Super-connectivity of Kronecker products of split graphs, powers of cycles, powers of paths and complete graphs

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\begin{abstract}
The Kronecker product of two connected graphs $G_1$, $G_2$, denoted by $G_1 \times G_2$, is the graph with vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1), v_1v_2 \in E(G_2)\}$. The $k$th power $G^k$ of $G$ is the graph with vertex set $V(G)$ such that two distinct vertices are adjacent in $G^k$ if and only if their distance apart in $G$ is at most $k$. A connected graph $G$ is called super-$\kappa$ if every minimal vertex cut of $G$ is the set of neighbors of some vertex in $G$. In this note, we consider the super-connectivity of the Kronecker products of several kinds of graphs and complete graphs. We show that $D = G \times K_m$ is super-$\kappa$ for $m \geq 3$ and $G$ satisfying one of the following conditions: (1) $G$ is a non-complete split graph with $|C| \geq 5$; (2) $G$ is a power graph of a path $P_n$ such that $n \geq 2k$; (3) $G$ is a power graph of a cycle $C_n$ such that $n \geq m$ and $n \geq 2r + 1$.
\end{abstract}

1. Introduction

We only consider undirected simple connected graphs without loops and multiple edges. Unless stated otherwise, we follow Bondy and Murty [1] for terminology and definitions.

Let $G = (V, E)$ be a connected graph. For $S \subseteq V$ and $x \in V$, $N(x)$ is the set of neighbors of $x$ in $G$, and $N_S(x) = N(x) \cap S$. The connectivity $\kappa(G)$ of a connected graph $G$ is the least positive integer $k$ such that there is $S \subseteq V$ with $|S| = k$ and $G - S$ is disconnected or reduces to the trivial graph $K_1$. In [2], Whitney showed that $\kappa(G) \leq \delta(G)$. A graph $G$ is called maximally connected if $\kappa(G) = \delta(G)$, or max-$\kappa$ for short. Furthermore, a maximally connected graph $G$ is super-connected, or simply super-$\kappa$, if every minimum vertex cut is the set of the neighbors of a vertex of $G$, that is every minimum vertex cut isolates a vertex; see the survey [3] for details of max-$\kappa$ and super-$\kappa$ graphs. The $k$th power $G^k$ of $G$ is the graph with vertex set $V(G)$ such that two distinct vertices are adjacent in $G^k$ if and only if their distance in $G$ is at most $k$.

A graph $G = (V, E)$ is called a split graph if its vertex set $V$ can be partitioned into a clique $C$ and an independent set $I$. Usually, the split graph $G$ is denoted by $G = (C, I, E)$. If $N(I) \neq C$, then by choosing a vertex $v \in C \setminus N(I)$, and replacing $C$ by $C - v$ and $I$ by $I \cup \{v\}$, $G$ can be rewritten as $G = (C - v, I \cup \{v\}, E)$, in which $N(I \cup \{v\}) = C - v$. Hence, in this work we always assume that $N(I) = C$ for any split graph $G = (C, I, E)$. Clearly, $\delta(G) \leq |C|$. In [4], the authors show that if $G = (C, I, E)$ is a non-complete connected split graph, then $\kappa(G) = \delta(G)$.

The Kronecker product (also named the direct product, tensor product or cross-product) of two nontrivial connected graphs $G_1$ and $G_2$, denoted by $G_1 \times G_2$, is the graph having the vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and the edge set $E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1), v_1v_2 \in E(G_2)\}$. Clearly the Kronecker product of two nontrivial graphs is

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connected if and only if at least one of the factors is not bipartite. The Kronecker product of graphs has been extensively investigated as regards graph colorings, graph recognition and decomposition, graph embeddings, matching theory and stability in graphs (see, for example, [5,6], and the references therein), and this graph product has several applications; for instance, it can be used in modeling concurrency in multiprocessor systems [7] and in the theory of automata [8].

Bresar and Spacapan [9] obtained an upper bound and a lower bound on the edge connectivity of the Kronecker products with some exceptions; they also obtained several upper bounds on the vertex connectivity of the Kronecker product of graphs. Mamut and Vumar obtained the value of the connectivity of the Kronecker product of two complete graphs [10]. And Guji and Vumar studied the connectivity of the Kronecker product of a bipartite graph and a complete graph [11]. We have shown that if $G$ is a bipartite graph with $\kappa(G) = \delta(G)$, then $G \times K_n (n \geq 3)$ is super-$\kappa$ [12]. In this note, we consider the super-connectivity of Kronecker products of several kinds of graphs and complete graphs and we show that $D = G \times K_n$ is super-$\kappa$ for $m \geq 3$ and $G$ satisfying one of the following conditions: (1) $G$ is a non-complete split graph with $|C| \geq 5$; (2) $G$ is a power graph of a path $P^m_n$ such that $n \geq 2k$; (3) $G$ is a power graph of a cycle $C^m_n$ such that $n \geq m$ and $n \geq 2r + 1$.

2. The main results

When considering the Kronecker product $G_1 \times G_2$ with $|V(G_1)| = m$, $|V(G_2)| = n$, we shall always use the labeling $V_1 = V(G_1) = \{u_1, \ldots, u_m\}$, $V_2 = V(G_2) = \{v_1, \ldots, v_n\}$, and set $S_i = V_1 \times \{v_i\}$, $i = 1, \ldots, n$. Moreover, for convenience, we shall abbreviate $(u_i, v_j)$ as $w_{ij}$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$. Then $S_i = \{w_{1i}, \ldots, w_{mi}\} (i = 1, 2, \ldots, n)$ is an independent set in $G_1 \times G_2$, and $V(G_1 \times G_2)$ has a partition $V_1 \times V_2 = S_1 \cup S_2 \cup \cdots \cup S_n$. And we denote by $K_n$ the complete graph, $C_n$ the cycle of $n$ vertices, and $P_m$ the path of $n$ vertices.

We first show that $D = (C, I, E) \times K_m$ is super-connected if $|C| \geq 5$.

Theorem 2.1. Let $G = (C, I, E)$ be a split graph with $|C| \geq 4$. Then $\kappa(G \times K_m) = \delta(G)(m - 1)$ for $m \geq 3$.

Proof. For convenience, we use $D$ for $G \times K_m$. Suppose $w_{ij}$ is a vertex such that $d(w_{ij}) = \delta(G)(m - 1)$. Clearly, $N(w_{ij})$ is a vertex cut. Suppose that there is a minimum vertex cut $S$ with $|S| < \delta(G)(m - 1)$ such that $D - S$ is not connected. Let $w_{kl}$ and $w_{pq}$ be two vertices in different components in $D - S$ with $t \leq q$.

Case 1. $k = p$ or $t = q$.

Without loss of generality we may assume that $t = q$.

Subcase 1.1. $w_{kl}$ and $w_{pq}$ are in $V(C \times K_m)$.

If one of $w_{kl}$ and $w_{pq}$, say $w_{kl}$, has no neighbors in $V(C \times K_m)$, then let $w_{k'l'}$ be a neighbor of $w_{kl}$ in $V(I \times K_m)$ and $w_{k'q'}$ be a neighbor of $w_{pq}$ in $V(C \times K_m)$. If $t' = q'$, then since $w_{kl}$ is a vertex of $I$ and $w_{pq}$ is a vertex of $K_m$, $w_{k'l'}$ is adjacent to $w_{kl}$, $t' = q'$. So the removed vertices are at least $(m - 1)(|C| - 1) + m - 2 = \delta(G)(m - 1) + m - 2 = \delta(G)(m - 1) - 1 = |S|$ and $\delta(G) = |C| \geq 4$. Note that $w_{k'l'}$ has a neighbor $w_{p'q'}$ in $V(C \times K_m)$ as $\delta(G) = |C| \geq 4$. And $w_{p'q'}$ is adjacent to $w_{kq'}$, that is there is a $(w_{pq}, w_{kq'})$-path, a contradiction. Hence $t' \neq q'$, but $w_{k'l'}$ is adjacent to $w_{kl}$ and there is a $(w_{pq}, w_{kq'})$-path, a contradiction.

So we assume that $w_{pq}$ has a neighbor $w_{kq'}$ in $V(C \times K_m)$. $w_{kq'}$ is not adjacent to $w_{pq}$ and thus $k' = p$. Also $w_{pq}$ has a neighbor, say $w_{p'q'}$ in $V(C \times K_m)$, and by our assumption the only possibility is that $p' = k$ and $q' = t'$. Clearly the vertices in $V(C \times K_m)$ other than these four must be in $S$. Thus $|C| \geq 4$, a contradiction.

Subcase 1.2. At least one of $w_{kl}$ and $w_{pq}$ is in $V(C \times K_m)$.

Without loss of generality, say that $w_{pq}$ is not in $V(C \times K_m)$. If $w_{pq}$ is in $V(C \times K_m)$, then $w_{pq}$ has a neighbor $w_{p'q'}$ in $V(C \times K_m)$. If $k \neq p'$ and $t' \neq q'$, then there is a $(w_{pq}, w_{kq'})$-path, a contradiction. If $k = p'$ or $t = q'$, then this is similar to Subcase 1.1, and we can get a contradiction.

Hence $w_{pq}$ and $w_{pq}$ are not in $V(C \times K_m)$. But they have neighbors in $V(C \times K_m)$. This is analogous to the above analysis and we can get the desired result.

Case 2. $t < q$.

$w_{kt}$ has a neighbor $w_{k't'}$ and $w_{pq}$ has a neighbor $w_{p'q'}$.

Subcase 2.1. $w_{k't'}$ and $w_{p'q'}$ are in $V(C \times K_m)$.

If $k' \neq p'$ and $t' \neq q'$, then there is a $(w_{pq}, w_{kq'})$-path, a contradiction. So $k' = p'$ or $t' = q'$, and by Case 1 we are done.

Subcase 2.2. At least one of $w_{k't'}$ and $w_{p'q'}$ is in $V(C \times K_m)$.

Without loss of generality, say that $w_{p'q'}$ is not in $V(C \times K_m)$ and $w_{k't'}$ is in $V(C \times K_m)$. Then $w_{pq}$ is in $V(C \times K_m)$. This is similar to Subcase 2.1 and we can get the desired result.

If $w_{k't'}$ and $w_{p'q'}$ are not in $V(C \times K_m)$, then $w_{kl}$ and $w_{pq}$ are in $V(C \times K_m)$. By Subcase 2.1 we are done. \qed

In fact, the result above can be strengthened to super-$\kappa$ if we replace the condition $|C| \geq 4$ by $|C| \geq 5$. The proof is similar to that of Theorem 2.1. And we have shown that $K_m \times K_n$ is super-$\kappa$ for $n \geq m \geq 2$ and $n + m > 5$ [13]; we consider $G = (C, I, E)$ to be a non-complete split graph.

Theorem 2.2. Let $G = (C, I, E)$ be a non-complete split graph with $|C| \geq 5$. Then $D = G \times K_m$ is super-$\kappa$ with $m \geq 3$. 

Subcase 1. If \( w_{ik} \) and \( w_{pq} \), say \( w_{ik} \), has no neighbors in \( V(C \times K_m) \), then let \( w_{ik}^{(1)} \) be a neighbor of \( w_{ik} \) in \( V(I \times K_m) \) and \( w_{ik}^{(2)} \) be a neighbor of \( w_{pq} \) in \( V(C \times K_m) \). If \( t' = q' \), then since \( w_{ik} \) (\( i = 1, \ldots, m - 1 \)) are the neighbors of \( w_{pq} \) in \( V(C \times K_m) \), \( w_{ik} \) is adjacent to \( w_{pq} \) (\( i = 1, \ldots, m - 1 \)) with \( t_1 \neq q' \). Note that \( |C| > \delta \) since otherwise a path \( w_{ik} w_{ik}^{(1)} w_{pq} \) will be induced. So the vertices are at least \( (m - 1)(|C| - 1) + m - 2 \geq (m - 1)(\delta(G) + 1 - 1) + m - 2 = \delta(G)(m - 1) + m - 2 > \frac{|S|}{m} \) since \( m \geq 3 \). Note that \( w_{ik}^{(2)} \) has a neighbor \( w_{ik}^{(1)} \) in \( V(C \times K_m) \) as \( |C| = |C| \geq 4 \). And \( w_{ik}^{(2)} \) is adjacent to \( w_{pq} \); that is there is a \((w_{ik}, w_{pq})\)-path, a contradiction. Hence \( t' \neq q' \), but \( w_{ik} \) is adjacent to \( w_{pq} \) and there is a \((w_{pq}, w_{ik})\)-path, a contradiction.

So we assume that \( w_{ik} \) has a neighbor \( w_{ik}^{(1)} \) in \( V(C \times K_m) \). \( w_{ik}^{(1)} \) is not adjacent to \( w_{pq} \) and thus \( k' = p \). Also \( w_{pq} \) has a neighbor, say, \( w_{pq}^{(1)} \) in \( V(C \times K_m) \), and by our assumption the only possibility is that \( p' = k \) and \( q' = t \). Clearly the vertices in \( V(C \times K_m) \) other than these four must be in \( S \). Thus \((m - 1)|C| > \delta(G)(m - 1) = |S| \geq m|C| - 4 \) which implies that \( |C| \leq 4 \), a contradiction.

Subcase 1.2. At least one of \( w_{ik} \) and \( w_{pq} \) is not in \( V(C \times K_m) \).

Without loss of generality, say that \( w_{pq} \) is not in \( V(C \times K_m) \). If \( w_{pq} \) is in \( V(C \times K_m) \), then \( w_{pq} \) has a neighbor \( w_{pq}^{(1)} \) in \( V(C \times K_m) \). If \( k' \neq p' \) and \( t' \neq q' \), then there is a \((w_{pq}, w_{ik})\)-path, a contradiction. If \( k' = p' \) or \( t' = q' \), then this is similar to Subcase 1.1, and we can get a contradiction.

Hence \( w_{ik} \) and \( w_{pq} \) are not in \( V(C \times K_m) \). But they have neighbors in \( V(C \times K_m) \). This is analogous to the above analysis and we can get the desired result.

Case 2. \( t < q \).

\( w_{ik} \) has a neighbor \( w_{ik}^{(1)} \) and \( w_{pq} \) has a neighbor \( w_{pq}^{(1)} \).

Subcase 2.1. \( w_{ik}^{(1)} \) and \( w_{pq}^{(1)} \) are in \( V(C \times K_m) \).

If \( k' \neq p' \) and \( t' \neq q' \), then there is a \((w_{pq}, w_{ik})\)-path, a contradiction. So \( k' = p' \) or \( t' = q' \), and by Case 1 we are done.

Subcase 2.2. At least one of \( w_{ik}^{(1)} \) and \( w_{pq}^{(1)} \) is not in \( V(C \times K_m) \).

Without loss of generality, say that \( w_{pq}^{(1)} \) is not in \( V(C \times K_m) \) and \( w_{ik}^{(1)} \) is in \( V(C \times K_m) \). Then \( w_{pq} \) is in \( V(C \times K_m) \). This is similar to Subcase 2.1 and we can get the desired result.

If \( w_{ik}^{(1)} \) and \( w_{pq}^{(1)} \) are not in \( V(C \times K_m) \), then \( w_{ik} \) and \( w_{pq} \) are in \( V(C \times K_m) \). By Subcase 2.1 we are done. \( \square \)

Like in Theorems 2.1 and 2.2, we next consider the maximal connectivity and super-connectivity of \( K_m \times P_n^k \) and \( K_m \times C_n^l \).

We will only give the complete proof for maximal connectivity for two kinds of graphs and omit the proof for super-connectivity, as the two proofs are very similar.

**Theorem 2.3.** Let \( m, n \) be two integers with \( n \geq 2k + 1, m \geq 3 \) and \( G = K_m \times P_n^k \). Then \( G \) is maximally connected.

**Proof.** Obviously, for any vertex \( w_{ij} \) with \( d(w_{ij}) = k(m - 1) \), the neighborhood \( N(w_{ij}) \) of \( w_{ij} \) is a vertex cut of \( G = K_m \times P_n^k \). So \( \kappa(G) \leq k(m - 1) \). It is sufficient to show that \( \kappa(G) = k(m - 1) \).

By way of contradiction we suppose that \( |S| < k(m - 1) \) and \( S \) be a minimal vertex set of \( G \). We shall prove that \( G - S \) is connected. Let \( w_{it} \) and \( w_{pq} \) be two vertices with \( t \leq q \) in two different components, say \( G_1 \) and \( G_2 \), of \( G - S \), respectively.

Case 1. \( t = q \). Then \( t - 1 \geq k \) or \( n - t \geq k \).

Without loss of generality, say that \( n - t \geq k \). If there is a vertex \( w_{it} \) in one of \( [S_{t+} : i = 1, \ldots, k] \) with \( r' \neq k, p \), then there is a \((w_{it}, w_{pq})\)-path, a contradiction. Hence \( A = [S_{t+} : i = 1, \ldots, k] - [w_{it+1}, w_{it+k+1}] : i = -1, \ldots, \max[-t + 1, -k], 1, \ldots, k] \subseteq S \). And \( |A| = (k + \min(t - 1, k))m - 2(k + \min(t - 1, k)) \leq km - k - 1 \). It is easy to see that \( D([w_{it+1}, w_{it+k+1}, w_{pq+1}, w_{pq}], w_{pq}, 1 = -1, \ldots, \max[-t + 1, -k], 1, \ldots, k]) \) (in fact such a subgraph is isomorphic to the Kronecker product of \( K_2 \) and \( P_n^k \)) contains \( k + t - 1 \) vertex disjoint paths between \( w_{it} \) and \( w_{pq} \); see Fig. 1. Thus \( |S| > km - k \) if \( t \geq 2 \). If \( t = 1 \), then it is also easy to find \( k \) vertex disjoint paths between \( w_{it} \) and \( w_{pq} \); see Fig. 1.

Case 2. \( t < q \)

If there are \( w_{it} \in V(G_1) \) and \( w_{pq} \in V(G_2) \) such that \( w_{it} \) and \( w_{pq} \) are in the same \( S_t \), that is \( t' = q' = l \), then by Case 1 there is a \((w_{it}, w_{pq})\)-path in \( G - S \), a contradiction.

Hence any vertex of \( G_1 \) and any vertex of \( G_2 \) are not in the same \( S_t \). Let \( S_t \) be the minimum subscripts of \( G_1 \) and \( S_p \) be the maximum subscripts of \( G_1 \) and \( S_p \) be the minimum subscripts of \( G_2 \) and \( S_p \) be the maximum subscripts of \( G_2 \). Since \( |S| < k(m - 1) \), we have \( r_2 + k < r_1 \). But we have \( |S| \geq k(m - 1) \), a contradiction. \( \square \)
Theorem 2.4. Let \( m, n \) be integers with \( n \geq 2k + 1, m \geq 3 \) and \( G = K_m \times P_n^k \). Then \( G \) is super-\( \kappa \).

Theorem 2.5. Let \( m, n \) be integers with \( n \geq m \geq 3, n \geq 2r + 1 \), and \( G = K_m \times \mathcal{C}_n^r \). Then \( \kappa(G) = 2r(m-1) \).

Proof. Obviously, we take a vertex \( w_{ij} \) with \( d(w_{ij}) = \delta(G) = 2r(m-1) \); then \( N(w_{ij}) \) is a vertex cut of \( G \). By contradiction, we suppose that \( S \) is a minimum vertex cut of \( G \) with \( |S| < 2r(m-1) \). We shall show that \( G - S \) is connected. Take arbitrary vertices \( w_{ik} \) and \( w_{pq} \) with \( t \leq q \) from two different components of \( G - S \).

Case 1. \( t = q \).

Firstly if there is a vertex \( w_{k1k2} \) in one of \( \{S_{q+1} : i = 1, \ldots, r\} \) with \( k_1 \neq k, p \), then there is a \((w_{kt}, w_{pq})\)-path, and \( G - S \) is connected. Secondly if there are no such vertices, we have \( A = \bigcup_{i=1}^r S_{q+i} \geq \{w_{k(t+1)}, w_{p(q+i)} : i = 1, \ldots, r\} \subseteq S \). Since \( |A| = 2mr - 4r \) and \( |S| \leq 2mr - 2r - 1 \), we have \(|N_{S_{q+i+r}}(w_{k(t+r)}) \cap N_{S_{q+i+r}}(w_{p(q+r)})| \geq 1, |N_{S_{q-i-r}}(w_{k(q-r)}) \cap N_{S_{q-i-r}}(w_{p(q-r)})| \geq 1 (i = 1, \ldots, r) \). Although we remove the last \( 2r - 1 \) vertex of \( S \), there is a \((w_{kt}, w_{pq})\)-path in \( G - S \). Hence \( G - S \) is again connected.

Case 2. \( t < q \).

Let \( G_1 \) be the connected component containing \( w_{kt} \) and \( G_2 \) be the connected component containing \( w_{pq} \). If there are \( w_{kt'} \in V(G_1) \) and \( w_{pq'} \in V(G_2) \) such that \( w_{kt'} \) and \( w_{pq'} \) are in the same \( S_i \), that is \( t' = q' = l \), then by Case 1 there is a \((w_{kt'}, w_{pq'})\)-path in \( G - S \), a contradiction.

Hence any vertex of \( G_1 \) and any vertex of \( G_2 \) are not in the same \( S_i \). Let \( S_{k1} \) be the minimum subscript of \( G_1 \) and \( S_{k2} \) be the maximum subscript of \( G_1 \). Let \( S_{p1} \) be the minimum subscript of \( G_2 \) and \( S_{p2} \) be the maximum subscript of \( G_2 \). Since \( |S| \geq 2r(m-1) \), we have \( k_2 + r < p_1, p_2 + r < k_1 \) (addition is modulo \( n \)). But we have \( |S| \geq 2r(m-1) \), a contradiction. \( \square \)

Similarly, we list a theorem below without proof.

Theorem 2.6. Let \( m, n \) be integers with \( n \geq m \geq 3, n \geq 2r + 1 \), and \( G = K_m \times \mathcal{C}_n^r \). Then \( G \) is super-\( \kappa \).

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