Power domination in block graphs

Guangjun Xu, Liying Kang*, Erfang Shan, Min Zhao

Department of Mathematics, Shanghai University, Shanghai 200444, China

Received 16 October 2005; received in revised form 17 April 2006; accepted 24 April 2006

Communicated by D.-Z. Du

Abstract

The problem of monitoring an electric power system by placing as few measurement devices in the system as possible is closely related to the well-known domination problem in graphs. In 2002, Haynes et al. considered the graph theoretical representation of this problem as a variation of the domination problem. They defined a set $S$ to be a power dominating set of a graph if every vertex and every edge in the system is monitored by the set $S$ (following a set of rules for power system monitoring). The power domination number $\gamma_p(G)$ of a graph $G$ is the minimum cardinality of a power dominating set of $G$. This problem was proved NP-complete even when restricted to bipartite graphs and chordal graphs. In this paper, we present a linear time algorithm for solving the power domination problem in block graphs. As an application of the algorithm, we establish a sharp upper bound for power domination number in block graphs and characterize the extremal graphs.

© 2006 Elsevier B.V. All rights reserved.

MSC: 05C85; 05C69

Keywords: Algorithm; Power dominating set; Block graphs; Electric power system

1. Introduction

All graphs considered here are simple, i.e., finite, undirected, and loopless. For standard graph theory terminology not given here we refer to [3]. Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$, the open neighborhood $N(v)$ of the vertex $v$ consists of vertices adjacent to $v$, i.e., $N(v) = \{u \in V \mid (u, v) \in E\}$, and the closed neighborhood of $v$ is $N[v] = \{v\} \cup N(v)$. For a subset $S \subseteq V$, we define $N[S] = \bigcup_{x \in S} N[x]$. The subgraph induced by $S$ is denoted by $G[S]$. The distance $d_G(u, v)$ of two vertices $u$ and $v$ is the minimum length of a path between $u$ and $v$. The degree of a vertex $v$ of $G$ is denoted by $d_G(v) = |N_G(v)|$, and a vertex with degree one is called a leaf. The minimum and maximum degrees of vertices of $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For any connected graph $G$, a vertex $v \in V(G)$ is called a cut-vertex of $G$ if $G - v$ is no longer connected. A maximal connected induced subgraph without a cut-vertex is called a block of $G$. A graph $G$ is a block graph if every block in $G$ is complete.

* Corresponding author.
E-mail address: lykang@staff.shu.edu.cn (L. Kang).

© 2006 Elsevier B.V. All rights reserved.
Electric power companies need to continually monitor their system’s state as defined by a set of variables, for example, the voltage magnitude at loads and the machine phase angle at generators [8]. One method of monitoring these variables is to place phase measurement units (PMUs) at selected locations in the system. For economical considerations, companies seek to minimize the number of PMUs needed to be placed and maintain the ability of monitoring the entire system. Recently, it has been shown that this problem can be viewed theoretically as power domination problem in graphs [7].

Let \( G = (V, E) \) be a graph representing an electric power system, where a vertex represents an electrical node (a substation bus where transmission lines, loads and generators are connected) and an edge represents a transmission line joining two electrical nodes. A PMU measures the state variables of the vertex at which it is placed and its incident edges and their endvertices. These vertices and edges are said to be observed. The other observation rules are as follows [7]:

1. Any vertex that is incident to an observed edge is observed.
2. Any edge joining two observed vertices is observed.
3. If a vertex is incident to a total of \( k \geq 1 \) edges and if \( k - 1 \) of these edges are observed, then all \( k \) of these edges are observed.

A set \( S \subseteq V \) is a dominating set in a graph \( G = (V, E) \) if every vertex in \( V \setminus S \) has at least one neighbor in \( S \). The domination number of \( G \), denoted by \( \gamma(G) \), is the minimum cardinality of a dominating set of \( G \). Considering the power system monitoring problem as a variation of the dominating set problem, we define a set \( S \) to be a power dominating set (PDS) if every vertex and every edge in \( G \) is observed by \( S \). The power domination number of \( G \), denoted by \( \gamma_p(G) \), is the minimum cardinality of a PDS of \( G \). A dominating set (respectively, PDS) of \( G \) with minimum cardinality is called a \( \gamma(G) \)-set (respectively, \( \gamma_p(G) \)-set).

The concept of power domination was introduced by Haynes et al. [7]. They showed that the power domination problem is NP-complete even when restricted to bipartite graphs and chordal graphs, and provided a linear time algorithm on the power domination problem for trees. The power domination problem is also studied in [2,4,6] and elsewhere. Some efficient algorithms for solving domination-related problems on block graphs were given in [5,10].

In this paper, we study power domination in block graphs, which is a superclass of trees, and present a linear algorithm for finding a minimum PDS of a block graph.

2. The power domination problem in block graphs

In this section we investigate the power domination of block graphs and present a linear algorithm for solving the power domination problem on this class of graphs. For block graphs, we first present the following result.

**Lemma 2.1.** Let \( G \) be a block graph, then there exists a \( \gamma_p(G) \)-set in which every vertex is a cut-vertex of \( G \).

**Proof.** Let \( S \) be a \( \gamma_p(G) \)-set. Suppose there exists a vertex \( v \in S \) such that \( v \) is not a cut-vertex of \( G \). If every cut-vertex that is adjacent to \( v \) is contained in \( S \), then \( S \setminus \{v\} \) is a smaller PDS than \( S \), which is a contradiction. So there exists a cut-vertex \( u \) such that \( u \) is adjacent to \( v \) and \( u \notin S \), then \( (S \setminus \{v\}) \cup \{u\} \) is a \( \gamma_p(G) \)-set, the result follows. \( \square \)

Our algorithm works on a tree-like decomposition structure, named refined cut-tree, of a block graph. Let \( G \) be a block graph with \( h \) blocks \( BK_1, \ldots, BK_h \) and \( p \) cut-vertices \( v_1, \ldots, v_p \). The cut-tree of \( G \), denoted by \( T^B(V^B, E^B) \), is defined as \( V^B = \{BK_1, \ldots, BK_h, v_1, \ldots, v_p\} \) and \( E^B = \{(BK_i, v_j) \mid v_j \in BK_i, 1 \leq i \leq h, 1 \leq j \leq p\} \). It is shown in [1] that the cut-tree of a block graph can be constructed in linear time by the depth-first-search method. For any block \( BK_i \) of \( G \), define \( B_i = \{v \in BK_i \mid v \) is not a cut-vertex\}, where \( 1 \leq i \leq h \). We can refine the cut-tree \( T^B(V^B, E^B) \) as \( V^B = \{B_1, \ldots, B_h, v_1, \ldots, v_p\} \) and \( E^B = \{(B_i, v_j) \mid v_j \in BK_i, 1 \leq i \leq h, 1 \leq j \leq p\} \), and each \( B_i \) is called a block-vertex. It should be noted that a block-vertex in the refined cut-tree of a block graph may be empty. A block graph \( G \) with five blocks, \( BK_1 = G[[a, b, d]], BK_2 = G[[c, e]], BK_3 = G[[d, e]], BK_4 = G[[d, g, h]] \) and \( BK_5 = G[[e, f, i, j]] \) is shown in Fig. 1, the corresponding cut-tree and refined cut-tree of \( G \) are shown in Fig. 2, where \( B_1 = [a, b], B_2 = [c], B_3 = \emptyset, B_4 = [g, h] \) and \( B_5 = \{f, i, j\} \).

In what follows, we consider the refined cut-tree \( T^B(V^B, E^B) \) of the original block graph \( G \) as the input of our problem. We just treat \( T^B(V^B, E^B) \) as an ordinary tree regardless of the fact that every block-vertex is actually subset of
vertices of the original block graph. For clarity, we denote a block-vertex $B_i$ by $v_i^B$ in $T^B(V^B, E^B)$, here the superscript $B$ of $v_i^B$ indicates that this vertex is a block-vertex. Furthermore, $v_i^B$ is corresponding to $B_i$ one by one. We first give some notations for a tree.

Let $T$ be a tree rooted at $r$ and $v$ is a vertex of $T$, the level number of $v$, denoted by $l(v)$, is the length of the unique $r$-$v$ path in $T$. If a vertex $v$ of $T$ is adjacent to $u$ and $l(u) > l(v)$, then $u$ is called a child of $v$ and $v$ is the parent of $u$. A vertex $w$ is a descendant of $v$ (and $v$ is an ancestor of $w$) if the level numbers of the vertices on the $v$-$w$ path are monotonically increasing. Let $D(v)$ denote the set of descendants of $v$, and define $D[v] = D(v) \cup \{v\}$. The maximal subtree of $T$ rooted at $v$ is the subtree of $T$ induced by $D[v]$ and is denoted by $T_v$.

We are now ready to give a linear algorithm for finding a minimum PDS in a block graph. Our algorithm is presented actually as a color-marking process and we give a brief overview on this algorithm.

Let $G$ be a connected block graph, and $T^B(V^B, E^B)$ the refined cut-tree of $G$. For notation convenience, set $T^B = T^B(V^B, E^B)$ and let $V_{vj} = \{v \in V(G) | v \in BK_i$ and $v_i^B$ is a vertex of $T^B_{vj}\}$ for a cut-vertex $vj$ of $G$. Suppose $T^B(V^B, E^B)$ is a rooted tree with vertex set $\{v_1, v_2, \ldots, v_n\}$ such that $l(v_i) \leq l(v_j)$ for $i > j$, and the root is a cut-vertex $v_n$ (in $G$). Let $H_i$ be the set of all vertices with level number $i$, and $k$ be the largest level number. By the definition of $T^B(V^B, E^B)$, it is clear that $H_0 = \{v_n\}$ and, $H_i$ contains only either cut-vertices of $G$ for even $i$ or block-vertices for odd $i$. Clearly, $k$ is odd. By Lemma 2.1, there exists a $\gamma_p(G)$-set that contains only cut-vertices for a block graph $G$. Our algorithm considers directly the cut-vertices in the rooted tree $T^B$. It starts from the largest even level of $T^B$ and works upward to the root of the tree. Initially, all vertices of $T^B$ are marked with white (which means all needed to be observed), and eventually, every white vertex will be marked with black or gray (except for the possibility that some will remain white and can also be observed by the Rules of power observation). In the end of the algorithm, all black-vertices form a minimum PDS of $G$, and each gray vertex is observed by some black cut-vertex. For a gray cut-vertex $v$, $\text{bound}(v) = 1$ means that $v$ is bounded, i.e., $v$ itself is already observed and there exists unique white vertex which is adjacent to $v$ in $G[V_v]$ (noting that such a white vertex can be observed by Rules of power observation in $G[V_v]$); $\text{bound}(v) = 0$ means that there does not exist any white vertex adjacent to $v$ in $G[V_v]$. 

![Fig. 1. A block graph $G$ with five blocks.](attachment://fig1.png)

![Fig. 2. (a) The cut-tree of $G$ in Fig. 1, (b) the refined cut-tree of $G$ in Fig. 1.](attachment://fig2.png)
Algorithm PDS-BG. Find a minimum PDS of a connected block graph \( G \).

**Input:** A block graph \( G \) of order \( n \geq 3 \).

**Output:** A minimum PDS of \( G \).

Construct a refined cut-tree \( T^B(V^B, E^B) \) of \( G \) with vertex set \{\( v_1, v_2, \ldots, v_n \)\} so that \( l(v_i) \leq l(v_j) \) for \( i > j \), and the root is a cut-vertex \( v_n \) (in \( G \)). For every vertex \( v_j \) that lies in the odd levels \( H_1, H_3, \ldots, H_k \), relabel \( v_j \) as \( v^B_j \) (the superscript \( B \) of \( v^B_j \) indicates that it is a block-vertex and \( v^B_j \) corresponds to \( B_j \) one by one).

Initialization: \( S := \emptyset \); for every vertex \( v \in V^B \), mark \( v \) with white and set \( bound(v) := 0 \).

for \( i := k - 1 \) down to 0 by step-length 2 do

for every \( v_j \in H_i \) do

if there exists a gray vertex \( v^B_a \in N(v_j) \cap H_{i+1} \) or a white vertex \( v^B_b \in N(v_j) \cap H_{i+1} \) so that \( B_b = \emptyset \) and all vertices of \( N(v^B_a) \cap H_{i+2} \) are gray and there exists at least one vertex \( v \in N(v^B_a) \cap H_{i+2} \) with \( bound(v) = 0 \) then

{ mark \( v_j \) with gray;

for every white \( v^B_z \in N(v_j) \cap H_{i+1} \) do

if \( N(v^B_z) \cap H_{i+2} \) contains at least one gray vertex \( v \) with \( bound(v) = 0 \) then

{ if \( |B_z| = 0 \) and \( N(v^B_z) \cap H_{i+2} \) contains at most one white vertex or \( |B_z| = 1 \) and \( N(v^B_z) \cap H_{i+2} \) contains no white vertex then mark \( v^B_z \) with gray;

if \( B_z = 0 \) and every vertex \( v \in N(v^B_z) \cap H_{i+2} \) is gray and \( bound(v) = 1 \) then

mark \( v^B_z \) with gray;

end for

end for

if \( i \geq 0 \) then

\( W := \{v^B \mid v^B \in N(v_j) \cap H_{i+1} \) and \( v^B \) is white\};

\( C^v := \{u \mid u \in N(W) \cap H_{i+2} \) and \( u \) is white\};

\( B^v := \bigcup_{v^B \in W} \{v^B \mid v^B \in B^v_m \} \);

if \( v_j \neq v_n \) then

{ if \( |B^v_j \cup C^v| \geq 2 \) then

mark \( v_j \) with black and all white vertices in \( N(v_j) \) with gray;

\( S := S \cup \{v_j\} \);

if \( |B^v_j \cup C^v| = 1 \) and \( v_j \) is gray then set \( bound(v_j) := 1 \) }

if \( v_j = v_n \) then

{ if \( |B^v_j \cup C^v| \geq 2 \) or \( v_j \) is white then

mark \( v_j \) with black and all white vertices in \( N(v_j) \) with gray;

\( S := S \cup \{v_j\} \) }

end for

end for

end for

output \( S \).

In order to prove that \( S \) is a PDS of \( G \), we first show that Algorithm PDS-BG maintains the following two invariants: after all vertices in \( H_i \) have been just processed, invariant 1, for any black cut-vertex \( v_j \in H_i \), \( S \) is a PDS of \( G[V_{v_j}] \); invariant 2, for any gray cut-vertex \( v_j \in H_i \), \( S \) is a PDS of \( G[V_{v_j}] \). Apparently, the invariants are true initially.

Suppose these two invariants hold for all \( l \geq i \in \{k - 1, k - 3, k - 5, \ldots, 4, 2\} \), in the following we show that these two invariants hold for \( l = i - 2 \).

Let \( l = i - 2 \), \( v_j \) be any black cut-vertex in \( H_i \), then any vertex in \( BK_l \) can be observed by \( v_j \) if \( v^B_i \in N(v_j) \cap H_{i+1} \). For any white \( z \in \bigcup_{v^B_j \in H_{i+2}} |B^v_z \cup C^v| \leq 1 \) by Algorithm PDS-BG. Since \( z \) is observed by \( v_j \) and \( |B^v_z \cup C^v| \leq 1 \), the vertex in \( B^v_z \cup C^v \) can be observed by \( v_j \). By the similar argument and the assumption that two invariants hold for \( H_i (l \geq i) \), then all vertices in \( V_z \) are observed. So \( V_{v_j} \) is eventually observed and invariant 1 holds for \( l = i - 2 \).

Next suppose \( v_j \) is any gray vertex in \( H_i \), where \( l = i - 2 \). We define the following sets:

\( Y_1 = \{v^B_a \mid v^B_a \in T_{v_j} \cap H_{i+1}, v^B_a \) is gray\};

\( Y_2 = \{v^B_a \mid v^B_a \in T_{v_j} \cap H_{i+1}, v^B_a \) is white\};

\( Y_{11} = \{v^B_a \in Y_1 \mid \) there exists a black vertex \( v \in H_{i+2} \cap N(v^B_a) \) or \( B_a = \emptyset \) and all vertices of \( N(v^B_a) \cap H_{i+2} \) are gray and there exists at least one gray vertex \( v \in N(v^B_a) \cap H_{i+2} \) with \( bound(v) = 0 \);
Theorem 2.3. Algorithm PDS-BG computes in linear time a minimum PDS of a given connected block graph \( G \).

Let \( G \) be a connected block graph, and \( T^B(\mathcal{V}^B, E^B) \) be the refined cut-tree of \( G \). Let \( T^B(\mathcal{V}^B, E^B) \) be a rooted tree with vertex set \( \{v_1, \ldots, v_n\} \) such that \( l(v_i) \leq l(v_j) \) for \( i > j \), and the root is a cut-vertex \( v_n \) (in \( G \)). Let \( S = \{v_1, v_2, \ldots, v_m\} \) be the set computed by Algorithm PDS-BG, where \( i_1 < i_2 < \cdots < i_m \). By Lemma 2.2, \( S \) is a PDS of \( G \). So \( \gamma_p(G) \leq |S| \). We now show that \( \gamma_p(G) = |S| \). Suppose that \( \gamma_p(G) < |S| \). Among all \( \gamma_p(G) \)-sets, let \( S^* \) be chosen so that the first integer \( j(1 \leq j \leq m) \) with \( v_{i_j} \notin S^* \) is as large as possible. Let \( T^B_{v_{i_j}} \) be the subtree induced by \( D[v_{i_j}] \). If \( (T^B_{v_{i_j}} \cap S^*) \setminus S \neq \emptyset \), then replace any vertex of \( (T^B_{v_{i_j}} \cap S^*) \setminus S \) by \( v_{i_j} \) to form a new \( \gamma_p(G) \)-set which contains all vertices in \( \{v_1, v_2, \ldots, v_j\} \), which contradicts our choice of \( S^* \). Thus, we have \( (T^B_{v_{i_j}} \cap S^*) \setminus S = \emptyset \). Furthermore, let \( w^B_{v_{i_j}} \) be the father of \( v_{i_j} \) in \( T^B \), then \( S^* \) contains no vertex of \( B_f \). Otherwise, we can also get a new \( \gamma_p(G) \)-set by replacing that vertex with \( v_{i_j} \), which contains all vertices in \( \{v_1, v_2, \ldots, v_j\} \). We distinguish between two cases.

Case 1: \( v_{i_j} \neq v_n \). By Algorithm PDS-BG, \( v_{i_j} \neq v_n \) and \( v_{i_j} \in S \) implies \( |B^n_{v_{i_j}} \cup C^{v_{i_j}}| \geq 2 \), i.e., there exist at least two white vertices in \( G[V_{v_{i_j}}] \) need to be observed by \( v_{i_j} \). Since \( (T^B_{v_{i_j}} \cap S^*) \setminus S = \emptyset \), it is clear that such white vertices in \( G[V_{v_{i_j}}] \) cannot be observed by \( S^* \).

Case 2: \( v_{i_j} = v_n \). Then there must be either \( |B^n_{v_n} \cup C^{v_n}| \geq 2 \) or \( v_n \) is white (before \( v_n \) is eventually marked with black). If \( |B^n_{v_n} \cup C^{v_n}| \geq 2 \), the argument is similar to case 1; if \( v_n \) is white (before \( v_n \) is eventually marked with black), clearly at least \( v_n \) cannot be observed by \( S^* \).

Both cases 1 and 2 contradict the assumption that \( S^* \) is a \( \gamma_p(G) \)-set, then \( \gamma_p(G) = |S| \).

The running time of Algorithm PDS-BG can be estimated as follows. The running time is linear to the size of refined cut-tree of \( G \), while the time for constructing a refined cut-tree of \( G \) is linear [1]. Therefore, the total time needed to perform Algorithm PDS-BG is linear. □

We now describe the running of Algorithm PDS-BG by using an example. Fig. 3 shows a block graph \( BG \) and its corresponding refined cut-tree \( T^B \) with block-vertices \( B_2 = B_4 = B_5 = B_7 = B_8 = B_{10} = B_{12} = \emptyset \), \( B_1 = \{v_{10}\}, B_3 = \{v_{15}\}, B_6 = \{v_4\}, B_9 = \{v_{13}\}, B_{11} = \{v_{20}\}, B_{13} = \{v_1, v_2\}, B_{14} = \{v_5, v_9\}, B_{15} = \{v_{23}, v_{24}\}, B_{16} = \{v_{18}\}.\)
Then each vertex in block graph. Since every vertex in $\text{H}_2$ would received color gray by whose one gray neighbor in $\text{H}_3$. Furthermore, the bound labels of vertices $v_7, v_{12}$ and $v_{16}$ would have been changed to 1 by Algorithm PDS-BG, i.e., $\text{bound}(v_7) = \text{bound}(v_{16}) = 1$. After that, Algorithm PDS-BG would process $v_{11}$ in $\text{H}_0$ and marked $v_{11}$ with gray. By Algorithm PDS-BG, for each black vertex $v_n$ is a cut-vertex of $\text{G}$, $\text{bound}(v_n) = 1$, let some vertex $v_B$ adjacent to $v$ in $T^B$ (when $v$ is gray) or the unique white vertex $w_B$ adjacent to $v$ in $T^B$ (when $v$ is white) and marked with color black.

We first claim that all cut-vertices in $\text{H}_{k-1}$ (where $k$ is the largest level number) are black. Suppose to the contrary that there exists a vertex $w \in \text{H}_{k-1}$ which is not black, then there must be exactly one vertex $w$ dangling at $w$. Let $S^*$ be a PDS obtained by Algorithm PDS-BG. Since every vertex in $S^*$ corresponds to at least two white vertices in $\text{G}$ and $u$ does not correspond to any vertex in $S^*$, therefore $|S^*| \leq (n - 1)/3$, and $S^*$ is a smaller PDS than $S$, which is a contradiction. Next, for any vertex $v \in \text{H}_{k-1}$, let $W = \{v_B \mid v_B \in N(v) \cap \text{H}_k\}$ and $B_W = \bigcup_{v \in W} B_{v_B}$, and let $w_B \in \text{H}_{k-2}$ be the father of $v$, then we claim that $|B_{v_B}| = 2$ and $|B_{w_B}| = 0$. We first show $|B_{v_B}| = 2$ for every vertex $v \in \text{H}_{k-1}$. Suppose to the contrary that there exists a vertex $v \in \text{H}_{k-1}$ such that $|B_{v_B}| \geq 3$. Let $G^*$ be obtained from $G$ by deleting a vertex in $B_{v_B}$, obviously $V(G^*) < V(G) = n$. Using Algorithm PDS-BG for graph $G^*$, we get a PDS $S'$.

**Theorem 2.4.** For any block graph $G$ with order $n \geq 3$, $\gamma_p(G) \leq n/3$ with equality if and only if $G$ is the corona $G' \circ H_2$, where $G'$ is any block graph and $H_2 \in \{K_2, \overline{K_2}\}$.

**Proof.** By Algorithm PDS-BG, for each black vertex $v \in S$, it is easy to see that $v$ corresponds to at least two white vertices in $\text{G}$ if $v \neq v_n$. We now show that $v_n$ also corresponds to at least two white vertices in $\text{G}$ if $v_n \in S$. Since $v_n \in S$, there must be either $|B_{v_n} \cup C(v_n)| \geq 2$ or $v_n$ is white (for which $v_n$ is eventually marked with black). Clearly $v_n$ corresponds to at least two white vertices in $\text{G}$ when $|B_{v_n} \cup C(v_n)| \geq 2$. Suppose $v_n$ is white (before $v_n$ is eventually marked with black), then each block-vertex $v_B^B$ of $T^B \cap H_1$ is white as well as either $|B_{v_n}| \geq 1$ or $|B_{v_n}| = 0$ and the vertex in $N(v_B^B) \cap H_2$ (there exists at least one) is white or gray with $\text{bound}(v) = 1$. For a block-vertex $v_B^B$ of $T^B_{v_n} \cap H_1$, if $|B_{v_n}| \geq 1$, then let some vertex $u \in B_a$ be corresponding to $v_B^B$; if $|B_{v_n}| = 0$, there exists at least one vertex $v \in N(v_B^B) \cap H_2$, let $v$ (when $v$ is white) or the unique white vertex $u$ adjacent to $v$ in $T^B$ (when $v$ is gray) and marked with color black.

We first claim that all cut-vertices in $\text{H}_{k-1}$ (where $k$ is the largest level number) are black. Suppose to the contrary that there exists a vertex $w \in \text{H}_{k-1}$ which is not black, then there must be exactly one vertex $u$ dangling at $w$. Let $S^*$ be a PDS obtained by Algorithm PDS-BG. Since every vertex in $S^*$ corresponds to at least two white vertices in $\text{G}$ and $u$ does not correspond to any vertex in $S^*$, therefore $|S^*| \leq (n - 1)/3$, and $S^*$ is a smaller PDS than $S$, which is a contradiction. Next, for any vertex $v \in \text{H}_{k-1}$, let $W = \{v_B \mid v_B \in N(v) \cap \text{H}_k\}$ and $B_W = \bigcup_{v \in W} B_{v_B}$, and let $w_B \in \text{H}_{k-2}$ be the father of $v$, then we claim that $|B_{v_B}| = 2$ and $|B_{w_B}| = 0$. We first show $|B_{v_B}| = 2$ for every vertex $v \in \text{H}_{k-1}$. Suppose to the contrary that there exists a vertex $v \in \text{H}_{k-1}$ such that $|B_{v_B}| \geq 3$. Let $G^*$ be obtained from $G$ by deleting a vertex in $B_{v_B}$, obviously $V(G^*) < V(G) = n$. Using Algorithm PDS-BG for graph $G^*$, we get a PDS $S'$.
of $G^*$, which is also a PDS of $G$ and $|S'| < |S|$, this is a contradiction to the assumption that $S$ is a $\gamma_p (G)$-set. So we have $|B^v_W| = 2$. Analogously we can get $|B^v_f| = 0$. For every vertex $v \in H_i$ where $i \in \{k - 3, k - 5, \ldots, 2, 0\}$, it is not difficult to prove $G\{v \cup B^v_W\} = K_1 \circ H_2$ and $|B^v_f| = 0$, where $H_2 \in \{K_2, \bar{K}_2\}$, $W = \{v^B \mid v^B \in N(v) \cap H_{i+1}\}$, $B^v_W = \bigcup_{v_B \in W} B_m$ and $w^B_f$ is the father of $v$. It follows that $G$ is the corona $G' \circ H_2$, where $G'$ is any block graph and $H_2 \in \{K_2, \bar{K}_2\}$. □

**Corollary 2.1** (Haynes [7]). For any tree $T$ with order $n \geq 3$, $\gamma_p (T) \leq n/3$ with equality if and only if $T$ is the corona $T' \circ \bar{K}_2$, where $T'$ is any tree.

**Remark.** Recently, Zhao et al. [11] proved that the bound $\gamma_p (G) \leq n/3$ holds for any connected graph $G$ with order $n \geq 3$.

3. Conclusions

In this paper, we investigate power domination in block graphs, and present a linear color-marking-based algorithm for finding a minimum power dominating set of a block graph. By using this algorithm we obtain a sharp upper bound on the power domination number of the class of block graphs. It would be interesting to study whether our algorithm could be adapted to solve the power domination problem on other class of graphs with tree-like decomposition structure.

**References**