Panpositionable Hamiltonian Graphs

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Abstract

A hamiltonian graph $G$ is panpositionable if for any two different vertices $x$ and $y$ of $G$ and any integer $k$ with $d_G(x, y) \leq k < |V(G)|/2$, there exists a hamiltonian cycle $C$ of $G$ with $d_C(x, y) = k$. A bipartite hamiltonian graph $G$ is bipanpositionable if for any two different vertices $x$ and $y$ of $G$ and for any integer $k$ with $d_G(x, y) \leq k < |V(G)|/2$ and $(k - d_G(x, y))$ is even, there exists a hamiltonian cycle $C$ of $G$ such that $d_C(x, y) = k$. In this paper, we prove that the hypercube $Q_n$ is bipanpositionable hamiltonian if and only if $n \geq 2$. The recursive circulant graph $G(n; 1, 3)$ is bipanpositionable hamiltonian if and only if $n \geq 6$ and $n$ is even; $G(n; 1, 2)$ is panpositionable hamiltonian if and

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only if \( n \in \{5, 6, 7, 8, 9, 11\} \), and \( G(n; 1, 2, 3) \) is panpositionable hamiltonian if and only if \( n \geq 5 \).

**Keywords**: hamiltonian, pancyclic, panconnected.

1 Introduction

For the graph definitions and notations we follow [3]. \( G = (V, E) \) is a graph if \( V \) is a finite set and \( E \) is a subset of \( \{(u, v) \mid (u, v) \) is an unordered pair of \( V\}\). We say that \( V \) is the vertex set and \( E \) is the edge set of \( G \). Two vertices \( u \) and \( v \) are adjacent if \( (u, v) \in E \). A path is represented by \( \langle v_0, v_1, v_2, \ldots, v_k \rangle \), where all vertices are distinct. The length of a path \( Q \) is the number of edges in \( Q \). We also write the path \( \langle u_0, u_1, u_2, \ldots, u_k \rangle \) as \( \langle v_0, Q_1, v_i, v_{i+1}, \ldots, v_j, Q_2, v_k, \ldots, v_k \rangle \), where \( Q_1 \) is the path \( \langle v_0, v_1, \ldots, v_{i-1}, v_i \rangle \) and \( Q_2 \) is the path \( \langle v_j, v_{j+1}, \ldots, v_{k-1}, v_k \rangle \) Hence, it is possible to write a path \( \langle v_0, v_1, Q_1, v_k \rangle \) if the length of \( Q \) is zero. We use \( d_G(u, v) \) to denote the distance between \( u \) and \( v \) in \( G \), i.e., the length of the shortest path joining \( u \) and \( v \) in \( G \). A cycle is a path of at least three vertices such that the first vertex is the same as the last vertex. A hamiltonian cycle of \( G \) is a cycle that traverses every vertex of \( G \) exactly once. We use \( d_G(u, v) \) to denote the distance between \( u \) and \( v \) in a hamiltonian cycle \( C \) of \( G \), i.e., the length of the path joining \( u \) and \( v \) in \( C \). A hamiltonian graph is a graph with a hamiltonian cycle.

Hamiltonian graphs is perhaps the most important outstanding materials in graph theory and has been defying solutions for more than a century. Further attempts at hamiltonian problems led researchers into the study of super-hamiltonian graphs, such as pancyclic graphs and panconnected graphs.

A graph is pancyclic if it contains a cycle of every length from 3 to \( |V(G)| \) inclusive. The concept of pancyclic graphs is proposed by Bondy [2]. A graph \( G = (V_0 \cup V_1, E) \) is bipartite if
\(V(G) = V_0 \cup V_1\) and \(E(G)\) is a subset of \(\{(u, v) \mid u \in V_0, v \in V_1\}\).

It is known that there is no odd cycle in any bipartite graph. Hence, any bipartite graph is not pancyclic. For this reason, the concept of bipancyclicity is proposed [8]. A bipartite graph is \textit{bipancyclic} if it contains a cycle of every even length from 4 to \(|V(G)|\) inclusive. It is proved that the hypercube is bipancyclic [5, 9].

A graph \(G\) is \textit{panconnected} if there exists a path of length \(l\) joining any two different vertices \(x\) and \(y\) with \(d_G(x, y) \leq l \leq |V(G)| - 1\). The concept of panconnected graphs is proposed by Alavi and Williamson [1]. It is obvious that any bipartite graph with at least 3 vertices is not panconnected. For this reason, we say a bipartite graph is \textit{bipanconnected} if there exists a path of length \(l\) joining any two different vertices \(x\) and \(y\) with \(d_G(x, y) \leq l \leq |V(G)| - 1\) and \((l - d_G(x, y))\) is even. It is proved that the hypercube is bipanconnected [5].

Here, we introduce a new concept, called panpositionable hamiltonian. A hamiltonian graph \(G\) is \textit{panpositionable} if for any two different vertices \(x\) and \(y\) of \(G\) and any integer \(k\) with \(d_G(x, y) \leq k < |V(G)|/2\), there exists a hamiltonian cycle \(C\) of \(G\) with \(d_G(x, y) = k\). Obviously, the complete graph \(K_n\) with \(n \geq 3\) is panpositionable. It is easy to see that the length of the shortest cycle for any panpositionable hamiltonian graph is 3. A hamiltonian bipartite graph \(G\) is \textit{bipanpositionable} if for any two different vertices \(x\) and \(y\) of \(G\) and for any integer \(k\) with \(d_G(x, y) \leq k < |V(G)|/2\) and \((k - d_G(x, y))\) is even, there exists a hamiltonian cycle \(C\) of \(G\) such that \(d_C(x, y) = k\). Obviously, the complete bipartite graph \(K_{n,n}\) with \(n \geq 2\) is bipanpositionable.

Let \(u = u_{n-1}u_{n-2}\ldots u_1u_0\) and \(v = v_{n-1}v_{n-2}\ldots v_1v_0\) be two \(n\)-bit binary strings. The Hamming distance \(h(u, v)\) between two vertices \(u\) and \(v\) is the number of different bits in the corresponding strings of both vertices. The \(n\)-dimensional \textit{hypercube}, \(Q_n\), consists of all \(n\)-bit binary strings as its vertices and two
vertices \( u \) and \( v \) are adjacent if and only if \( h(u, v) = 1 \). Let \( Q^i_n \) be the subgraph of \( Q_n \) induced by \( \{u_{n-1}u_{n-2} \ldots u_1u_0 \mid u_{n-1} = i\} \) for \( i = 0, 1 \). Obviously, \( Q_n \) can be constructed recursively by taking two copies of \( Q_{n-1} \), \( Q^0_{n-1} \) and \( Q^1_{n-1} \), and adding a perfect matching. We will prove that \( Q_n \) is bipanpositionable hamiltonian.

Assume that \( n, s_1, s_2, \ldots, s_r \) are integers with \( 1 \leq s_1 < s_2 < \ldots < s_r \leq \frac{n}{2} \). The circulant graph \( G(n; s_1, s_2, \ldots, s_r) \) is the graph with the vertex set \( \{0, 1, \ldots, n-1\} \). Two vertices \( i \) and \( j \) are adjacent if and only if \( i - j = \pm s_k \pmod{n} \) for some \( k \) where \( 1 \leq k \leq r \). We will prove that \( G(n; 1, 3) \) is bipanpositionable for any even integer with \( n \geq 6 \), and \( G(n; 1, 2) \) is panpositionable if and only if \( n \in \{5, 6, 7, 8, 9, 11\} \). Moreover, \( G(n; 1, 2, 3) \) is panpositionable for \( n \geq 6 \).

2 Some bipanpositionable hamiltonian graphs

Theorem 1 \( Q_n \) is bipanpositionable hamiltonian for \( n \geq 2 \).

Proof. Obviously, the theorem is true for \( Q_2 \). Now, we assume that the theorem is true for \( Q_{n-1} \) for some \( n \geq 3 \). Let \( u \) and \( v \) be two distinct vertices of \( Q_n \) with \( h(u, v) = r \). It is known that \( h(u, v) = d_{Q_n}(u, v) \). We need to show that for any integer \( i \) with \( r \leq i \leq 2^n - 1 \) and \( i - r \) is even, there exists a hamiltonian cycle \( C \) of \( Q_n \) such that \( d_C(u, v) = i \). Since \( Q_n \) is edge symmetric, \( Q_n \) can be split into \( Q^0_{n-1} \) and \( Q^1_{n-1} \) such that \( u \in Q^0_{n-1} \) and \( v \in Q^1_{n-1} \). Let \( y = y_{n-1}y_{n-2} \ldots y_1y_0 \in V(Q_n) \). We use \( y^k \) to denote the vertex \( y_{n-1}y_{n-2} \ldots y_k \ldots y_1y_0 \) for some \( 0 \leq k \leq n-1 \). Let \( z = v^{n-1} \). Obviously, \( d_{Q^0_{n-1}}(u, z) = r - 1 \) and \( z = u \) if \( d_{Q_n}(u, v) = 1 \). By induction assumption, there exists a hamiltonian cycle \( C = \{x_1, x_2, \ldots, x_{2^n - 1}, x_1\} \) of \( Q^0_{n-1} \) such that \( d_C(u, z) = r - 1 \). Without loss of generality, we assume that \( x_1 = u \) and \( x_r = z \). Note that \( r \leq \frac{1 + r}{2} \) and \( x_r^{n-1} = v \). Let
Figure 1: The hamiltonian cycle in Theorem 1.

\[ P_1 = \langle x_1, x_2, \ldots, x_r, \ldots, x_{i+r} \rangle \text{ and} \]
\[ P_2 = \langle x_{\frac{i+r}{2}+1}, x_{\frac{i+r}{2}+2}, \ldots, x_{2n-1}, x_1 \rangle. \] We set
\[ P_1^* = \langle x_{i+r}^{-1}, x_{i+r}^{n-1}, \ldots, x_{n-1}^{n-1}, x_1^{n-1} \rangle \text{ and} \]
\[ P_2^* = \langle x_1^{n-1}, x_{2n-1}^{n-1}, x_{2n-1-i}^{n-1}, \ldots, x_{\frac{i+r}{2}+1}^{n-1} \rangle. \]

Let \[ C_i = \langle x_1, P_1, x_{i+r}, x_i^{-1}, P_1^*, x_1^{-1}, P_2, x_{\frac{i+r}{2}+1}, x_{i+r+1}, P_2, x_1 \rangle. \]

Obviously, \( C_i \) be a hamiltonian cycle of \( Q_n \) and \( d_C(u, v) = i \).
See Figure 2 as an illustration. \( \square \)

**Theorem 2** \( G(n; 1, 3) \) is bipanpositionable hamiltonian if and only if \( n \) is an even integer and \( n \geq 6 \).

**Proof.** Let \( H = G(n; 1, 3) \). Obviously, \( H \) is bipartite if and only if \( n \) is even. Thus, \( H \) is not bipanpositionable hamiltonian
if $n$ is odd. Assume that $n$ is an even integer with $n \geq 6$. With the symmetric property of $H$, it suffices to show that there exists a hamiltonian cycle $C$ such that $d_C(0, u) = k$ for any vertex $u$ of $H$ with $1 \leq u \leq \frac{n}{2}$, and any integer $k$ with $d_H(0, u) \leq k \leq \frac{n}{2}$ and $k - d_H(0, u)$ is even. It is easy to see that $d_H(0, u) = \lceil \frac{n}{3} \rceil$. We set $r = \lceil \frac{n}{3} \rceil$. To describe the required hamiltonian cycles, we define some path patterns:

$$p(i, j) = \langle i, i + 1, i + 2, \ldots, j - 1, j \rangle;$$

$$q(i, i + 3) = \langle i, i + 3 \rangle;$$

$$q^{-1}(i, i - 3) = \langle i, i - 3 \rangle.$$

Then we define the path pattern $q^t$ by executing the path pattern $q$ for $t$ times. Similarly for $(q^{-1})^t$. More precisely,

$$q^t(i, i + 3t) = \langle i, q(i, i + 3), i + 3, q(i + 3, i + 6), \ldots, i + 3(t - 1), q(i + 3(t - 1), i + 3t), i + 3t \rangle;$$

$$(q^{-1})^t(i, i - 3t) = \langle i, q^{-1}(i, i - 3), i - 3, q^{-1}(i - 3, i - 6), \ldots, i - 3(t - 1), q^{-1}(i - 3(t - 1), i - 3t), i - 3t \rangle.$$

There are three cases:

**Case 1.** $u \equiv 0 \pmod{3}$.

(1.1) $r \leq k \leq u$. Let $l = \frac{k - r}{2}$. The hamiltonian cycle is

$$C = \langle 0, p(0, 3l), 3l, q^\frac{u - 3l}{3}(3l, u), u, u + 1, (q^{-1})^\frac{u - 3l}{3}(u + 1, 3l + 1), 3l + 1, 3l + 2, q^\frac{u - 3l}{3}(3l + 2, u + 2), u + 2, p(u + 2, n - 1), n - 1, 0 \rangle.$$

(1.2) $u < k \leq \frac{n}{2}$. Let $l = \frac{k - u}{2}$. The hamiltonian cycle is

$$C = \langle 0, p(0, u - 1), u - 1, q^t(u - 1, u + 3l - 1), u + 3l - 1, u + 3l - 2, (q^{-1})^{t-1}(u + 3l - 2, u + 1), u + 1, u, q^t(u, u + 3l), u + 3l, p(u + 3l, n - 1), n - 1, 0 \rangle.$$
Case 2. $u \equiv 1 \pmod{3}$.

(2.1) $r \leq k \leq u$. Let $l = \frac{k-r}{2}$. The Hamiltonian cycle is

$$C = \langle 0, 1, p(1, 3l + 1), 3l + 1, q^{\frac{u-3l-1}{3}}(3l + 1, u), u, u + 1, (q^{-1})^{\frac{u-3l-1}{3}}(u + 1, 3l + 2), 3l + 2, 3l + 3, q^{\frac{u-3l-1}{3}}(3l + 3, u + 2), u + 2, p(u + 2, n - 1), n - 1, 0 \rangle.$$

(2.2) $u < k \leq \frac{n}{2}$. Let $l = \frac{k-u}{2}$. The Hamiltonian cycle is

$$C = \langle 0, p(0, u - 1), u - 1, q^l(u - 1, u + 3l - 1), u + 3l - 1, u + 3l, (q^{-1})^l(u + 3l, u), u, u + 1, q^l(u + 1, u + 3l + 1), u + 3l + 1, p(u + 3l + 1, n - 1), n - 1, 0 \rangle.$$

Case 3. $u \equiv 2 \pmod{3}$.

(3.1) $r \leq k < u$. Let $l = \frac{k-r}{2}$. The Hamiltonian cycle is

$$C = \langle 0, p(0, 3l + 2), 3l + 2, q^{\frac{u+3l-2}{3}}(3l + 2, u), u, u - 1, (q^{-1})^{\frac{u+3l-2}{3}}(u - 1, 3l + 4), 3l + 4, 3l + 3, q^{\frac{u-3l-2}{3}}(3l + 3, u + 1), u + 1, p(u + 1, n - 1), n - 1, 0 \rangle.$$

(3.2) $k = u$. The Hamiltonian cycle is

$$C = \langle 0, p(0, n - 1), n - 1, 0 \rangle.$$

(3.3) $u < k \leq \frac{n}{2}$. Let $l = \frac{k-u}{2}$. The Hamiltonian cycle is

$$C = \langle 0, p(0, u - 2), u - 2, q^l(u - 2, u + 3l - 2), u + 3l - 2, u + 3l - 1, (q^{-1})^l(u + 3l - 1, u - 1), u - 1, u, q^l(u, u + 3l), u + 3l, p(u + 3l, n - 1), n - 1, 0 \rangle.$$

The theorem is proved. \qed
3 Some panpositionable hamiltonian graphs

Theorem 3 $G(n; 1, 2)$ is panpositionable hamiltonian if and only if $n \in \{5, 6, 7, 8, 9, 11\}.$

Proof. Let $H = G(n; 1, 2).$ We first show that $H$ is panpositionable if $n \in \{5, 6, 7, 8, 9, 11\}$. With the symmetric property of $H$, it suffices to show that for any vertex $u$ with $1 \leq u \leq \frac{n}{2}$ and for any integer $k$ with $d_H(0, u) \leq k \leq \frac{n}{2}$, there exists a hamiltonian cycle $C$ such that $d_C(0, u) = k$. It is easy to see that $d_H(0, u) = \lceil \frac{n}{2} \rceil$. We set $r = \lceil \frac{n}{2} \rceil$. To describe the required hamiltonian cycles, we define some path patterns:

\[ p(i, j) = (i, i + 1, i + 2, \ldots, j - 1, j); \]
\[ q(i, j) = (i, i + 2, i + 4, \ldots, j - 2, j); \]
\[ q^{-1}(j, i) = (j, j - 2, j - 4, \ldots, i + 2, i). \]

Case 1. $n \in \{5, 7, 9, 11\}$.

<table>
<thead>
<tr>
<th>${0, u}$</th>
<th>$d_C(0, u)$</th>
<th>Hamiltonian cycle $C$</th>
</tr>
</thead>
<tbody>
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<td>$4, n \in {9, 11}$</td>
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<tr>
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<tr>
<td>{0, u}</td>
<td>(d_C(0, u))</td>
<td>Hamiltonian cycle (C)</td>
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Case 2. \(n \in \{6, 8\}\).
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<th>Hamiltonian cycle $C$</th>
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<td>${0, 4}$</td>
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<td>${0, 4}$</td>
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</table>

To show that $H$ is not panpositionable hamiltonian if $n = 10$ or $n \geq 12$, we prove that there exists no hamiltonian cycle in $H$ such that the distance between 0 and 2 is 5. Suppose that $C$ is a hamiltonian cycle of $H$ with $d_C(0, 2) = 5$. Obviously, $P_1 = \langle 0, n - 2, n - 1, 1, 3, 2 \rangle$, $P_2 = \langle 0, n - 1, 1, 3, 4, 2 \rangle$ and $P_3 = \langle 0, 1, 3, 5, 4, 2 \rangle$ are all the possible paths of length 5 joining 0 and 2. Then $C$ contains exactly one of $P_1$, $P_2$ and $P_3$.

If $C$ contains $P_1$, then $\{(0, 1), (0, n - 1)\} \not\subseteq C$. Thus, $C$ contains $\langle n - 2, 0, 2 \rangle$. This means $C$ contains a cycle $\langle 0, P_1, 2, 0 \rangle$, which is impossible. If $C$ contains $P_2$ or $P_3$, then $\{(2, 1), (2, 3)\} \not\subseteq C$. Thus, $C$ contains $\langle 0, 2, 4 \rangle$. This means that $C$ contains a cycle $\langle 0, P_2, 2, 0 \rangle$ or $\langle 0, P_3, 2, 0 \rangle$, respectively, which is impossible. The theorem is proved.

**Theorem 4** $G(n; 1, 2, 3)$ is panpositionable hamiltonian for $n \geq 5$.

**Proof.** Let $H = G(n; 1, 2, 3)$ and $u$ be any vertex of $H$ with $1 \leq u \leq \frac{n}{2}$. Since $G(n; 1, 2)$ is a spanning subgraph of $H$, with
Theorem 3, $H$ is panpositionable hamiltonian when $n = 5$. It is easy to see that $d_H(0,u) = \lceil \frac{n}{2} \rceil$. We set $r = \lceil \frac{n}{2} \rceil$. With the symmetric property of $H$, it suffices to show that there exists a hamiltonian cycle $C$ such that $d_C(0,u) = k$ for any integer $k$ with $r \leq k \leq \frac{n}{2}$. Suppose that $k - r$ is even. Since $G(n,1,3)$ is a spanning subgraph of $H$, we can use the similar argument as in Theorem 2, no matter $n$ is odd or even, to prove that there exists a hamiltonian cycle $C$ of $H$ such that $d_C(0,u) = k$. Therefore, we only consider the cases $k - r$ is odd. To describe the required hamiltonian cycles, we define some path patterns:

\[
\begin{align*}
    p(i,j) & = \langle i, i+1, i+2, \ldots, j-1, j \rangle; \\
    q(i,i+3) & = \langle i, i+3 \rangle; \\
    q^{-1}(i,i-3) & = \langle i, i-3 \rangle; \\
    q^t(i,i+3t) & = \langle i, q(i,i+3), i+3, q(i+3,i+6), \ldots, \\
                     & i+3(t-1), q(i+3(t-1), i+3t), \\
                     & i+3t; \\
    (q^{-1})^t(i,i-3t) & = \langle i, q^{-1}(i,i-3), i-3, q^{-1}(i-3,i-6), \ldots, \\
                       & i-3(t-1), q^{-1}(i-3(t-1), i-3t), \\
                       & i-3t; \\
    r^t_1(0,3t) & = \langle 0, p(0,3t-3), 3t-3, 3t-2, 3t \rangle; \\
    s^t_1(u-1,u+1) & = \langle u-1, q^t(u-1,u+3t-1), u+3t-1, \\
                        & u+3t+1, (q^{-1})^t(u+3t+1,u+1), \\
                        & u+1; \\
    r^t_2(0,3t+1) & = \langle 0, p(0,3t-1), 3t-1, 3t+1 \rangle; \\
    s^t_2(u-1,u) & = \langle u-1, q^t(u-1,u+3t-1), u+3t-1, \\
                      & u+3t-3, (q^{-1})^{t-1}(u+3t-3,u), u \rangle; \\
    r^t_3(0,3t+2) & = \langle 0, p(0,3t), 3t, 3t+2 \rangle; \\
    s^t_3(u-1,u) & = \langle u-1, q^{t+1}(u-1,u+3t+2), u+3t+2, \\
                      & u+3t, (q^{-1})^t(u+3t,u), u \rangle.
\end{align*}
\]

There are three cases:
Case 1. $u \equiv 0 \pmod{3}$.

(1.1) $r < k < u$. Let $l = \frac{k-r+1}{2}$.

The Hamiltonian cycle is

$$C = \langle 0, r^l_1(0, 3l), 3l, q^{\frac{u-3l-1}{3}}(3l, u), u, u+1,$$

$$(q^{-1})^{\frac{u-3l-1}{3}}(u+1, 3l+1), 3l+1, 3l-1,$$

$$q^{\frac{u-3l-1}{3}}(3l-1, u+2), u+2, p(u+2, n-1), n-1, 0 \rangle.$$

(1.2) $u < k \leq \frac{n}{2}$. Let $l = \frac{k-u-1}{2}$.

The Hamiltonian cycle is

$$C = \langle 0, p(0, u-1), u-1, s^l_1(u-1, u+1), u+1, u,$$

$$q^{l+1}(u, u+3l+3), u+3l+3, u+3l+4, u+3l+2,$$

$$u+3l+5, p(u+3l+5, n-1), n-1, 0 \rangle.$$

Case 2. $u \equiv 1 \pmod{3}$.

(2.1) $r < k < u$. Let $l = \frac{k-r+1}{2}$.

The Hamiltonian cycle is

$$C = \langle 0, r^l_2(0, 3l+1), 3l+1, q^{\frac{u-3l-1}{3}}(3l+1, u), u, u-1,$$

$$(q^{-1})^{\frac{u-3l-1}{3}}(u-1, 3l), 3l, 3l+2, q^{\frac{u-3l-1}{3}}(3l+2, u+1),$$_

$$u+1, p(u+1, n-1), n-1, 0 \rangle.$$

(2.2) $u < k \leq \frac{n}{2}$. Let $l = \frac{k-u+1}{2}$.

The Hamiltonian cycle is

$$C = \langle 0, p(0, u-1), u-1, s^l_2(u-1, u), u, u+1,$$

$$q^{l-1}(u+1, u+3l-2), u+3l-2, u+3l, p(u+3l, n-1),$$

$$n-1, 0 \rangle.$$

Case 3. $u \equiv 2 \pmod{3}$.

(3.1) $r < k < u$. Let $l = \frac{k-r+1}{2}$.

The Hamiltonian cycle is

$$C = \langle 0, p(0, 3l-1), 3l-1, q^{\frac{u-3l+1}{3}}(3l-1, u), u, u-1,$$

$$(q^{-1})^{\frac{u-3l-2}{3}}(u-1, 3l+1), 3l+1, 3l, q^{\frac{u-3l+1}{3}}(3l, u+1),$$_

$$u+1, p(u+1, n-1), n-1, 0 \rangle.$$

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(3.2) \( k = u \). The hamiltonian cycle is
\[
C = (0, p(0, n - 1), n - 1, 0).
\]

(3.3) \( u < k \leq \frac{n}{2} \). Let \( l = \frac{k - u}{2} \). The hamiltonian cycle is
\[
C = (0, p(0, u - 2), u - 2, q^l(u - 2, u - 2 + 3l), u - 2 + 3l, \\
   u - 1 + 3l, q^{-l}(u - 1 + 3l, u - 1), u - 1, u, q^l(u, u + 3l), \\
   u + 3l, p(u + 3, n - 1), n - 1, 0).
\]

The theorem is proved. \( \square \)

4 Concluding Remark

A \( k \)-container \( C(x, y) \) in a graph \( G \) is a set of \( k \) internal vertex-disjoint paths between \( x \) and \( y \). Based on Menger's Theorem [7], there exists a \( k \)-container between any pair of vertices in a \( k \)-connected graph. The length of a \( k \)-container \( C(x, y) \), written as \( l(C(x, y)) \), is the length of the longest path in \( C(x, y) \). Suppose that \( G \) is a \( k \)-connected graph. The \( k \)-distance between \( x \) and \( y \), denoted by \( d_k(x, y) \), is defined as \( \min \{ l(C(x, y)) \mid C(x, y) \) is a \( k \)-container\}. The \( k \)-diameter of \( G \), denoted by \( D_k(G) \), is defined as \( \max \{ d_k(x, y) \mid x \neq y; x, y \in V(G) \} \). The \( k \)-diameter, proposed by Hsu [4], measures the performance of multigraph communication.

Now, we introduce another type of containers. A \( k^* \)-container \( C(x, y) \) is a \( k \)-container such that every vertex of \( G \) is incident with a path in \( C(x, y) \). A graph \( G \) is \( k^* \)-connected if there exists a \( k^* \)-container between any two vertices \( x \) and \( y \) with \( x \neq y \). Obviously, a graph \( G \) is \( 1^* \)-connected if and only if it is hamiltonian connected. Moreover, a graph \( G \) is \( 2^* \)-connected if it is hamiltonian. The concept of \( k^* \)-connected graphs is proposed by Lin et. al.[6].

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Suppose that $G$ is a $k^*$-connected graph. Similar to the definitions of $k$-distance and $k$-diameter, we can define the $k^*$-distance, $d^*_k(x, y)$, as $\min\{C(x, y) \mid C(x, y) \text{ is a } k^*-\text{container}\}$. The $k^*$-diameter, denoted by $D^*_k(G)$, is defined by $\max\{d^*_k(x, y) \mid x \neq y; x, y \in V(G)\}$.

Assume that $G$ is a panpositionable hamiltonian graph with $n$ vertices. Obviously, $d^*_2(u, v) = \lceil \frac{n}{2} \rceil$ if $u$ and $v$ are two different vertices in $G$. Hence $D^*_2(G) = \lceil \frac{n}{2} \rceil$. Similarly, let $G$ be a bipanpositionable hamiltonian graph with $n$ vertices. Obviously, $d^*_2(u, v)$ is either $\lceil \frac{n}{2} \rceil + 1$ or $\lceil \frac{n}{2} \rceil$ depending on the parity of $d(u, v)$. (Note that $d^*_2(u, v) = d(u, v)$.) Thus, $D^*_2(G) = \lceil \frac{n}{2} \rceil + 1$. In particular, $D^*_2(Q_n) = 2^{n-1} + 1$ for $n \geq 2$.

Let $f(n)$ denote the minimum number of edges among any panpositionable hamiltonian graph with $n$ vertices. With Theorem 4, we know that $f(n) \leq 3n$ if $n \geq 6$. It is interesting to find the asymptotic value of $f(n)$ as $n$ is large. Similarly, let $f_b(n)$ be the minimum number of edges among any bipanpositionable hamiltonian graph with $n$ vertices. Obviously, $f(n) = 0$ if $n$ is odd. With Theorem 2, $f_b(n) \leq 2n$ if $n$ is an even integer with $n \geq 6$. It is interesting to find the asymptotic value of $f_b(n)$ as $n$ is large and $n$ is even.

References


