Some new perturbation bounds of generalized polar decomposition

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1. Introduction

Let \( \mathbb{C}^{m,n} \) be the set of all \( m \times n \) complex matrices, subset of \( \mathbb{C}^{m,n} \) with rank \( r \), set of the Hermitian positive semidefinite matrices of order \( m \), subset of \( \mathbb{C}^m_+ \) comprising positive definite matrices and set of all unitary matrices of order \( m \), respectively. Without mentioning, we always assume \( m \geq n > r \). For a matrix \( A \in \mathbb{C}^{m,n} \), \( A^* \), \( \| A \|_2 \) and \( \| A \| \) denote the conjugate transpose, range space, spectral norm and unitarily invariant norm of \( A \), respectively.

For \( A \in \mathbb{C}^{m,n} \), there are a subunitary matrix \( Q \) (\( Q \) is called a subunitary matrix if \( \| Qx \|_2 = \| x \|_2 \) for any \( x \in R(Q^*) \)) and a Hermitian positive semidefinite matrix \( H \) such that

\[
A = QH.
\]

(1.1)

The decomposition (1.1) is called a generalized polar decomposition (GPD) (see [22]) of \( A \). In this case, \( Q \) and \( H \) are respectively called the subunitary polar factor and positive semidefinite polar factor of \( A \) associated with the decomposition (1.1).

In general, the decomposition (1.1) is not unique, but when \( Q \) and \( H \) satisfy (see [22])

\[
R(Q^*) = R(H),
\]

(1.2)

the GPD (1.1) of \( A \) is unique. So from now on, we always assume that the generalized polar decomposition (GPD) of \( A \) with \( m \geq n > r \) satisfies the condition (1.2). We note that when the matrix \( A \) is of full column rank, i.e., \( m \geq n = r \), the GPD of \( A \) is actually its partial decomposition (PD) which is always unique. The GPD (1.1) of matrix \( A \) satisfied condition (1.2) can be easily obtained by using the singular value decomposition (SVD). To illustrate this, we now give a lemma which concerns the singular value decomposition of a matrix \( A \).

Lemma 1.1 [4]. Let \( A \in \mathbb{C}^{m,n}_+ \). Then there exist unitary matrices \( U \in \mathcal{U}_m \) and \( V \in \mathcal{U}_n \), i.e., \( U^*U = I_m \) and \( V^*V = I_n \) such that...
\[ A = U \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} V^* = U_1 \Sigma_1 V_1^*, \]

(1.3)

where \( U = (U_1, U_2), \ V = (V_1, V_2), \ U_1 \in \mathbb{C}^{m \times r}, \ V_1 \in \mathbb{C}^{n \times r}, \ \Sigma_1 = \text{diag}(\sigma_1, \ldots, \sigma_r) \) with \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0, \ \sigma_i^2 (i = 1, 2, \ldots, r) \) is the nonzero eigenvalue of \( A' A \) and \( \sigma_1, \sigma_2, \ldots, \sigma_r \) are called the nonzero singular values of \( A \).

As seen in Lemma 1.1, if we take \( Q = U_1 V_1^* \) and \( H = V_1^* S_1 V_1 \), then \( A = (U_1 V_1^*) (V_1^* S_1 V_1) = QH \) is the unique generalized polar decomposition of matrix \( A \) which satisfies condition (1.2).

The generalized polar decomposition or polar decomposition of a matrix plays the important roles in many fields such as scientific computation, optimization theory, aerospace and even psychometrics, see [10,11,9]. For this reason, the perturbation theory of the generalized polar decomposition or polar decomposition of a matrix was paid more attentions by many researchers, see [17,6,2,21,14,18,5,1,12,13,15,19,20,16,22,7]. Many works were done in the Frobenius norm in the literatures.

Let \( A, \ \tilde{A} = A + E \) have the (generalized) polar decompositions

\[ A = QH \quad \text{and} \quad \tilde{A} = \tilde{Q} \tilde{H}. \]

(1.4)

When \( A \) and \( \tilde{A} \) are nonsingular, the perturbation bound for the positive definite polar factor \( H \) in the unitarily invariant norm was given in [2]. Recently, Chen and Li [5] further presented a bound for the positive definite polar factor \( H \) in the unitarily invariant norm when \( A \) and \( \tilde{A} = A + E \) are of full column ranks with \( \|E\|_2 \leq \sigma_n \). Li [13] and Li [17] studied the perturbation bounds for the unitary polar factor \( Q \) when the matrices \( A \) and \( \tilde{A} \) are nonsingular or of full column ranks. For the rank-deficient case, Li and Sun [19,20] provided two perturbation bounds for the subunitary polar factor in the unitarily invariant norm as well as some bounds in the spectral norm. In this paper, we will continue to consider these problems and establish some new perturbation bounds for both the polar decomposition and generalized polar decomposition. Our new bounds are sharper than the known results.

The rest of this paper is organized as follows. Section 2 gives some lemmas which are needed in the later discussion. For the rank-deficient case, the perturbation bound for the positive semidefinite polar factor \( H \) under the unitarily invariant norm is derived in Section 3. Section 4 presents the new and sharper perturbation bounds for the (sub) unitary polar factor \( Q \) under the unitarily invariant norm and the spectral norm. Finally, in Section 5, we will provide new perturbation bounds of the generalized nonnegative polar factor and weighted unitary polar factor for the weighted polar decomposition under the weighted unitarily invariant norm and weighted spectral norm by applying our results for the (generalized) polar decomposition in previous sections to the case of the weighted polar decomposition.

2. Lemmas

In this section we present some lemmas which will be used when we estimate the perturbation bounds in the following sections.

Let \( A \in \mathbb{C}^{m \times n} \) have the SVD as in Lemma 1.1 and \( \tilde{A} \in \mathbb{C}^{m \times n} \) have the following SVD

\[ \tilde{A} = \tilde{U} \begin{pmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & 0 \end{pmatrix} \tilde{V}^* = \tilde{U}_1 \tilde{\Sigma}_1 \tilde{V}_1^*. \]

(2.1)

Take \( S = \tilde{U}^* U, \ T = \tilde{V}^* V, \) and partition \( S \) and \( T \) into the following block forms:

\[ S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}. \]

(2.2)

where \( S_{11} \in \mathbb{C}^{r \times r}, \ T_{11} \in \mathbb{C}^{r \times r}, \) then \( S' S = I_m \) and \( T' T = I_n, \) i.e., \( S \) and \( T \) are unitary matrices.

Lemma 2.1 [6]. Let \( W \in \mathbb{C}^{n \times n} \) be a unitary matrix partitioned as

\[ W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \]

where \( W_{11} \in \mathbb{C}^{r \times r}, \ W_{22} \in \mathbb{C}^{(n-r) \times (n-r)}, \) \( 1 \leq r < n. \) Then \( \|W_{12}\| = \|W_{21}\| \) for any unitary invariant norm.

Lemma 2.2 [16,17]. Let \( A, \tilde{A} = A + E \in \mathbb{C}_r^{m \times n} \) have the GPDs in (1.4). Then

\[ \|T_{11} - S_{11}\| \leq \frac{2}{\sigma_r + \sigma_r} \|E\|. \]

(2.3)

\[ \|S_{12}\| = \|S_{21}\| \leq \frac{1}{\max\{\sigma_r, \sigma_r\}} \|E\|. \]

(2.4)

\[ \|T_{12}\| = \|T_{21}\| \leq \frac{1}{\max\{\sigma_r, \sigma_r\}} \|E\|. \]

(2.5)
and

$$\| \bar{Q} - Q \| = \left\| \begin{pmatrix} S_{11} - T_{11} & -T_{12} \\ S_{21} & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} S_{11} - T_{11} & S_{12} \\ -T_{21} & 0 \end{pmatrix} \right\|,$$

where $S_i, T_i, (i, j = 1, 2)$ are defined as in (2.2) and $\sigma_r, \bar{\sigma}_r$, are the smallest singular values of $A$ and $\bar{A}$, respectively.

In Lemma 2.2, the inequality (2.3) can be found in Lemma 3.1 in [16]. Lemma 3.4 in [17] gives the inequalities (2.4) and (2.5) and the equality (2.6) is the identity (2.5) in [17]. $\|S_{12}\| = \|S_{21}\|$ and $\|T_{12}\| = \|T_{21}\|$ are the direct results from Lemma 2.1.

**Lemma 2.3** [8]. Let $F, \bar{F}$ be two Hermitian matrices, $S$ be a complex matrix with proper dimension and

$$\Delta = [\alpha, \beta] \subset R, \quad \Delta' = R \setminus (\alpha - \delta, \beta + \delta), \quad \delta > 0.$$

If $\lambda(F) \in \Delta$, $\lambda(\bar{F}) \in \Delta'$, then the equation $FX - \bar{F}S = H$ has a unique solution $X$, where $\lambda(F)$ and $\lambda(\bar{F})$ denote the eigenvalue sets of $F$ and $\bar{F}$, respectively. Moreover, $\|X\| \leq \|S\|/\delta$ for any unitarily invariant norm.

**Lemma 2.4** [23]. Partition $A$ into the block form $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, and let $B = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}, C = (A_{11} A_{12})$ and $D = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, then for any unitarily invariant norm, we have

$$\|A_{ij}\| \leq \|A\|, \quad \|B\| \leq \|A\|, \quad \|C\| \leq \|A\| \quad \text{and} \quad \|D\| \leq \|A\|,$$

where $i, j = 1, 2$.

**Lemma 2.5.** [16]. Let $A$ have the following block form $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$. Then

$$\|A_{11}\|^2 \leq \|A_{11}\|^2 + \|A_{12}\|^2 + \|A_{21}\|^2 + \|A_{22}\|^2.$$

### 3. The perturbation bound for positive semidefinite polar factor

In [2], the perturbation bound for the positive definite polar factor $H$ of a nonsingular matrix $A$ under the unitarily invariant norm was obtained. Chen and Li [5] provided a bound for the positive definite polar factor $H$ under the same norm as when the matrix $A$ is of full column rank. To our knowledge, the perturbation bound for the positive semidefinite polar factor has not been discussed in the literature when matrix $A$ is rank deficient. The following theorem will fill this gap.

**Theorem 3.1.** Let $A, \bar{A} = A + E \in \mathbb{C}^{m \times n}$ have the GPDs in (1.4). Then

$$\| \bar{H} - H \| \leq \left( 2 + \frac{2 \min \{\sigma_r, \bar{\sigma}_r\}}{\sigma_r + \bar{\sigma}_r} \right) \|E\|,$$

where $\sigma_r, \bar{\sigma}_1$ and $\sigma_r, \bar{\sigma}_1$ are the largest and smallest nonzero singular values of $A, \bar{A}$, respectively.

**Proof.** Let $A, \bar{A}$ have SVDs (1.3) and (2.1), then from $E = \bar{A} - A$, we have

$$\|E\| = \|\bar{U}_1 \bar{\Sigma}_1 \bar{V}_1 - U_1 \Sigma_1 V_1\| = \|\bar{U}_1^T \bar{U}_1 \bar{\Sigma}_1 \bar{V}_1 - U_1 \Sigma_1 V_1\| V = \left\| \begin{pmatrix} \bar{\Sigma}_1 V_1 V_1 - \bar{U}_1 U_1 \Sigma_1 & \bar{\Sigma}_1 V_1 V_2 \\ -\bar{U}_2 U_1 \Sigma_1 & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} S_1 \Sigma_1 - T_11 & \bar{\Sigma}_1 T_12 \\ -T_21 \Sigma_1 & 0 \end{pmatrix} \right\|.$$

Similarly, we also have

$$\|E\| = \|E^*\| = \left\| \begin{pmatrix} \bar{\Sigma}_1 S_{11} - T_{11} \Sigma_1 & \bar{\Sigma}_1 S_{12} \\ -T_{21} \Sigma_1 & 0 \end{pmatrix} \right\|.$$

Now applying Lemma 2.2 and Lemma 2.4 to (3.2) and (3.3), we have

$$\| \bar{H} - H \| = \| \bar{V}_1 \bar{\Sigma}_1 \bar{V}_1 - V_1 \Sigma_1 V_1\| V = \|\bar{V}_1^T \bar{\Sigma}_1 \bar{V}_1 - V_1 \Sigma_1 V_1\| V = \left\| \begin{pmatrix} \bar{\Sigma}_1 V_1 V_1 - \bar{V}_1 V_1 \Sigma_1 & \bar{\Sigma}_1 V_1 V_2 \\ -\bar{V}_2 V_1 \Sigma_1 & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} \bar{\Sigma}_1 T_{11} - T_{11} \Sigma_1 & \bar{\Sigma}_1 T_{12} \\ -T_{21} \Sigma_1 & 0 \end{pmatrix} \right\|$$

$$\leq \left\| \begin{pmatrix} \bar{\Sigma}_1 T_{11} - \bar{\Sigma}_1 S_{11} & \bar{\Sigma}_1 T_{12} \\ -T_{21} \Sigma_1 & 0 \end{pmatrix} \right\| + \left\| \begin{pmatrix} \bar{\Sigma}_1 S_{11} - T_{11} \Sigma_1 & 0 \\ -T_{21} \Sigma_1 & 0 \end{pmatrix} \right\| \leq \|\bar{\Sigma}_1 T_{11} - \bar{\Sigma}_1 S_{11}\| + \|\bar{\Sigma}_1 T_{12}\|$$

$$\leq \|E\| \leq \bar{\sigma}_1 \|T_{11} - S_{11}\| + \|E\| + \|E\| \leq \left( \frac{2 \bar{\sigma}_1}{\sigma_r + \bar{\sigma}_r} + 2 \right) \|E\|.$$
On the other hand, from (3.4) we have
\[ \| \tilde{H} - H \| \leq \left\| \begin{pmatrix} S_{11} \Sigma_1 - T_{11} \Sigma_1 & 0 \\ -T_{21} \Sigma_1 & 0 \end{pmatrix} \right\| + \left\| \begin{pmatrix} \Sigma_1 T_{11} - S_{11} \Sigma_1 & \Sigma_1 T_{12} \\ 0 & 0 \end{pmatrix} \right\| \leq \left( \frac{2 \sigma_1}{\sigma_r + \sigma_r} + 2 \right) \| E \|, \]
which together with (3.5), we immediately get
\[ \| \tilde{H} - H \| \leq \left( 2 + \frac{2 \min(\sigma_1, \bar{\sigma}_1)}{\sigma_r + \sigma_r} \right) \| E \|. \]
The proof of this theorem is completed. □

Corollary 3.1. Let \( A, \tilde{A} = A + E \in \mathbb{C}_n^{m \times n} \) have the GPDs in (1.4). Then
\[ \| \tilde{H} - H \| \leq \left( \frac{2}{\sigma_r + \sigma_r} \right) \| E \|, \] (3.6)
where \( \sigma_r, \sigma_n \) and \( \bar{\sigma}_1, \bar{\sigma}_n \) are the largest and smallest singular values of \( A \) and \( \tilde{A} \), respectively.

We remark that when matrix \( A \) and \( \tilde{A} = A + E \) are nonsingular, the perturbation bound (3.6) is same as that in [2]. This implies that Corollary 3.1 extends the result in [2] for nonsingular matrix to the case of full column rank.

4. Perturbation bounds for the subunitary polar factor

In this section we focus on the perturbation bounds of the subunitary polar factor \( Q \) under the unitarily invariant norm and the spectral norm. When \( A, \tilde{A} = A + E \in \mathbb{C}_r^{m \times n} \) and \( r \leq n \leq m \), Li [17] provided the perturbation bounds for the (sub) unitary polar factor \( Q \), which are stated in the following proposition.

Proposition 4.1 [17]. Let \( A, \tilde{A} = A + E \in \mathbb{C}_r^{m \times n} \) where \( r \leq n \leq m \) have the GPDs in (1.4). Then
\[ \| \tilde{Q} - Q \| \leq \frac{3}{\sigma_r + \sigma_r} \| E \|, \] (4.1)
where \( \sigma_r \) and \( \bar{\sigma}_r \) denote the smallest nonzero singular values of \( A \) and \( \tilde{A} \), respectively. In particular, when \( r = n \leq m \), we have
\[ \| \tilde{Q} - Q \| \leq \frac{2}{\max(\sigma_n, \bar{\sigma}_n)} \| E \|, \] (4.2)
where \( \sigma_n \) and \( \bar{\sigma}_n \) denote the smallest singular values of \( A \) and \( \tilde{A} \), respectively.

When \( r = n \leq m \), we found that if \( \sigma_n > 2 \bar{\sigma}_n \) or \( \bar{\sigma}_n > 2 \sigma_n \), the bound (4.2) is sharper than the bound (4.1); while the bound (4.1) is sharper than the bound (4.2) when \( \sigma_n < 2 \bar{\sigma}_n \) or \( \bar{\sigma}_n < 2 \sigma_n \).

In the following theorem, we will present a new perturbation bound for the unitary polar factor \( Q \) when \( r = n \leq m \), which is sharper than the known bounds (4.1) and (4.2).

Theorem 4.1. Let \( A \) and \( \tilde{A} = A + E \in \mathbb{C}_r^{m \times n} \) have the GPDs in (1.4). Then
\[ \| \tilde{Q} - Q \| \leq \frac{1}{\sigma_n + \sigma_n} \left( 2 + \frac{\min(\sigma_n, \bar{\sigma}_n)}{\max(\sigma_n, \bar{\sigma}_n)} \right) \| E \|, \] (4.3)
where \( \sigma_n \) and \( \bar{\sigma}_n \) denote the smallest singular values of \( A \) and \( \tilde{A} \), respectively.

Proof. When \( r = n \leq m \), from (2.6), (3.2) and (3.3), we can get
\[ \| E \| = \left\| \begin{pmatrix} S_{11} \Sigma_1 - \Sigma_1 T \\ S_{21} \Sigma_1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} \Sigma_1 S_{11} - T \Sigma_1 \\ \Sigma_1 S_{12} \end{pmatrix} \right\| \] (4.4)
and
\[ \| \tilde{Q} - Q \| = \left\| \begin{pmatrix} \tilde{S}_{11} - T \\ \tilde{S}_{21} \end{pmatrix} \right\| = \| (S_{11} - T \quad S_{12}) \|.
\]
If we take
\[ X = \begin{pmatrix} S_{11} - T \\ S_{21} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \bar{\sigma}_n I \end{pmatrix}. \]
then \( \| \tilde{Q} - Q \| = \| X \| \) and

\[
X \Sigma_1 + DX = \begin{pmatrix} S_{11} \Sigma_1 - \tilde{\Sigma}_1 T \\ S_{21} \Sigma_1 \end{pmatrix} + \begin{pmatrix} \tilde{\Sigma}_1 S_{11} - T \Sigma_1 \\ \sigma_s S_{21} \end{pmatrix}.
\]

We note that \( \lambda(D) \subseteq [\sigma_n, \sigma_1] \) and \( \lambda(-\Sigma_1) \subseteq [-\sigma_1, -\sigma_n] \). So by Lemma 2.3, Lemma 2.4, (2.4) and (4.4) we have

\[
\| \tilde{Q} - Q \| = \| X \| \leq \frac{1}{\sigma_n + \sigma_n} \left( \| E \| + \left( \left\| \tilde{\Sigma}_1 S_{11} - T \Sigma_1 \right\| \right) \right)
\]

\[
\leq \frac{1}{\sigma_n + \sigma_n} \left( \| E \| + \| \tilde{\Sigma}_1 S_{11} - T \Sigma_1 \| + \| \sigma_s S_{21} \| \right) \leq \frac{1}{\sigma_n + \sigma_n} \left( 2 + \frac{\sigma_n}{\max\{\sigma_n, \sigma_1\}} \right) \| E \|.
\]

On the other hand, if we take

\[
X = \begin{pmatrix} S_{11} & S_{12} \end{pmatrix} \text{ and } D_1 = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \sigma_n I \end{pmatrix},
\]

then \( \| \tilde{Q} - Q \| = \| X \| \) and

\[
XD_1 + \tilde{\Sigma}_1 X = \begin{pmatrix} \tilde{\Sigma}_1 S_{11} - T \Sigma_1 \tilde{\Sigma}_1 S_{12} + (S_{11} \Sigma_1 - \tilde{\Sigma}_1 T \sigma_s S_{12}).
\]

Similar to the previous discussion, we can get another bound

\[
\| \tilde{Q} - Q \| \leq \frac{1}{\sigma_n + \sigma_n} \left( \| E \| + \left( \left\| S_{11} \Sigma_1 - \tilde{\Sigma}_1 T \sigma_n S_{12} \right\| \right) \right)
\]

\[
\leq \frac{1}{\sigma_n + \sigma_n} \left( 2 + \frac{\sigma_n}{\max\{\sigma_n, \sigma_1\}} \right) \| E \|.
\]

Combining (4.6) with (4.8), we obtain the perturbation bound (4.3). \( \square \)

Note that

\[
\frac{1}{\sigma_n + \sigma_n} \left( 2 + \frac{\sigma_n}{\max\{\sigma_n, \sigma_1\}} \right) \leq \frac{3}{\sigma_n + \sigma_n}
\]

and

\[
\frac{1}{\sigma_n + \sigma_n} \left( 2 + \frac{\sigma_n}{\max\{\sigma_n, \sigma_1\}} \right) < \frac{2}{\max\{\sigma_n, \sigma_1\}},
\]

our new perturbation bound (4.3) for the unitary polar factor \( Q \) of \( A \) with full column rank is smaller than the known bounds (4.1) and (4.2).

Under the spectral norm, Li and Sun [19,20] present two perturbation bounds for the subunitary polar factor \( Q \) of matrix \( A \in \mathbb{C}_{r \times n} \).

**Proposition 4.2.** ([19,20]) Let \( A, A = A + E \in \mathbb{C}_{r \times n} \) have the GPDs in (1.4). Then

\[
\| \tilde{Q} - Q \|_2 \leq \sqrt{\left( \frac{2}{\sigma_r + \sigma_r} \right)^2 + \frac{2}{\max\{\sigma_r^2, \sigma_r^2}\}} \| E \|_2
\]

and

\[
\| \tilde{Q} - Q \|_2 \leq \sqrt{\delta} \| E \|_2.
\]

where

\[
\delta = \frac{1}{2} \left[ \left( \frac{2}{\sigma_r + \sigma_r} \right)^2 + \frac{1}{\sigma_r^2} + \frac{1}{\sigma_r^2} + \sqrt{\left( \frac{4}{(\sigma_r + \sigma_r)^2} + \frac{1}{\sigma_r^2} + \frac{1}{\sigma_r^2} \right)^2 - \frac{4}{\sigma_r^2 \sigma_r^2}} \right].
\]

In the next theorem, we present a new sharper bound than those in Proposition 4.2.

**Theorem 2.** Let \( A, A = A + E \in \mathbb{C}_{r \times n} \) have the GPDs in (1.4). Then

\[
\| \tilde{Q} - Q \|_2 \leq \frac{1}{\sigma_r + \sigma_r} \left( 1 + \sqrt{\frac{\sigma_r^2 + \sigma_r^2}{\max\{\sigma_r^2, \sigma_r^2\}} + \frac{\sigma_r^2 + \sigma_r^2}{\max\{\sigma_r^2, \sigma_r^2\}} \| E \|_2} \right)
\]

where \( \sigma_r \) and \( \sigma_r \) denote the smallest nonzero singular values of \( A \) and \( \tilde{A} \), respectively.
**Proof.** Let

\[ X = \begin{pmatrix} S_{11} - T_{11} & -T_{12} \\ S_{21} & 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{pmatrix}, \quad D_2 = \begin{pmatrix} \bar{\Sigma}_1 & 0 \\ 0 & \bar{\sigma} I \end{pmatrix}. \]

Then

\[ XD_1 + D_2 X = \begin{pmatrix} S_{11} \Sigma_1 - \bar{\Sigma}_1 T_{11} & -\bar{\Sigma}_1 T_{12} \\ S_{21} \Sigma_1 & 0 \end{pmatrix} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{pmatrix} + \begin{pmatrix} \bar{\Sigma}_1 S_{11} - T_{11} \Sigma_1 & -\sigma T_{12} \\ \bar{\sigma} S_{21} & 0 \end{pmatrix} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \sigma I \end{pmatrix}. \]

(4.12)

It follows from Lemma 2.3, Lemma 2.4, (3.2), (3.3), and (4.12) that

\[ \| X \| \leq \frac{1}{\sigma_r + \sigma} \left( \| E \| + \left\| \begin{pmatrix} \bar{\Sigma}_1 S_{11} - T_{11} \Sigma_1 & -\sigma T_{12} \\ \bar{\sigma} S_{21} & 0 \end{pmatrix} \right\| \right). \]

(4.13)

From (2.4), (2.5), Lemma 2.4 and Lemma 2.5, we have

\[ \left( \frac{\bar{\Sigma}_1 S_{11} - T_{11} \Sigma_1}{\bar{\sigma} S_{21}} - \sigma T_{12} \right) \right\|_2 \leq \left\| \frac{\bar{\Sigma}_1 S_{11} - T_{11} \Sigma_1}{\bar{\sigma} S_{21}} \right\|_2 + \| \sigma T_{12} \|_2 + \| \sigma S_{21} \|_2 \leq \left( 1 + \frac{\sigma^2 + \bar{\sigma}^2}{\max(\sigma^2, \bar{\sigma}^2)} \right) \| E \|_2, \]

which together with (4.13) and (2.6) gives

\[ \| Q - Q \|_2 = \| X \|_2 \leq \frac{1}{\sigma_r + \sigma} \left( 1 + \sqrt{\frac{\sigma^2 + \bar{\sigma}^2}{\max(\sigma^2, \bar{\sigma}^2)}} \right) \| E \|_2. \]

The proof is completed. □

Since

\[ \frac{1}{\sigma_r + \sigma} \left( 1 + \sqrt{\frac{\sigma^2 + \bar{\sigma}^2}{\max(\sigma^2, \bar{\sigma}^2)}} \right) < \sqrt{\left( \frac{2}{\sigma_r + \sigma} \right)^2 + \frac{2}{\max(\sigma^2, \bar{\sigma}^2)}}, \]

so the bound (4.11) is sharper than the known bound (4.9). On the other hand, we notice that the inequality \( \frac{1}{\sigma_r} + \frac{1}{\bar{\sigma}_r} \geq \frac{8}{(\sigma_r + \bar{\sigma}_r)^2} \) always holds, and thus for the quantity \( \delta \) in Proposition 4.2, we have

\[ \delta > \frac{1}{2} \left[ \frac{12}{(\sigma_r + \bar{\sigma}_r)^2} + \sqrt{\left( \frac{2}{\sigma_r + \sigma} \right)^4 + \frac{1}{\sigma_r^2} + \frac{1}{\bar{\sigma}_r^2}} - \frac{4}{\sigma_r^2} \right] \geq \frac{1}{2} \left[ \frac{12}{(\sigma_r + \bar{\sigma}_r)^2} + \frac{4}{(\sigma_r + \bar{\sigma}_r)^2} \right] = \frac{8}{(\sigma_r + \bar{\sigma}_r)^2}, \]

which results in the following inequality

\[ \frac{1}{\sigma_r + \sigma} \left( 1 + \sqrt{\frac{\sigma^2 + \bar{\sigma}^2}{\max(\sigma^2, \bar{\sigma}^2)}} \right) \leq \frac{1 + \sqrt{3}}{\sigma_r + \bar{\sigma}_r} < \frac{\sqrt{8}}{\sigma_r + \bar{\sigma}_r} < \sqrt{\delta}. \]

This means that the bound (4.11) is also smaller than the bound (4.10).

When \( A, \bar{A} = A + E \in \mathbb{C}^{m \times n} \), Li [16,17] studied the perturbation bounds of the unitary polar factor \( Q \) of \( A \) under Q-norm. A unitarily invariant norm \( \| \cdot \| \) is called a Q-norm (see [2,3]) if there exists another unitarily invariant norm \( \| \cdot \| \) such that \( \| Y \| = (\| Y Y^* \|^{1/2}) \) for any \( Y \in \mathbb{C}^{m \times n} \). The spectral norm is known as a special Q-norm. Next proposition gives two perturbation bounds for the unitary polar factor \( Q \) of \( A \) in the spectral norm, which stem from those in [16,17] instead of Q-norm.

**Proposition 4.3 (16,17).** Let \( A, \bar{A} = A + E \in \mathbb{C}^{m \times n} \) have the GPA in (1.4). Then

\[ \| \bar{Q} - Q \|_2 \leq \sqrt{\frac{2}{\sigma_n + \sigma_n} \| E \|_2} \]

and

\[ \| \bar{Q} - Q \|_2 \leq \frac{1 + \sqrt{2}}{\sigma_n + \sigma_n} \| E \|_2. \]

(4.14)

In the following theorem we will provide a new bound for the unitary polar factor \( Q \).

**Theorem 4.3.** Let \( A, \bar{A} = A + E \in \mathbb{C}^{m \times n} \) have the GPA in (1.4). Then
\[ \| \tilde{Q} - Q \|_2 \leq \frac{1}{\sigma_n + \sigma_n} \left( 1 + \sqrt{1 + \frac{\min(\sigma_n^2, \tilde{\sigma}_n^2)}{\max(\sigma_n^2, \tilde{\sigma}_n^2)}} \right) \| E \|_2. \]  

(4.16)

where \( \sigma_n \) and \( \tilde{\sigma}_n \) denote the smallest singular values of \( A \) and \( \tilde{A} \), respectively.

**Proof.** By (2.4), Lemma 2.4 and Lemma 2.5, we get

\[ \left\| \left( \tilde{\Sigma}_1 \Sigma_1 - T \Sigma_1 \right) \frac{\sigma_n S_{11}}{\sigma_n S_{21}} \right\|_2^2 \leq \left\| \tilde{\Sigma}_1 \Sigma_1 - T \Sigma_1 \right\|_2^2 \left\| \sigma_n S_{11} \right\|_2^2 \leq \left( 1 + \frac{\sigma_n^2}{\max(\sigma_n^2, \tilde{\sigma}_n^2)} \right) \| E \|_2^2. \]  

(4.17)

Combining (4.5) with (4.17), we have

\[ \| \tilde{Q} - Q \|_2 = \| X \|_2 \leq \frac{1}{\sigma_n + \sigma_n} \left( 1 + \sqrt{1 + \frac{\sigma_n^2}{\max(\sigma_n^2, \tilde{\sigma}_n^2)}} \right) \| E \|_2. \]  

(4.18)

Similarly, from (2.4), Lemma 2.4 and Lemma 2.5, we have

\[ \left\| \left( S_{11} \tilde{\Sigma}_1 - \Sigma_1 T \sigma_n S_{12} \right) \right\|_2^2 \leq \left\| S_{11} \Sigma_1 - \tilde{\Sigma}_1 T \right\|_2^2 + \left\| \sigma_n S_{12} \right\|_2^2 \leq \left( 1 + \frac{\sigma_n^2}{\max(\sigma_n^2, \tilde{\sigma}_n^2)} \right) \| E \|_2^2. \]  

(4.19)

Hence starting from (4.7) with the spectral norm and using (4.19), we have

\[ \| \tilde{Q} - Q \|_2 \leq \frac{1}{\sigma_n + \sigma_n} \left( 1 + \sqrt{1 + \frac{\sigma_n^2}{\max(\sigma_n^2, \tilde{\sigma}_n^2)}} \right) \| E \|_2. \]  

(4.20)

Combining (4.18) with (4.20) finally results in the consequence of this theorem.

We note that

\[ \frac{1}{\sigma_n + \sigma_n} \left( 1 + \sqrt{1 + \frac{\min(\sigma_n^2, \tilde{\sigma}_n^2)}{\max(\sigma_n^2, \tilde{\sigma}_n^2)}} \right) < \sqrt{\left( \frac{2}{\sigma_n + \sigma_n} \right)^2 + \frac{1}{\max(\sigma_n^2, \tilde{\sigma}_n^2)}} \]

and

\[ \frac{1}{\sigma_n + \sigma_n} \left( 1 + \sqrt{1 + \frac{\min(\sigma_n^2, \tilde{\sigma}_n^2)}{\max(\sigma_n^2, \tilde{\sigma}_n^2)}} \right) \leq \frac{1 + \sqrt{2}}{\sigma_n + \tilde{\sigma}_n}, \]

our new bound (4.16) is smaller than the bounds (4.14) and (4.15).

**Remark 1.** If \( r < n \leq m \), then

\[ \frac{1}{\sigma_r + \sigma_r} \left( 1 + \sqrt{1 + \frac{\sigma_r^2 + \tilde{\sigma}_r^2}{\max(\sigma_r^2, \tilde{\sigma}_r^2)}} \right) < \frac{3}{\sigma_r + \sigma_r}. \]

If \( r = n \leq m \), then

\[ \frac{1}{\sigma_n + \sigma_n} \left( 1 + \sqrt{1 + \frac{\min(\sigma_n^2, \tilde{\sigma}_n^2)}{\max(\sigma_n^2, \tilde{\sigma}_n^2)}} \right) \leq \frac{1}{\sigma_n + \sigma_n} \left( 2 + \frac{\min(\sigma_n, \tilde{\sigma}_n)}{\max(\sigma_n, \tilde{\sigma}_n)} \right). \]

This implies that the perturbation bounds in Theorem 4.2 and Theorem 4.3 are respectively smaller than the bounds (4.1) and (4.3) where, instead of the general unitarily invariant norm, we take the spectral norm.

**5. Perturbation results of weighted polar decomposition**

We now investigate the perturbation bounds of the weighted polar decomposition. Let \( A \in \mathbb{C}^{m \times n} \), and \( M \) and \( N \) be Hermitian positive definite matrices of order \( m \) and \( n \), respectively. Then there are an \((M, N)\) weighted partial isometric matrix \( Q \) [26] and a generalized Hermitian positive semidefinite matrix \( H \) satisfying \( NH \in \mathbb{C}^{n \times n} \) such that

\[ A = QH. \]  

(5.1)

This decomposition is called the \((M, N)\) weighted polar decomposition (MN-WPD) (see [26]) of \( A \). \( Q \) and \( H \) are respectively called the \((M, N)\) weighted unitary polar factor and generalized nonnegative polar factor of this decomposition. In general, the decomposition (5.1) is not unique. But when \( Q \) and \( H \) satisfy (see [27])

\[ R(Q^*) = R(H), \]  

(5.2)
the MN-WPD (5.1) of $A$ is unique, where $Q^* = N^{-1}Q'M$ denotes the weighted conjugate transpose of $Q$. So from now on, we always assume that the MN-WPD satisfies the condition (5.2). The unique MN-WPD (5.1) satisfying the condition (5.2) can be got from the weighted singular value decomposition (MN-SVD) (see [24]) of $A$.

Recently, Yang and Li [25] studied the perturbation problem for the weighted polar decomposition (MN-WPD) of a matrix $A$ and presented the perturbation bounds for the weighted unitary polar factor $Q$ and generalized nonnegative polar factor $H$ in the weighted unitarily invariant norm. However, we found that their results can be improved by applying our new bounds for the (generalized) polar decomposition in the previous section to the weighted case.

Let $\|A\|_{(MN)} = \|M^{1/2}AN^{-1/2}\|$ and $\|A\|_{(2MN)} = \|M^{1/2}AN^{-1/2}\|_2$ be respectively the weighted unitarily invariant norm and the weighted spectral norm of $A$ with $M \in C^\infty_+$, $N \in C^n_+$. We note that if $A$ has the MN-WDP, i.e., $A = QH$, then

$$M^{1/2}AN^{-1/2} = (M^{1/2}QN^{-1/2})(N^{1/2}HN^{-1/2})$$

is the generalized polar decomposition of $M^{1/2}AN^{-1/2}$, and vice versa. Therefore, all perturbation bounds for the generalized polar decomposition of $A$ can be naturally extended to the case of the weighted polar decomposition of $A$. So without their proofs here, we list some new bounds for the weighted polar decomposition only corresponding to ones we have given in the previous sections. These new bounds are found to be sharper than those given by Yang and Li [25].

**Theorem 5.1.** Let $A, \tilde{A} = A + E \in C^{m \times n}$ have the MN-WPDs $A = QH$ and $\tilde{A} = \tilde{Q}\tilde{H}$. Then

$$\|\tilde{H} - H\|_{(MN)} \leq \left(2 + 2 \min\{\sigma_1, \tilde{\sigma}_1\}\right)\|E\|_{(MN)},$$

where $\sigma_1, \tilde{\sigma}_1$ and $\sigma_r, \tilde{\sigma}_r$ are the largest and smallest $(M, N)$ nonzero singular values of $A$ and $\tilde{A}$, respectively.

**Theorem 5.2.** Let $A, \tilde{A} = A + E \in C^{m \times n}$ have the MN-WPDs $A = QH$ and $\tilde{A} = \tilde{Q}\tilde{H}$. Then

$$\|\tilde{Q} - Q\|_{(MN)} \leq \frac{3}{\sigma_r + \tilde{\sigma}_r}\|E\|_{(MN)},$$

where $\sigma_r$ and $\tilde{\sigma}_r$ denote the smallest $(M, N)$ nonzero singular values of $A$ and $\tilde{A}$, respectively.

**Theorem 5.3.** Let $A$ and $\tilde{A} = A + E \in C^{m \times n}$ have the MN-WPDs $A = QH$ and $\tilde{A} = \tilde{Q}\tilde{H}$. Then

$$\|\tilde{Q} - Q\|_{(MN)} \leq \frac{1}{\sigma_n + \tilde{\sigma}_n}\left(2 + \min\{\sigma_n, \tilde{\sigma}_n\}\right)\|E\|_{(MN)},$$

where $\sigma_n$ and $\tilde{\sigma}_n$ denote the smallest $(M, N)$ singular values of $A$ and $\tilde{A}$, respectively.

**Remark 2.** Since

$$2 + 2 \min\{\sigma_1, \tilde{\sigma}_1\} \leq 2 + \frac{\sigma_1 + \tilde{\sigma}_1}{\sigma_r + \tilde{\sigma}_r},$$

$$\frac{3}{\sigma_r + \tilde{\sigma}_r} < \frac{2}{\sigma_r + \tilde{\sigma}_r} + \frac{2}{\max\{\sigma_r, \tilde{\sigma}_r\}},$$

$$\frac{3}{\sigma_r + \tilde{\sigma}_r} < \frac{2}{\sigma_r + \tilde{\sigma}_r} + \frac{1}{\min\{\sigma_r, \tilde{\sigma}_r\}}$$

and

$$\frac{1}{\sigma_n + \tilde{\sigma}_n}\left(2 + \min\{\sigma_n, \tilde{\sigma}_n\}\right) \leq \frac{3}{\sigma_n + \tilde{\sigma}_n} < \frac{2}{\sigma_n + \tilde{\sigma}_n} + \frac{1}{\max\{\sigma_n, \tilde{\sigma}_n\}}.$$  

Therefore, our bounds (5.3)–(5.5) respectively improve the corresponding perturbation bounds (27), (46), (67) and (58) in [25].

The next two theorems give the new perturbation bounds under the weighted spectral norm for the weighted unitary polar factor $Q$.

**Theorem 5.4.** Let $A, \tilde{A} = A + E \in C^{m \times n}$ have the MN-WPDs $A = QH$ and $\tilde{A} = \tilde{Q}\tilde{H}$. Then

$$\|\tilde{Q} - Q\|_{(2MN)} \leq \frac{1}{\sigma_r + \tilde{\sigma}_r}\left(1 + \sqrt{1 + \frac{\sigma_1^2 + \tilde{\sigma}_1^2}{\max\{\sigma_1^2, \tilde{\sigma}_1^2\}}}\right)\|E\|_{(2MN)},$$

where $\sigma_r$ and $\tilde{\sigma}_r$ denote the smallest $(M, N)$ nonzero singular values of $A$ and $\tilde{A}$, respectively.
Theorem 5.5. Let $A$ and $\tilde{A} = A + E \in \mathbb{C}^{m \times n}$ have the MN-WPDs $A = QH$ and $\tilde{A} = \tilde{Q}H$. Then

$$
\|\tilde{Q} - Q\|_{2(MN)} \leq \frac{1}{\sigma_n + \tilde{\sigma}_n} \left( 1 + \sqrt{1 + \frac{\min\{\sigma_n^2, \tilde{\sigma}_n^2\}}{\max\{\sigma_n^2, \tilde{\sigma}_n^2\}}} \right) \|E\|_{2(MN)},
$$

(5.7)

where $\sigma_n$ and $\tilde{\sigma}_n$ denote the smallest $(M, N)$ nonzero singular values of $A$ and $\tilde{A}$, respectively.

It is easy to see

$$
\frac{1}{\sigma_n + \tilde{\sigma}_n} \left( 1 + \sqrt{1 + \frac{\sigma_n^2 + \tilde{\sigma}_n^2}{\max\{\sigma_n^2, \tilde{\sigma}_n^2\}}} \right) < \frac{3}{\tilde{\sigma}_n + \sigma_n},
$$

and

$$
\frac{1}{\sigma_n + \tilde{\sigma}_n} \left( 1 + \sqrt{1 + \frac{\min\{\sigma_n^2, \tilde{\sigma}_n^2\}}{\max\{\sigma_n^2, \tilde{\sigma}_n^2\}}} \right) < \frac{1}{\sigma_n + \tilde{\sigma}_n} \left( 2 + \frac{\min\{\sigma_n, \tilde{\sigma}_n\}}{\max\{\sigma_n, \tilde{\sigma}_n\}} \right)
$$

hold. So the perturbation bounds (5.6) and (5.7) for the weighted unitary polar factor $Q$ are respectively smaller than (5.4) and (5.5) where, instead of the general weighted unitarily invariant norm, we take the weighted spectral norm.

6. Conclusion

In this paper, we investigate the perturbation bounds of the positive (semi) definite polar factor and the (sub) unitary polar factor of the (generalized) polar decomposition of a matrix. Some known results are improved. Furthermore, by applying our new bounds to the weighted cases, we also improved the known perturbation bounds for the weighted polar decomposition.

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