \hfill (2.6)

In what follows, we denote $\mathcal{E}$ as the expectation operator of a random variable.
Assumption 1. Assume that

1. The operator \( A : \mathcal{D}(A) \rightarrow \mathcal{H} \) is the infinitesimal generator of a \( C_0 \)-group of unitary operators \( T(t), t \in \mathbb{R} \), in \( \mathcal{H} \), i.e., \( \|T(t)\|_{\mathcal{L}(\mathcal{H})} = 1 \), for every \( t \in \mathbb{R} \).

2. \( X_0 \) is an \( \mathcal{H} \)-valued, \( \mathcal{F}_0 \) measurable, square integrable random variable, i.e., \( \mathbb{E}(\|X_0\|_{\mathcal{H}}^2) < \infty \).

Under the above assumption, [9] proved the following well-posedness result.

Theorem 1. [9]
Suppose that setting in Assumption 1 is fulfilled. If \( X \in C([0,T]; \mathcal{H}) \), then the problem (2.6) has a unique mild solution.

2.3. Energy conservation law

As well-known, the deterministic Maxwell equations, i.e., the case of \( \lambda = 0 \), have the following invariant

\[
\int_{\Theta} (|E(x,y,z,t)|^2 + |H(x,y,z,t)|^2) d\Theta = \text{Constant}. \tag{2.7}
\]

In the deterministic case, this invariant is called Poynting theorem in electromagnetism and can be easily verified. Similarly, based on the equivalent Itô equation (2.6), we can also obtain the energy conservation law a.s. for the stochastic Maxwell equations with multiplicative noise (2.1). This result shows that the electromagnetic energy is still invariant at different times in the influence of this kind of multiplicative noise. This is stated in the following Theorem.

Theorem 2. Let \( E \) and \( H \) be the solutions of the problem (2.1) under PEC boundary condition (2.2). Then for any \( t \in [0,T] \),

\[
Q(t) = \int_{\Theta} (|E(x,y,z,t)|^2 + |H(x,y,z,t)|^2) d\Theta = \int_{\Theta} (|E(x,y,z,t_0)|^2 + |H(x,y,z,t_0)|^2) d\Theta = Q(t_0), \text{ a.s.} \tag{2.8}
\]

Hereafter, \( |f(t)|^2 = |f_1(t)|^2 + |f_2(t)|^2 + |f_3(t)|^2 \), with \( f = E, H \).

Proof : The equation (2.6) can be written into the componentwise formula

\[
dE = (\nabla \times H - \frac{1}{2} \lambda^2 \Psi E) dt - \lambda H dW,
\]

\[
dH = (- \nabla \times E - \frac{1}{2} \lambda^2 \Psi H) dt + \lambda E dW.
\]

Introducing the following functions

\[
F_1(E) = \int_{\Theta} |E(x,y,z,t)|^2 d\Theta,
\]

\[
F_2(H) = \int_{\Theta} |H(x,y,z,t)|^2 d\Theta.
\]
It is easy to verify that $\mathbf{E}$ and $\mathbf{H}$ satisfy the following first and second Fréchet derivative

$$
DF_1(\mathbf{E})(\varphi) = 2 \int_\Theta \langle \mathbf{E}, \varphi \rangle d\Theta, \quad D^2 F_1(\mathbf{E})(\varphi, \psi) = 2 \int_\Theta \langle \psi, \varphi \rangle d\Theta,
$$

$$
DF_2(\mathbf{H})(\varphi) = 2 \int_\Theta \langle \mathbf{H}, \varphi \rangle d\Theta, \quad D^2 F_2(\mathbf{H})(\varphi, \psi) = 2 \int_\Theta \langle \psi, \varphi \rangle d\Theta,
$$

where $\varphi, \psi \in L^2(\Theta)^3$, and $\langle \cdot, \cdot \rangle$ denotes the Euclid inner product.

By using the infinite dimensional Itô formula for $F_1(\mathbf{E}(t))$ and $F_2(\mathbf{H}(t))$, we can obtain

$$
F_1(\mathbf{E}(t)) = F_1(\mathbf{E}(0)) + \int_0^t \langle DF_1(\mathbf{E}(s)), -\lambda \mathbf{H}(s) dW(s) \rangle
+ \int_0^t \langle DF_1(\mathbf{E}(s)), \nabla \times \mathbf{H}(s) - \frac{1}{2} \lambda^2 \Psi \mathbf{E}(s) \rangle ds
+ \frac{\lambda^2}{2} \int_0^t \text{Tr}(D^2 F_1(\mathbf{E}(s))(\mathbf{H}(s) Q^{\frac{1}{2}} (\mathbf{H}(s) Q^{\frac{1}{2}})^*) ds.
$$

Similarly, we apply Itô formula to function $F_2(\mathbf{H}(t))$ and obtain

$$
F_2(\mathbf{H}(t)) = F_2(\mathbf{H}(0)) + \int_0^t \langle DF_2(\mathbf{H}(s)), \lambda \mathbf{E}(s) dW(s) \rangle d\Theta
- 2 \int_0^t \langle \mathbf{H}(s), \nabla \times \mathbf{E}(s) + \frac{1}{2} \lambda^2 \Psi \mathbf{H}(s) \rangle ds d\Theta
+ \lambda^2 \int_0^t \text{Tr}(\langle \mathbf{E}(s), \mathbf{E}(s) \rangle Q^{\frac{1}{2}} (Q^{\frac{1}{2}})^*) ds d\Theta.
$$

Summing (2.11) and (2.12), we find that $A$ and $B$ extinguish, then we have the following
equality
\[ \int_{\Theta} (|E(t)|^2 + |H(t)|^2) d\Theta = \int_{\Theta} (|E(0)|^2 + |H(0)|^2) d\Theta + 2 \int_{\Theta} \left[ \int_0^t \left( \langle E(s), \nabla \times H(s) \rangle - \langle H(s), \nabla \times E(s) \rangle \right) ds d\Theta \right] \]

By Green’s formula and PEC boundary condition (2.2), the part \( C \) satisfies the following equality
\[ C = - \int_{\Theta} \nabla \cdot (E \times H) d\Theta = - \int_{\partial \Theta} (E \times H) \cdot n d\Theta = 0. \]

It follows from the definition of \( \Psi \), see (2.5). It is easy to verify that
\[ D + P = 0. \]

Combining all these results, (2.8) is obtained in the sense of almost surely (a.s.). The proof is thus finished.

**Remark 1.** (i). The equality (2.8) is an important criteria in constructing efficient numerical schemes for computing the propagation of electromagnetic waves and in measuring whether a numerical simulation method is good or not;

(ii). For 3D stochastic Maxwell equations with additive noise, the energy is not conserved (for more details, see [6]). This is very different with the result in this paper, i.e., the case of multiplicative noise. In many others circumstances, the stochastic Maxwell equations (2.1) are an idealized model in which many random effects for the energy evolution have been neglected.

3. Stochastic Hamiltonian Maxwell equations

A stochastic partial differential equation is called a stochastic Hamiltonian partial differential equation if it can be written in the form
\[ F d_t z + K_z dt = \nabla S_1(z) dt + \nabla S_2(z) \circ dW(t), \quad z \in \mathbb{R}^d, \quad (3.1) \]
where, $F$ and $K$ are skew-symmetric matrices and $S_1$ and $S_2$ are real smooth functions of the variable $z$.

Based on the definition of stochastic Hamiltonian partial differential equation (3.1), we can write stochastic Maxwell equations with multiplicative noise (2.1) into the following structure:

$$Fd_tu + K_1u_xdt + K_2u_ydt + K_3u_zdt = \nabla_uS(u) \circ dW, \ u \in \mathbb{R}^6.$$  

(3.2)

Here,

$$u = (H_1, H_2, H_3, E_1, E_2, E_3)^T,$$

$$S(u) = \frac{\lambda}{2}(|E|^2 + |H|^2),$$

$$F = \begin{pmatrix} 0 & -I_{3 \times 3} \\ I_{3 \times 3} & 0 \end{pmatrix}, K_i = \begin{pmatrix} \partial_i & 0 \\ 0 & \partial_i \end{pmatrix}, \forall i = 1, 2, 3.$$

The sub-matrix $I_{3 \times 3}$ is a 3 $\times$ 3 identity matrix and

$$\partial_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \partial_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \partial_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

According to the mathematical definition of the stochastic multi-symplectic conservation law [6, 7], we have the following theorem directly.

**Theorem 3.** The stochastic Hamiltonian Maxwell equations (3.2) have the stochastic multi-symplectic conservation law locally

$$d_t \omega + \partial_x \kappa_1 dt + \partial_y \kappa_2 dt + \partial_z \kappa_3 dt = 0, \text{ a.s.}$$

i.e.,

$$\int_{t_0}^{t_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} \omega(t_1, x, y, z) dx dy dz + \int_{t_0}^{t_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} \kappa_1(t, x_1, y, z) dt dy dz$$

$$+ \int_{t_0}^{t_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} \kappa_2(t, x_1, y, z) dt dx dy + \int_{t_0}^{t_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} \kappa_3(t, x_1, y, z_1) dt dx dy$$

$$= \int_{t_0}^{t_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} \omega(t_0, x, y, z) dx dy dz + \int_{t_0}^{t_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} \kappa_1(t, x_0, y, z) dt dy dz$$

$$+ \int_{t_0}^{t_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} \kappa_2(t, x_0, y, z) dt dx dy + \int_{t_0}^{t_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} \kappa_3(t, x_0, y, z_0) dt dx dy,$$

where $\omega(t, x, y, z) = \frac{1}{2} du \wedge F du$, $\kappa_i(t, x, y, z) = \frac{1}{2} du \wedge K_i du$ are the differential 2-forms associated with the skew-symmetric matrices $F$ and $K_i$ ($i = 1, 2, 3$), respectively, and $(t_0, t_1) \times (x_0, x_1) \times (y_0, y_1) \times (z_0, z_1)$ is the local definition domain of $u(t, x, y, z)$.
4. Stochastic multi-symplectic wavelet collocation method for stochastic Maxwell equations

In this section, we introduce a stochastic multi-symplectic wavelet collocation method to numerically solve stochastic Hamiltonian Maxwell equations (3.2). Based on the autocorrelation functions of Daubechies compactly supported scaling functions, wavelet collocation method is conducted for the space discretization. In temporal direction, a symplectic scheme is employed for the integration of the semi-discrete system. Thus we can obtain the full-discrete stochastic multi-symplectic numerical method.

4.1. Stochastic semi-discrete symplectic method

We rewrite (2.4) into

\[ \text{Apply the implicit midpoint scheme to the equation (4.1) in the temporal direction, we get} \]

\[ X_{n+1} = X_n + a(t_n + \frac{h}{2}, X_n + \frac{X_{n+1}}{2})h + b(t_n, \frac{X_n + X_{n+1}}{2})\Delta W^n, \]  \hspace{1cm} (4.2)

where \( h \) is the temporal step-size. Hereafter, we use the notations \( \Delta \beta_m^n(\omega) := \beta_{m+1}^n(\omega) - \beta_m^n(\omega) \) and

\[ \Delta W^n = W(t_{n+1}) - W(t_n) = \sum_{m=1}^M \sqrt{\Omega_m} \Delta \beta_m^n(\omega) e_m, \text{ for } \omega \in \Omega, \] \hspace{1cm} (4.3)

where \( M \) is a positive integer.

We rewrite (4.2) in the following component formula

\[ H_1^{n+1} = H_1^n + h \left( \frac{\partial}{\partial \sigma} E_2^{n+1/2} - \frac{\partial}{\partial \lambda} E_3^{n+1/2} \right) + \lambda E_1^{n+1/2} \Delta W^n, \]
\[ H_2^{n+1} = H_2^n + h \left( \frac{\partial}{\partial \sigma} E_3^{n+1/2} - \frac{\partial}{\partial \lambda} E_1^{n+1/2} \right) + \lambda E_2^{n+1/2} \Delta W^n, \]
\[ H_3^{n+1} = H_3^n + h \left( \frac{\partial}{\partial \sigma} E_1^{n+1/2} - \frac{\partial}{\partial \lambda} E_2^{n+1/2} \right) + \lambda E_3^{n+1/2} \Delta W^n, \]
\[ E_1^{n+1} = E_1^n + h \left( \frac{\partial}{\partial \sigma} H_2^{n+1/2} - \frac{\partial}{\partial \lambda} H_3^{n+1/2} \right) - \lambda H_2^{n+1/2} \Delta W^n, \] \hspace{1cm} (4.4)
\[ E_2^{n+1} = E_2^n + h \left( \frac{\partial}{\partial \sigma} H_3^{n+1/2} - \frac{\partial}{\partial \lambda} H_1^{n+1/2} \right) - \lambda H_3^{n+1/2} \Delta W^n, \]
\[ E_3^{n+1} = E_3^n + h \left( \frac{\partial}{\partial \sigma} H_1^{n+1/2} - \frac{\partial}{\partial \lambda} H_2^{n+1/2} \right) - \lambda H_1^{n+1/2} \Delta W^n. \]

In order to present the following mean-square convergence order result, our basic idea consists in replacement of \( \Delta W^n \) by \( \sqrt{h} \sum_{m=1}^M \sqrt{\Omega_m} e_m(x)(\zeta_m)_m \), where \( (\zeta_m)_m \) are mutually independent \( \mathcal{N}(0,1) \)-distributed random variables, and the parameters \( (\zeta_m)_m \) are
defined by the following way with \( A_h = \sqrt{2|\ln h|} \) for definiteness

\[
(\zeta_h)_m = \begin{cases} 
A_h, & \xi > A_h \\
|\xi|, & |\xi| \leq A_h \\
-A_h, & \xi < -A_h 
\end{cases}
\]

where \( \xi_m \) are mutually independent \( \mathcal{N}(0, 1) \)-distributed random variables. The main properties of parameter \( \zeta_h \) are summarized in the following lemma (see [10]).

**Lemma 4.** For any \( m = 1, 2, \ldots, M \), the parameter \( \zeta_h \) has the following properties

1. \( \mathbb{E}[(\zeta_h)_m^2 - \xi_m^2] \leq h^k \), \( k \geq 1 \);
2. \( 0 \leq \mathbb{E}(\xi_m^2) \leq (1 + 2\sqrt{2k|\ln h|})h^k \);
3. \( \mathbb{E}(\zeta_h)_m^4 < \mathbb{E}^{\xi_m^4} = 3 \).

The above lemma implies the following useful result.

**Lemma 5.** Let \( \Psi(x) \) is a bounded function, then we have

\[
\mathbb{E} \left[ \sum_{m=1}^{M} (\sqrt{\eta_m} \epsilon_m(x)(\zeta_h)_m)^2 - \sum_{m=1}^{M} (\sqrt{\eta_m} \epsilon_m(x)(\xi_m)_m)^2 \right] \leq K(1 + 2\sqrt{2k|\ln h|})h,
\]

where \( K \) is a positive constant depending on the bound of function \( \Psi \).

In the following theorem, we will give the mean-square convergence order for our semi-discrete implicit midpoint scheme.

**Theorem 6.** Suppose that there exists a constant \( L \) such that

\[
|a(t, y) - a(t, x)| \leq L|y - x|, \quad \left| \frac{\partial b}{\partial x}(t, x) \right| \leq L,
\]

\[
\left| \frac{\partial a}{\partial t}(t, x) \right| \leq L(1 + |x|), \quad \left| \frac{\partial \tilde{b}}{\partial t}(t, x) \right| \leq L(1 + |x|),
\]

for \( t \in [0, T] \), \( x \in \mathbb{R}^d \), \( \tilde{b}(t, x) = \frac{1}{2} b^T(t, x) \frac{\partial b(t, x)}{\partial x} \), then the semi-discrete implicit midpoint method (4.4) has the first mean-square order of convergence.

**Proof:** Let \( \tilde{X} \) be the approximation of solution to (4.1), we have

\[
\tilde{X} = X + a(t + \frac{h}{2}, X)h + b(t, X)\Delta W(h).
\]

It is known that the method based on this one-step approximation has the first mean-square order of convergence.

Denote by \( \bar{X} \) the one step approximation of the midpoint method

\[
\bar{X} = X + a(t + \frac{h}{2}, \frac{X + \bar{X}}{2})h + \sum_{m=1}^{M} b(t, \frac{X + \bar{X}}{2})\sqrt{\eta_m} \epsilon_m(x)(\zeta_h)_m \sqrt{h}.
\]
Expanding the right-hand side of in X, it follows

\[
X = X + ha(t + \frac{h}{2}, X) + \sqrt{h}b(t, X) \sum_{m=1}^{M} \sqrt{\eta_m e_m(x)}(\zeta_h)_m \\
+ \frac{h}{2} b(t, X)^T \frac{\partial b}{\partial x}(t, X) \left( \sum_{m=1}^{M} \sqrt{\eta_m e_m(x)}(\zeta_h)_m \right)^2 + \rho.
\]

Notice that the conditions (4.5) and (4.6) imply

\[
\mathcal{E}(\rho^2) = \mathcal{E}|X - \bar{X}| \\
\leq L \left( \mathcal{E}\left( \frac{|X - \bar{X}|^2}{2} h^2 \right) + \mathcal{E}\left( \frac{|X - \bar{X}|}{2}^2 \left( \sum_{m=1}^{M} \sqrt{\eta_m e_m(x)}(\zeta_h)_m \right)^2 h + (1 + |X|^2) h^3 \right) \right),
\]

where $|\cdot|$ denotes the Euclidean norm. Since, $|X - \bar{X}| \leq L(1 + |X|)(h + |(\zeta_h)|\sqrt{h})$, and from the above lemma, we have

\[
\mathcal{E}(\rho^2) = O(h^3).
\] (4.7)

Similarly,

\[
\mathcal{E}(\rho) = O(h^2).
\] (4.8)

Denoting $R = X - \bar{X}$, we obtain

\[
R = X - \bar{X} \\
= b(t, X) \left( \sqrt{h} \sum_{m=1}^{M} \sqrt{\eta_m e_m(x)}(\zeta_h)_m - \Delta W(h) \right) \\
+ b(t, X)^T \frac{\partial b}{\partial x}(t, X)^T \left[ h \left( \sum_{m=1}^{M} \sqrt{\eta_m e_m(x)}(\zeta_h)_m \right)^2 - (\Delta W(h))^2 \right] + \rho.
\]

Therefore, from Lemma 5, we obtain that

\[
|\mathcal{E}(R)| = O(h^2), \quad \mathcal{E}(R^2) = O(h^3).
\]

Based on the fundamental convergence theorem [10], which establishes the first mean-square order of convergence to the semi-discrete scheme in temporal direction (4.4). Thus the proof is finished.

4.2. Stochastic multi-symplectic wavelet collocation method

In this subsection, we apply wavelet collocation method to discretize (4.4) in the spatial direction and obtain the full-discrete stochastic multi-symplectic wavelet collocation method. Firstly, we give some preliminary results of wavelet method, for more details, see [12] and references therein.
A Daubechies scaling function \( \phi(x) \) of order \( M \) satisfies the scaling relation:

\[
\phi(x) = \sum_{k=0}^{M-1} h_k \phi(2x - k),
\]

where \( M \) is a positive even integer and \( \{h_k\}_{k=0}^{M-1} \) are \( M \) non-vanishing “filter coefficients”. The function \( \phi \) has its support in the interval \([0, M - 1]\) and it has \((M/2 - 1)\) vanishing wavelet moments. Furthermore, a multiresolution analysis can be conducted on \( L^2(R) \). Define the autocorrelation function \( \theta(x) \) of \( \phi(x) \) as

\[
\theta(x) = \int \phi(x) \phi(t - x) dt.
\]

Suppose that \( V_f \) is the linear span of \( \{\theta_k(x) = 2^{j/2} \theta(2^j x - k), k \in \mathbb{Z}\} \), then it can be proved that \( (V_f)_{f \in \mathbb{Z}} \) form a multiresolution analysis where \( \theta(x) \) plays the role of scaling function (see [2]).

Based on the autocorrelation function \( \theta(x) \) of Daubechies scaling function, we construct the stochastic multi-symplectic wavelet collocation method for the semi-discrete stochastic Maxwell equations (4.4). Consider periodic boundary conditions in \([0, L_1] \times [0, L_2] \times [0, L_3] \) with \( N_1 \times N_2 \times N_3 \) grids points, where \( N_1 = L_1 \cdot 2^l, N_2 = L_2 \cdot 2^l, N_3 = L_3 \cdot 2^l \).

Without loss of generality, we take \( E_1(x, y, z, t) \) as an example. The interpolation operator \( IE_1(x, y, z, t) \),

\[
IE_1(x, y, z, t) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \sum_{k=1}^{N_3} \theta(2^j x - i) \theta(2^j y - j) \theta(2^j z - k).
\]

Making partial differential in the \( y \)-direction with (4.9) and evaluating the resulting expression at collocation points \((x_i, y_j, z_k) = (i/2^l, j/2^l, k/2^l)\) for \( i = 1, ..., N_1, j = 1, ..., N_2, k = 1, ..., N_3 \), we obtain

\[
\frac{\partial IE_1(x_i, y_j, z_k, t)}{\partial y} = \sum_{l=1}^{N_l} \sum_{k=1}^{N_k} \sum_{l_k=1}^{N_{l_k}} E_{1,i,j,k} \theta(2^j x_i - i) \theta(2^j z_k - k) \frac{d \theta(2^j y - j)}{dy} |_{y_j} = \sum_{j=1}^{N_2} E_{1,i,j,k} (B^\prime)_{j,k} \frac{d \theta(2^j y - j)}{dy} |_{y_j} = ((I_{N_1} \otimes B^\prime \otimes I_{N_3}) E_1)_{i,j,k},
\]

where \( \otimes \) means Kronecker inner product, \( I_{N_l} \) is the \( N_l \times N_l \) identity matrix. The differentiation matrix \( B^\prime \) for the first-order partial differentiation operator \( \partial y \) is an \( N_2 \times N_2 \) sparse skew-symmetric circulant matrix:

\[
(B^\prime)_{m,m'} = \begin{cases} 2^j \theta'(m - m'), & m - (M - 1) \leq m' \leq m + (M - 1); \\ 2^j \theta'(-l), & m - m' = N_2 - l, \ 1 \leq l \leq M - 1; \\ 2^j \theta'(l), & m' - m = N_2 - l, \ 1 \leq l \leq M - 1; \\ 0, \ & \text{otherwise}. \end{cases}
\]
Using the similar manner, we can obtain the discrete form of partial differential to $E_2, E_3, H_1, H_2, H_3$ and the differentiation matrices $B^x$ and $B^z$, respectively.

In summary, we obtain the following full-discrete numerical method to equations (4.10)

$$
(E_1)_{i,j,k}^{n+1} - (E_1)_{i,j,k}^n = h \left( A_1 (H_1)_{i,j,k}^{n+1/2} - A_2 (H_2)_{i,j,k}^{n+1/2} - A_3 (H_3)_{i,j,k}^{n+1/2} \right) - \lambda (H_1)_{i,j,k}^{n+1/2} \Delta W_i^n,
$$

$$
(E_2)_{i,j,k}^{n+1} - (E_2)_{i,j,k}^n = h \left( A_1 (H_1)_{i,j,k}^{n+1/2} - A_2 (H_2)_{i,j,k}^{n+1/2} - A_3 (H_3)_{i,j,k}^{n+1/2} \right) - \lambda (H_2)_{i,j,k}^{n+1/2} \Delta W_i^n,
$$

$$
(E_3)_{i,j,k}^{n+1} - (E_3)_{i,j,k}^n = h \left( A_1 (H_1)_{i,j,k}^{n+1/2} - A_2 (H_2)_{i,j,k}^{n+1/2} - A_3 (H_3)_{i,j,k}^{n+1/2} \right) - \lambda (H_3)_{i,j,k}^{n+1/2} \Delta W_i^n,
$$

$$
(H_1)_{i,j,k}^{n+1} - (H_1)_{i,j,k}^n = -h \left( A_1 (E_1)_{i,j,k}^{n+1/2} - A_2 (E_2)_{i,j,k}^{n+1/2} - A_3 (E_3)_{i,j,k}^{n+1/2} \right) + \lambda (E_1)_{i,j,k}^{n+1/2} \Delta W_i^n,
$$

$$
(H_2)_{i,j,k}^{n+1} - (H_2)_{i,j,k}^n = -h \left( A_1 (E_1)_{i,j,k}^{n+1/2} - A_2 (E_2)_{i,j,k}^{n+1/2} - A_3 (E_3)_{i,j,k}^{n+1/2} \right) + \lambda (E_2)_{i,j,k}^{n+1/2} \Delta W_i^n,
$$

$$
(H_3)_{i,j,k}^{n+1} - (H_3)_{i,j,k}^n = -h \left( A_1 (E_1)_{i,j,k}^{n+1/2} - A_2 (E_2)_{i,j,k}^{n+1/2} - A_3 (E_3)_{i,j,k}^{n+1/2} \right) + \lambda (E_3)_{i,j,k}^{n+1/2} \Delta W_i^n,
$$

where $A_1 = B^x \otimes I_{N_2} \otimes I_{N_3}$, $A_2 = I_{N_1} \otimes B^y \otimes I_{N_3}$ and $A_3 = I_{N_1} \otimes I_{N_2} \otimes B^z$. And $\Delta W_i^n$ is an approximation of $\Delta W_i$ in spatial direction with respect to $x$, see (5.3).

**Remark 2.** In (4.10), $A_i$ satisfies the following skew-symmetric property

$$
A_i^T = -A_i, \quad i = 1, 2, 3.
$$

The following discussions are all based on the full-discrete numerical method (4.10).

### 4.3. Main results

In this subsection, we present the main results on method (4.10) for 3D stochastic Maxwell equations (2.1).

#### 4.3.1. Discrete stochastic multi-symplectic conservation law

**Theorem 7.** The full-discrete numerical method (4.10) has the following discrete stochastic multi-symplectic conservation law

$$
\frac{\omega_{i,j,k}^{n+1} - \omega_{i,j,k}^n}{h} + \sum_{i' = -(M-1)}^{i + (M-1)} (B^x)_{i,i'} \left( \kappa_{i,j,k}^{n+1/2} \right)' + \sum_{j' = -(M-1)}^{j + (M-1)} (B^y)_{j,j'} \left( \kappa_{i,j,k}^{n+1/2} \right)' + \sum_{k' = -(M-1)}^{k + (M-1)} (B^z)_{k,k'} \left( \kappa_{i,j,k}^{n+1/2} \right)' = 0,
$$

where $\omega_{i,j,k}$ are the discrete analogues of the skew-symmetric property.

13
\[ \omega_{i,j,k}^n = \frac{1}{2} d \mathbf{u}_{i,j,k}^n \wedge F d \mathbf{u}_{i,j,k}^n, \quad (\kappa_1)_{i,j,k}^n = d \mathbf{u}_{i,j,k}^n \wedge K_1 d \mathbf{u}_{i,j,k}^n, \quad (\kappa_2)_{i,j,k}^n = d \mathbf{u}_{i,j,k}^n \wedge K_2 d \mathbf{u}_{i,j,k}^n, \]
\[ (\kappa_3)_{i,j,k}^n = d \mathbf{u}_{i,j,k}^n \wedge K_3 d \mathbf{u}_{i,j,k}^n. \]

Therefore, (4.10) is called a stochastic multi-symplectic wavelet collocation method.

\textbf{Proof}: Let \( \mathbf{u} = (H_1, H_2, H_3, E_1, E_2, E_3)^T \), then the stochastic multi-symplectic wavelet collocation method (4.10) can be rewritten as follows

\[ F \frac{\mathbf{u}_{i,j,k}^{n+1} - \mathbf{u}_{i,j,k}^n}{h} + \sum_{i'} \left( \mathbf{B}^x \right)_{i',i} (K_1 \mathbf{u}_{i',j,k}^{n+1/2}) + \sum_{j'} \left( \mathbf{B}^y \right)_{j',j} (K_2 \mathbf{u}_{i,j',k}^{n+1/2}) + \]
\[ + \sum_{k'} \left( \mathbf{B}^z \right)_{k',k} (K_3 \mathbf{u}_{i,j,k'}^{n+1/2}) = \nabla_\mathbf{u} S(\mathbf{u}_{i,j,k}^{n+1/2}) \Delta W_i^n. \]

(4.12)

The corresponding variational equation of (4.12) is

\[ F \frac{d \mathbf{u}_{i,j,k}^{n+1} - d \mathbf{u}_{i,j,k}^n}{h} + \sum_{i'} \left( \mathbf{B}^x \right)_{i',i} (K_1 d \mathbf{u}_{i',j,k}^{n+1/2}) + \sum_{j'} \left( \mathbf{B}^y \right)_{j',j} (K_2 d \mathbf{u}_{i,j',k}^{n+1/2}) + \]
\[ + \sum_{k'} \left( \mathbf{B}^z \right)_{k',k} (K_3 d \mathbf{u}_{i,j,k'}^{n+1/2}) = \nabla^2 S(\mathbf{u}_{i,j,k}^{n+1/2}) d \mathbf{u}_{i,j,k}^{n+1/2} \Delta W_i^n. \]

(4.13)

Taking the wedge product with \( d \mathbf{u}_{i,j,k}^{n+1/2} \) on both sides of (4.13), we obtain the discrete stochastic multi-symplectic conservation law (4.11). The proof is thus completed.

4.3.2 Discrete energy conservation law

\textbf{Theorem 8}. Under periodic boundary conditions, the stochastic multi-symplectic wavelet collocation method (4.10) has the following discrete energy conservation law in the sense of almost surely, that is,

\[ |E^n|^2 + |H^n|^2 = \text{Constant, a.s.}, \]

(4.14)

where

\[ |E^n|^2 = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \sum_{k=1}^{N_3} \left( (E_{i,j,k}^n)^2 + (E_{i,j,k}^n)^2 + (E_{i,j,k}^n)^2 \right), \]
\[ |H^n|^2 = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \sum_{k=1}^{N_3} \left( (H_{i,j,k}^n)^2 + (H_{i,j,k}^n)^2 + (H_{i,j,k}^n)^2 \right). \]
Proof: For convenience, we omit the subscripts. Then, we make inner product for equations (4.10) with $E_1^{n+1/2}, E_2^{n+1/2}, E_3^{n+1/2}, H_1^{n+1/2}, H_2^{n+1/2}, H_3^{n+1/2}$, respectively, it yields

\[
\begin{align*}
\frac{(E_1^{n+1})^2 - (E_1^n)^2}{2} &= (A_2 H_3^{n+1/2} - A_3 H_2^{n+1/2}) E_1^{n+1/2} h - \lambda H_1^{n+1/2} E_1^{n+1/2} \Delta W_i, \\
\frac{(E_2^{n+1})^2 - (E_2^n)^2}{2} &= (A_3 H_1^{n+1/2} - A_1 H_3^{n+1/2}) E_2^{n+1/2} h - \lambda H_2^{n+1/2} E_2^{n+1/2} \Delta W_i, \\
\frac{(E_3^{n+1})^2 - (E_3^n)^2}{2} &= (A_1 H_2^{n+1/2} - A_2 H_1^{n+1/2}) E_3^{n+1/2} h - \lambda H_3^{n+1/2} E_3^{n+1/2} \Delta W_i, \\
\frac{(H_1^{n+1})^2 - (H_1^n)^2}{2} &= -(A_2 E_3^{n+1/2} - A_3 E_2^{n+1/2}) H_1^{n+1/2} h + \lambda E_1^{n+1/2} H_1^{n+1/2} \Delta W_i, \\
\frac{(H_2^{n+1})^2 - (H_2^n)^2}{2} &= -(A_3 E_1^{n+1/2} - A_1 E_3^{n+1/2}) H_2^{n+1/2} h + \lambda E_2^{n+1/2} H_2^{n+1/2} \Delta W_i, \\
\frac{(H_3^{n+1})^2 - (H_3^n)^2}{2} &= -(A_1 E_2^{n+1/2} - A_2 E_1^{n+1/2}) H_3^{n+1/2} h + \lambda E_3^{n+1/2} H_3^{n+1/2} \Delta W_i.
\end{align*}
\]

Summing over subscripts $i, j, k$, exploiting the skew-symmetric property of $A_i$, and using the periodic boundary conditions, we have

\[
\left( |E_i^{n+1}|^2 + |H_i^{n+1}|^2 \right) - \left( |E_i^n|^2 + |H_i^n|^2 \right) = 0, \ a.s.,
\]
which leads to the discrete energy conservation law (4.14).

This completes the proof.

The result of this theorem is evidently consistent with the energy conservation law (2.8), which means that the energy conservation law can be preserved by the proposed stochastic multi-symplectic wavelet collocation method in the sense of almost surely. In the next section, we will show this conservation law by several numerical experiments.

5. Numerical results

This section will provide numerical experiments to test the new derived stochastic multi-symplectic wavelet collocation method (4.10). We will present some numerical results for 3D stochastic Maxwell equations with multiplicative noise. The main work focuses on the energy conservation law, and stability in long-time computations as well.

When we take no account of the noise term, i.e., $\lambda = 0$, we come back to the 3D deterministic Maxwell equations. Choosing the initial values as

\[
E_{i0} = \cos(2\pi(x + y + z)), \ E_{20} = -2E_{i0}, \ E_{30} = E_{i0},
\]

\[
H_{i0} = \sqrt{3}E_{i0}, \ H_{20} = 0, \ H_{30} = -\sqrt{3}E_{i0}.
\]

(5.1)

In our numerical calculations, the periodic boundary conditions are considered. Here, the numerical spatial domain $\Theta = [0, 1] \times [0, 1] \times [0, 1]$, and the initial value is given by (5.1).
In the following experiments, we take the temporal step-size \( h = 0.005 \), the spatial meshgrid-size \( \Delta x = \Delta y = \Delta z = 1/2^5 \) and the grid points \( N_1 = N_2 = N_3 = 32 \). We use the order of the Daubechies scaling function \( M = 10 \) to solve the problem till time \( T = 20 \) and \( T = 200 \), respectively. Moreover, we choose various sizes of noise, such as \( \lambda = 0, \lambda = 0.5, \lambda = 1 \) and \( \lambda = 5 \).

According to the precise mathematics definition of the \( Q \)-Wiener process (2.3), we take the orthonormal basis \((e_m)_{m \in \mathbb{N}}\) and eigenvalue \( \eta_m \) as

\[
e_m(x) = \sqrt{2} \sin(m \pi x), \quad \eta_m = \frac{1}{m^2},
\]

then \( \Delta W^n_i \) can be regarded as an approximation of integral

\[
\Delta W^n_i = \frac{1}{h \Delta x} \int_{(i+1)\Delta x}^{i\Delta x} \int_{t_n}^{t_{n+1}} \sum_{m=1}^{M} \sqrt{\eta_m} e_m(x) d\beta_m(s) d\tau.
\]

Substituting (5.2) into (5.3)

\[
\Delta W^n_i = \frac{1}{h \Delta x} \sum_{m=1}^{M} \sqrt{\eta_m} \frac{1}{m \pi} \left[ \cos(m \pi i \Delta x) - \cos(m \pi (i + 1) \Delta x) \right] \left[ \beta_m(t_{n+1}) - \beta_m(t_n) \right],
\]

where \( (\beta_m(t_{n+1}) - \beta_m(t_n)) / \sqrt{h} \) are independent random variables with \( \mathcal{N}(0, 1) \) distribution.

In the sequel, we will use (5.4) as the discretization of \( Q \)-Wiener process. And we truncate the increment of the \( Q \)-Wiener process (4.3) until \( M = 100 \).

Fig. 5.1 shows the evolution of the discrete energy of one trajectory with different size of noise, \( \lambda = 0, \lambda = 0.5, \lambda = 1 \), and \( \lambda = 5 \). We show the time step in X-axis, and the discrete energy \( Q^n \) in Y-axis. As was proved in Theorem 8, the stochastic multisymplectic wavelet collocation method preserves the discrete energy conservation law in the sense of almost surely. Although different sizes of noise are chosen, the figures of the energy conservation law remain to be straight horizontal lines approximately.

Fig. 5.2 illustrates the global residuals of the discrete energy of one trajectory, i.e., \( (Err)^n := Q^n - Q^0 \), all reach the magnitude of \( 10^{-11} \) for various \( \lambda \). Here, \( Q^n \) denotes the discrete energy at time-step \( t_n \). We show the time step in X-axis, and the global residuals of the discrete energy \( (Err)^n \) in Y-axis. All these indicate that, the stochastic multi-symplectic wavelet collocation method preserves the discrete energy conservation law in the sense of almost surely no matter how big the size of noise is. Especially, the energy preserve invariant until \( T = 20 \) when \( \lambda = 0 \), this matches with the classical energy conservation law in deterministic case (2.7).

The stability in long-time computations related to the discrete energy conservation law is illustrated in Fig 5.3 from which it implies that the discrete energy preserves invariant until \( T = 200 \) with various size of noise. The error of the discrete energy can be still controlled under \( 10^{-10} \). It can be seen that the magnitude of the global residuals become bigger than the case of \( T = 20 \). This phenomenon is due to the effect of cumulative errors caused by the long time computations.
Figure 5.1: Evolution of the energy conservation law over one trajectory until $T = 20$ with $\Delta t = h = 0.005$ as $\lambda = 0$, $\lambda = 0.5$, $\lambda = 1$ and $\lambda = 5$, respectively.

Figure 5.2: The global errors of discrete energy conservation law over one trajectory until $T = 20$ with $\Delta t = h = 0.005$ as $\lambda = 0$, $\lambda = 0.5$, $\lambda = 1$ and $\lambda = 5$, respectively.
6. Conclusions and remarks

In conclusion, for 3D stochastic Maxwell equations with multiplicative noise, we find that the equations have an energy conservation law in the sense of almost surely. In particular, we propose a stochastic multi-symplectic wavelet collocation numerical method for 3D stochastic Maxwell equations with multiplicative noise which has not discovered in the existing literatures. The numerical method preserves the discrete stochastic multi-symplectic conservation law and the discrete energy conservation law in the sense of almost surely. Moreover, based on the fundamental convergence theorem, we give the first mean-square order of convergence to the semi-discrete scheme in temporal direction. Our numerical experiments match theoretical results well and also show that our method has the stability in long-time computations.

The convergence order we consider in the paper is merely the semi-discrete scheme, namely, in temporal direction. However, it is difficult to get the convergence order of the full-discrete scheme. We will discuss it in the further work. Certainly, it is also very important to widen the application of the method to other context, such as, stochastic Schrödinger equation, stochastic Korteweg-de Vries equation, etc.

7. References

References

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