Mutually Independent Hamiltonian Paths in Star Networks

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Two Hamiltonian Paths

Two Hamiltonian paths \( P_1 = \langle u_1, u_2, \ldots, u_{n(G)} \rangle \) and \( P_2 = \langle v_1, v_2, \ldots, v_{n(G)} \rangle \) of \( G \) from \( u \) to \( v \) are independent if \( u = u_i \), \( v = v_i \), and \( v_j \neq u_i \) for every \( 1 < i < n(G) \). A set of Hamiltonian paths, \( \{ P_1, P_2, \ldots, P_k \} \), of \( G \) from \( u \) to \( v \) are mutually independent if any two different Hamiltonian paths are independent from \( u \) to \( v \). A bipartite graph \( G \) is Hamiltonian laceable if there exists a Hamiltonian path joining any two nodes from different partite sets. A bipartite graph is \( k \)-mutually independent Hamiltonian laceable if there exists \( k \)-mutually independent Hamiltonian paths between any two nodes from distinct partite sets. The mutually independent Hamiltonian laceability of a bipartite graph \( G \), \( \text{IHP}_k(G) \), is the maximum integer \( k \) such that \( G \) is \( k \)-mutually independent Hamiltonian laceable. Let \( S_n \) denote the \( n \)-dimensional star graph. We prove that \( \text{IHP}_k(S_2) = 1 \), \( \text{IHP}_k(S_3) = 0 \), and \( \text{IHP}_k(S_n) = n - 2 \) if \( n \geq 4 \). © 2005 Wiley Periodicals, Inc.

**KEYWORDS:** Hamiltonian; Hamiltonian connected; Hamiltonian laceable; star networks; interconnection networks

1. INTRODUCTION

For definitions and notation, we follow [3]. \( G = (V, E) \) is a graph if \( V \) is a finite set and \( E \) is a subset of \( \{(u, v) \mid (u, v) \in E \} \) is an unordered pair of \( V \). We say that \( V \) is the node set and \( E \) is the edge set. For a node \( u \), \( N(u) \) denotes the neighborhood of \( u \), which is the set \( \{v \mid (u, v) \in E\} \). For any node \( u \) of \( V \), we denote the degree of \( u \) by \( \deg_G(u) = |N(u)| \).

Two nodes \( u \) and \( v \) are adjacent if \( (u, v) \in E \). A path is a sequence of adjacent nodes, written as \( \langle v_1, v_2, \ldots, v_k \rangle \), in which the nodes \( v_1, v_2, \ldots, v_k \) are distinct except that possibly \( v_1 = v_k \). We use \( Q(i) \) to denote \( v_i \), the \( i \)-th node \( v_i \) of \( Q = \langle v_1, v_2, \ldots, v_k \rangle \). We also write the path \( \langle v_1, v_2, \ldots, v_k \rangle \) as \( \langle v_1, Q_1, v_2, Q_2, v_3, \ldots, v_k \rangle \) where \( Q_1 \) is the path \( \langle v_1, v_2, \ldots, v_t \rangle \) and \( Q_2 \) is the path \( \langle v_{t+1}, v_{t+2}, \ldots, v_k \rangle \). We use \( d(u, v) \) to denote the distance between \( u \) and \( v \). A path is a Hamiltonian path if it contains all nodes of \( G \). A graph \( G \) is Hamiltonian connected if there exists a Hamiltonian path between any two different nodes of \( G \).

A graph \( G \) is bipartite if its node set can be partitioned into two subsets \( V_1 \) and \( V_2 \) such that every edge joins nodes between \( V_1 \) and \( V_2 \). It is easy to see that any bipartite graph with at least three nodes is not Hamiltonian connected. A bipartite graph \( G \) is Hamiltonian laceable if there exists a Hamiltonian path joining any two nodes from different partite sets.

An interconnection network connects the processors of parallel computers. Its architecture can be represented as a graph, in which the nodes correspond to processors and the edges correspond to connections. Hence, we use graphs and networks interchangeably. There are many mutually conflicting requirements in designing the topology for computer networks. The \( n \)-cube is one of the most popular topologies [11]. The \( n \)-dimensional star network \( S_n \) was proposed in [1] as “an attractive alternative to the \( n \)-cube” topology for interconnecting processors in parallel computers. Since its introduction, the network \( S_n \) has received considerable attention. Akers and Krishnamurthy [1] showed that the star graphs are node transitive and edge transitive. Iwo et al. [9] showed that the star graphs are bipartite. The diameter and fault diameters were computed in [1, 10, 13, 14]. In particular, Fragopoulou and Akl [6, 7] studied the embedding of \( (n - 1) \) directed edge-disjoint spanning trees into the star network \( S_n \).
These spanning trees are used in communication algorithms for star networks.

In this article, we are interested in another property of the star graphs. We say two Hamiltonian paths \( P_1 = \langle u_1, u_2, \ldots, u_n(G) \rangle \) and \( P_2 = \langle v_1, v_2, \ldots, v_n(G) \rangle \) of \( G \) from \( u \) to \( v \) are independent if \( u = u_1 = v_1, v = v_n(G) = u_n(G), \) and \( v_i \neq u_i \) for every \( 1 < i < n(G) \). A set of Hamiltonian paths, \( \{P_1, P_2, \ldots, P_k\} \), of \( G \) from \( u \) to \( v \) are mutually independent if any two different Hamiltonian paths are independent from \( u \) to \( v \). The concept of mutually independent Hamiltonian paths arises from the following application. If there are \( k \) pieces of data needed to be sent from \( u \) to \( v \), and the data needed to be processed at every node (and the process takes time), then we want mutually independent Hamiltonian paths from \( u \) to \( v \) so that there will be no waiting time at a processor. For this reason, a graph is \( k \)-mutually independent Hamiltonian connected if there exists \( k \)-mutually independent Hamiltonian paths between any two distinct nodes. Moreover, the mutually independent Hamiltonian connectivity of a graph \( G \), \( \text{IHP}(G) \), is the maximum integer \( k \) such that \( G \) is \( k \)-mutually independent Hamiltonian connected if \( G \) is Hamiltonian connected, and 0 if otherwise. It is easy to check that \( \text{IHP}(K_2) = 1 \) for \( n = 2 \) and \( \text{IHP}(K_n) = n - 2 \) if \( n \geq 3 \) where \( K_n \) is the complete graph with \( n \) nodes.

Assume that \( G \) is a graph with at least three nodes. Suppose that \( u \) and \( v \) are adjacent in \( G \). Let \( P = \{u = v_1, v_2, \ldots, v_n(G) = v\} \) be any Hamiltonian path of \( G \) from \( u \) to \( v \), because \( n(G) \geq 3 \), \( v_2 \neq v \). Thus, there are at most \( \deg_G(u) - 1 \) mutually independent Hamiltonian paths in \( G \) from \( u \) to \( v \). Thus, \( \text{IHP}(G) \leq \delta(G) - 1 \), where \( \delta(G) = \min[\deg_G(v) \mid v \in V] \).

Obviously, the concept of mutually independent Hamiltonian connectivity is not suitable for bipartite graphs. For this reason, we say a bipartite graph is \( k \)-mutually independent Hamiltonian laceable if there exists \( k \)-mutually independent Hamiltonian paths between any two nodes from distinct partite sets. Moreover, the mutually independent Hamiltonian laceability of a bipartite graph \( G \), \( \text{IHP}_L(G) \), is the maximum integer \( k \) such that \( G \) is \( k \)-mutually independent Hamiltonian laceable if \( G \) is Hamiltonian laceable, and 0 if otherwise. It is easy to check that \( \text{IHP}_L(K_{1,1}) = 1 \) and \( \text{IHP}_L(K_{n,n}) = n - 1 \) if \( n \geq 2 \) where \( K_{n,n} \) is the complete bipartite graph with \( n \) nodes in each partite set. Again, we have \( \text{IHP}_L(G) \leq \delta(G) - 1 \) for any bipartite graph with at least three vertices.

In this article, we prove that \( \text{IHP}_L(S_2) = 1, \text{IHP}_L(S_3) = 0, \) and \( \text{IHP}_L(S_4) = n - 2 \) if \( n \geq 4 \). In the following section, we give the definition of the star networks and summarize previous relevant work. In Section 3, we prove our main result. In the final section, we give our concluding remarks.  

2. BASIC PROPERTIES OF THE STAR NETWORKS

Assume that \( n \geq 2 \). We use \( \langle n \rangle \) to denote the set \( \{1, 2, \ldots, n\} \), where \( n \) is a positive integer. A permutation on \( \langle n \rangle \) is a sequence of \( n \) distinct element of \( u_i \in \langle n \rangle \), \( u_1u_2 \ldots u_i \ldots u_n \). An inversion of \( u_1u_2 \ldots u_i \ldots u_n \) is a pair \((i, j)\) such that \( u_i < u_j \) and \( i > j \). An even permutation is a permutation with an even number of inversions, and an odd permutation is a permutation with an odd number of inversions. The \( n \)-dimensional star network, denoted by \( S_n \), is a graph with the node set \( V(S_n) = \{u_1u_2 \ldots u_n \mid u_i \in \langle n \rangle \) and \( u_i \neq u_j \) for \( i \neq j \)\). The edges are specified as follows: \( u_1u_2 \ldots u_i \ldots u_n \) is adjacent to \( v_1v_2 \ldots v_i \ldots v_n \) by an edge in dimension \( i \) with \( 2 \leq i \leq n \) if \( v_j = u_j \) for \( j \neq \{1, i\}, v_1 = u_1 \) and \( v_i = u_i \). By definition, \( S_n \) is an \( (n-1) \)-regular graph with \( n! \) nodes. Moreover, it is node transitive and edge transitive. The star graphs \( S_2, S_3, \) and \( S_4 \) are shown in Figure 1 for illustration.

![Figure 1](image-url)
We use boldface letters to denote nodes in $S_n$. Hence, $u_1, u_2, \ldots, u_n$ is a sequence of $n$ nodes in $S_n$. We use $e$ to denote the element $12\ldots n$. It is known that the connectivity of $S_n$ is $n - 1$. Moreover, $S_n$ is a bipartite graph with one partite set containing those nodes corresponding to odd permutations and the other partite set containing those nodes corresponding to even permutations. We will use white nodes to represent nodes for even permutations and black nodes to represent nodes for odd permutations. Let $u = u_1 u_2 \ldots u_n$ be any node of the star network $S_n$. We say that $u_i$ is the $i$-th coordinate of $u$, denoted by $(u)_i$, for $1 \leq i \leq n$. By the definition of $S_n$, there is exactly one neighbor $v$ of $u$ such that $u$ and $v$ are adjacent through an edge in the $i$-th dimension for $2 \leq i \leq n$. For this reason, we use $u^{(i)}$ to denote the unique $i$-neighbor of $u$. Obviously, $(u^{(i)})^i = u$. For $1 \leq i \leq n$, let $S_n^i$ denote the subgraph of $S_n$ induced by those nodes $u$ with $(u)_i = i$. Obviously, $S_n$ can be decomposed into $n$ subgraphs $S_n^i$, $1 \leq i \leq n$, and each $S_n^i$ is isomorphic to $S_{n-1}$. This furnishes a recursive definition (construction) for star networks. For $1 \leq i \neq j \leq n$, we use $E^{ij}$ to denote the set of edges between $S_n^i$ and $S_n^j$.

**Lemma 1.** Assume that $n \geq 3$. $|E^{ij}| = (n - 2)!$ for any $1 \leq i \neq j \leq n$. Moreover, there are $(n-2)!/2$ edges joining black nodes of $S_n^i$ to white nodes of $S_n^j$.

**Proof.** For each of the $(n - 2)!$ permutations of $(n) - \{i, j\}$, there is exactly one transposition that represents an edge between $S_n^i$ and $S_n^j$. Hence, $|E^{ij}| = (n - 2)!$. Exactly half of these correspond to transpositions of an odd permutation in $S_n^i$ to an even permutation in $S_n^j$.

**Lemma 2.** Suppose that $u$ and $v$ are any two distinct nodes of $S_n$ with $(u)_n = (v)_n$ such that $1 \leq d(u, v) \leq 2$. Then $(u^b)_n \neq (v^b)_n$.

**Proof.** By definition, $(u)_1 \neq (v)_1$, $((u^b)_n)_n = (u)_1$, and $((v^b)_n)_n = (v)_1$. Thus, $(u^b)_n \neq (v^b)_n$.

**Theorem 1.** $S_n$ is hamiltonian laceable if and only if $n \neq 3$.

**Theorem 2.** Assume that $n \geq 5$. Let $\{a_1, a_2, \ldots, a_r\}$ be any subset of $(n)$ for some $2 \leq r \leq n$. For any white node $u$ of $S_n^{a_1}$ and any black node $v$ of $S_n^{a_r}$, there exists a path $P = \langle u = x_1, P_1, y_1, x_2, P_2, y_2, \ldots, x_r, P_r, y_r = v \rangle$ joining $u$ and $v$ such that $P_i$ is a hamiltonian path of $S_n^{a_i}$ joining $x_i$ to $y_i$ for $1 \leq j \leq r$. Moreover, $P$ is a hamiltonian path of the subgraph induced by $\cup_{i=1}^r S_n^{a_i}$ joining $u$ to $v$.

**Proof.** We set $x_1 = u$ and $y_r = v$. By Lemma 1, we choose $(y_1, x_{r+1}) \in E_n^{a_1, a_{r+1}}$ with $y_1$ a black node of $S_n^{a_1}$ and $x_{r+1}$ a white node of $S_n^{a_{r+1}}$ for $1 \leq i \leq r - 1$. By Theorem 1, there is a hamiltonian path $P_i$ of $S_n^{a_i}$ joining $x_i$ to $y_i$ for every $1 \leq i \leq r$. The path $\langle x_1, P_1, y_1, x_2, P_2, y_2, \ldots, x_r, P_r, y_r \rangle$ forms the required path.

**Lemma 3.** For any black node $w$ and any two distinct white nodes $u, v$ of $S_n$ with $n \geq 4$, there exists a hamiltonian path of $S_n - \{w\}$ joining $u$ to $v$.

**Proof.** We prove this lemma by induction. Because $S_4$ is node transitive, we can set $w = 1234$. The corresponding hamiltonian paths between $u$ and $v$ in $S_4 - \{w\}$ are listed below.

(continues)
Suppose that \( n \geq 5 \). Without loss of generality, we may assume that \( (u)_n = n \) and \( (w)_n = n - 1 \).

**Case 1.** \( v \in S^n_i \) for some \( 1 \leq i \leq n - 2 \). Let \( \{a_1, a_2, \ldots, a_{n-2}\} = (n-2) \) with \( a_{n-2} = (v)_n \). By Lemma 1, we choose two distinct white nodes, \( s \) and \( t \), in \( S^{n-1}_n - \{w\} \) with \( (s, (s)_n) \in E^{n-1,n} \) and \( (t, (t)_n) \in E^{n,n-1} \). Obviously, \( (s)_n \) is a black node of \( S^n_i \) and \( (t)_n \) is a black node of \( S^n_i \).

Because \( S^n_i \) is isomorphic to \( S_{n-1} \), by Theorem 1, there exists a Hamiltonian path \( P \) of \( S^n_i \) joining \( u \) to \( (s)_n \). Because \( S^{n-1}_n \) is isomorphic to \( S_{n-1} \), by induction, there exists a Hamiltonian path \( Q \) of \( S^{n-1}_n - \{w\} \) joining \( u \) to \( (s)_n \). Then \( (u, P, s, (s)_n, Q, t, (t)_n, R, v) \) forms the desired path.

**Case 2.** \( v \in S^{n-1}_n \). Let \( \{a_1, a_2, \ldots, a_{n-2}\} = (n-2) \) with \( a_{n-2} = (v)_n \). By Lemma 1, we choose a black node \( s \) in \( S^{n-1}_n \), and a white node \( t \) in \( S^{n-1}_n - \{w,v\} \) with \( (s, t) \in E^{n-1,n} \). Because \( S^n_i \) is isomorphic to \( S_{n-1} \) with \( i \in \{a_1, a_2, \ldots, a_{n-2}\} \), by Theorem 2, there exists a Hamiltonian path \( P \) of the subgraph induced by \( \cup_{i=1}^{n-1} S^n_i \) joining \( u \) to \( s \). Because \( S^{n-1}_n \) is isomorphic to \( S_{n-1} \), by induction, there exists a Hamiltonian path \( Q \) of \( S^{n-1}_n - \{w\} \) joining \( t \) to \( v \). Then \( (u, P, s, t, Q, v) \) forms the desired path.

**Case 3.** \( v \in S^n_i \). Note that \( |N(v) - (v)_n| \geq 3 \). We can choose \( x \) as a neighbor of \( v \) with \( (x)_n \neq n - 1 \). Because \( S^n_i \) is isomorphic to \( S_{n-1} \), there exists a Hamiltonian path \( P \) of \( S^n_i - \{x\} \) joining \( u \) to \( v \). Without loss of generality, we can write \( P \) as \( (u, P', s, v) \). Because \( d(s,x) \leq 2 \), by Lemma 2, \( (s')_n \neq (s')_n \). Obviously, \( x \) and \( s \) both are black nodes.

**Case 3a.** \( (s')_n \in S^{n-1}_n \). Let \( \{a_1, a_2, \ldots, a_{n-2}\} = (n-2) \) with \( a_{n-2} = (x)_n \). We choose a white node \( t \) of \( S^{n-1}_n - \{s'\} \) with \( (t)_1 = a_1 \). Obviously, \( (t)_n \) is a black node of \( S^{n-1}_n \).

Because \( S^{n-1}_n \) is isomorphic to \( S_{n-1} \), by induction, there exists a Hamiltonian path \( Q \) of \( S^{n-1}_n - \{w\} \) joining the white node \( (s)_n \) to \( t \).

By Theorem 2, there exists a Hamiltonian path \( R \) of the subgraph induced by \( \cup_{i=1}^{n-1} S^n_i \) joining the black node \( (t)_n \) to the white node \( (x)_n \). Then \( (u, P', s, (s')_n, Q, t, (t)_n, R, (x)_n, x, v) \) forms the desired path.

**Case 3b.** \( (s')_n \notin S^{n-1}_n \). Let \( \{a_1, a_2, \ldots, a_{n-2}\} = (n-2) \) with \( a_{n-2} = (s)_n \). We choose two distinct white nodes \( t \) and \( z \) in \( S^{n-1}_n - \{w\} \) with \( (t)_1 = a_1 \) and \( (z)_1 = a_1 \). Obviously, \( (t)_n \) is a black node of \( S^{n-2}_n \) and \( (z)_n \) is a black node of \( S^{n-2}_n \). Because \( S^{n-2}_n \) is isomorphic to \( S_{n-1} \), by Theorem 1, there exists a Hamiltonian path \( Q \) of \( S^{n-2}_n \) joining the white node \( (s')_n \) to \( (t)_n \).

Because \( S^{n-1}_n \) is isomorphic to \( S_{n-1} \), by induction, there exists a Hamiltonian path \( H \) of \( S^{n-1}_n - \{w\} \) joining \( t \) to \( z \).

By Theorem 2, there exists a Hamiltonian path \( R \) of the subgraph induced by \( \cup_{i=1}^{n-1} S^n_i \) joining \( (z)_n \) to the white node \( (x)_n \).

Then \( (u, P', s, (s')_n, Q, (t)_n, t, H, z, (z)_n, R, (x)_n, x, v) \) forms the desired path.

**Theorem 3.** Let \( u \) be any white node of \( S_n \), with \( n \geq 4 \) and \( \{a_1, a_2, \ldots, a_{n-1}\} \) be an \( n - 1 \) subset of \( \{n\} \), Then there exist Hamiltonian paths \( P_1, P_2, \ldots, P_{n-1} \) of \( S_n \) such that \( P_i \) is a path joining \( u \) to a black node \( z_i \), with

1. \( z_i = P_i(n) \) and \( (z_i)_1 = a_i \) for \( 1 \leq i \leq n - 1 \), and
2. \( |P_i(i), P_{i+1}(i), \ldots, P_{n-1}(i)| = n - 1 \) for \( 2 \leq i \leq n \).

**Proof.** Because \( S_4 \) is node transitive, we may assume that \( u = e \). We prove this theorem holds on \( S_4 \) by exhibiting the three required Hamiltonian paths as follows:
Assume that the theorem holds on $S_k$ for every $4 \leq k < n$. Without loss of generality, we suppose that $a_1 < a_2 < \cdots < a_{n-3} < a_{n-1} < a_n$. Obviously, $a_i \neq i+2$ for $1 \leq i \leq n-3$, $a_{n-2} \neq 2$, and $a_{n-1} \neq n$. By the induction hypothesis, there exist hamiltonian paths $H_1, H_2, \ldots, H_{n-2}$ of $S_n^a$ such that $H_i$ is a path joining $e$ to a black node $v_i$ with

\[
(1) \quad (v_1) = (H_1((n-1)!)_1) = i + 3 \quad \text{for} \quad 1 \leq i \leq n - 4, \\
(\nu_{n-3}) = (H_{n-3}((n-1)!-1)_1) = 1, (\nu_{n-2}) = (H_{n-2}((n-1)!)) = 3,
\]

and

\[
(2) \quad |H_1(i), H_2(i), \ldots, H_{n-2}(i)| = n - 2 \quad \text{for} \quad 2 \leq i \leq (n-1)!).
\]

For any $i, j \in \{n\}$ with $i \neq j$, $|v \in S_n^a| v$ is a black node with $(v_1) = j$ \iff $(j-3)! \geq 2$. We choose a black node $z_i \in S_n^a$ with $(z_i) = a_i$ for $1 \leq i \leq n-3$, a black node $z_{n-2} \in S_n^a$ with $(z_{n-2}) = a_{n-2}$, and a black node $z_{n-1} \in S_n^a$ with $(z_{n-1}) = a_{n-1}$. We let $B$ be the $(n-2) \times (n-1)$ matrix with

\[
b_{ij} = \begin{cases} 
i+j+2 & \text{if } i \leq n-3 \text{ and } i+j+2 \leq n-1, \\
i+j-n+3 & \text{if } i \leq n-3 \text{ and } n-1 \leq i+j+2, \\
j+2 & \text{if } i = n-2 \text{ and } j \leq n-3, \\
-j+n+3 & \text{if } i = n-2 \text{ and } n-2 \leq j \leq n-1. \end{cases}
\]

More precisely,

\[
B = \begin{bmatrix} 4 & 5 & \cdots & n-2 & n-1 & 1 & 2 & 3 \\ 5 & 6 & \cdots & n-1 & 1 & 2 & 3 & 4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & \cdots & n-5 & n-4 & n-3 & n-2 & n-1 \\ 3 & 4 & \cdots & n-3 & n-2 & n-1 & 1 & 2 \end{bmatrix}
\]

Obviously, $b_{i,j}, b_{j,1}, b_{j,2}, \ldots, b_{j,n-1}$ forms a permutation of $\{1, 2, \ldots, n-1\}$ for every $i$ with $1 \leq i \leq n-2$. Moreover, $b_{j,i} \neq b_{j,j}$ for $1 \leq i < j \leq n-2$ and $1 \leq j \leq n-2$. In other words, $B$ forms a rectangular Latin square with entries in $\{1, 2, \ldots, n-1\}$.

Let $i$ be any index with $1 \leq i \leq n-2$. By Theorem 2, there exists a path $W_i = (x_{i}^{1}, T_{i}^{1}, y_{i}^{1}, x_{i}^{2}, T_{i}^{2}, y_{i}^{2}, \ldots, x_{i}^{n-1}, T_{i}^{n-1}, y_{i}^{n-1})$ joining the white node $(v_1)_i$ to $z_i$ such that $x_{i}^{1} = (v_1)^{n}, y_{i}^{n-1} = z_{i}$, and $T_{i}^{j}$ is a hamiltonian path of $S_n^a$ joining $y_{i}^{j}$ to $y_{i}^{j}$ for every $1 \leq j \leq n-1$. Moreover, there exists a path $W = (x_{1}^{1}, T_{1}^{1}, y_{1}^{1}, x_{2}^{1}, T_{2}^{1}, y_{2}^{1}, x_{3}^{1}, T_{3}^{1}, y_{3}^{1}, \ldots, x_{n-1}^{1}, T_{n-1}^{1}, y_{n-1}^{1})$ joining the black node $(e)^{n-1}$ to the white node $(e)^{n-1}$ such that $x_{1}^{n-1} = (e)^{n}, y_{n-1}^{1} = ((e)^{n-1})^{n}$, and $T_{i}^{n-1}$ is a hamiltonian path of $S_n^a$ joining $x_{i}^{n-1}$ to $y_{i}^{n-1}$ for $1 \leq i \leq n-1$. Because $S_n^a$ is isomorphic to $S_n^a$, by Lemma 3, there exists a hamiltonian path $R$ of $S_n^a$ joining the black node $(e)^{n-1}$ to $z_{i-1}$.

We set $P_i = (e, H_{i}, v_i, (v_1)^{n}, W_i, z_i)$ for every $1 \leq i \leq n-2$, and $P_{n-1} = (e, W, ((e)^{n-1})^{n}, (e)^{n-1}, R, z_{n-1})$. Then $P_1, P_2, \ldots, P_{n-1}$ form the desired paths.

Hence, the proof is proved.

**Example.** We illustrate the proof of Theorem 3 with $n = 5$ as follows:

Because $a_1 < a_2 < a_4 < a_3, a_1 \neq 3, a_2 \neq 4, a_3 \neq 2$, and $a_4 \neq 5$. By the induction hypothesis, there exist hamiltonian paths $H_1, H_2, H_3$ of $S_n^a$ such that $H_i$ is a path joining $e$ to a black node $v_i$ with $(1) \quad (v_1)_1 = (H_1(24)!)_1 = 4, (v_2)_1 = (H_2(24)!)_1 = 1, (v_3)_1 = (H_3(24)!)_1 = 3$, and $(2) \quad |H_1(i), H_2(i), H_3(i)| = 3$ for $2 \leq i \leq 24$.

We choose a black node $z_1 \in S_5^a$ with $(z_1)_1 = a_1$, a black node $z_2 \in S_5^a$ with $(z_2)_1 = a_2$, a black node $z_3 \in S_5^a$ with $(z_3)_1 = a_3$, and a black node $z_4 \in S_5^a$ with $(z_4)_1 = a_4$. We set

\[
B = \begin{bmatrix} 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \end{bmatrix}
\]

By Theorem 2, there exists a path $W_1 = (x_1^1, T_1^1, y_1^1, x_2^1, T_2^1, y_2^1, \ldots, x_5^1, T_5^1, y_5^1)$ joining the white node $(v_1)_1$ to the black node $z_1$ such that $x_1^1 = (v_1)^{1}, y_5^1 = z_1$, and $T_1^1$ is a hamiltonian path of $S_5^a$ for every $1 \leq j \leq 4$. Similarly, there exists a path $W_2 = (x_1^1, T_1^1, y_1^1, x_2^1, T_2^1, y_2^1, x_3^1, T_3^1, y_3^1, x_4^1, T_4^1, y_4^1)$ joining the white node $(v_2)_1$ to the black node $z_2$ such that $x_1^1 = (v_2)^{1}, y_4^1 = z_2$, and $T_2^1$ is a hamiltonian path of $S_5^a$ for every $1 \leq j \leq 4$. Furthermore, there exists a path $W_3 = (x_1^1, T_1^1, y_1^1, x_2^1, T_2^1, y_2^1, x_3^1, T_3^1, y_3^1, x_4^1, T_4^1, y_4^1)$ joining the white node $(v_3)_1$ to the black node $z_3$ such that $x_1^3 = (v_3)^{1}, y_3^1 = z_3$, and $T_3^1$ is a hamiltonian path of $S_5^a$ for every $1 \leq j \leq 4$. Moreover, there exists a path $W = (x_1^1, T_1^1, y_1^1, x_2^1, T_2^1, y_2^1, x_3^1, T_3^1, y_3^1, x_4^1, T_4^1, y_4^1)$ joining the black node $(e)^{4}$ to the white node $(e)^{5}$ such that $x_1^4 = (e)^{5}, y_4^1 = (e)^{4}$, $y_4^1 = (e)^{4}$, and $T_4^1$ is a hamiltonian path of $S_5^a$ for $1 \leq i \leq 4$. Because $S_5^a$ is isomorphic to $S_5$, by Lemma 3,
there exists a Hamiltonian path $R$ of $S_5^3 - \{e\}$ joining the
black node $(e)^3$ to $z_4$.

Let $P_1 = \langle e, H_1, v_1, (v_i)^n, W_i, z_i \rangle$ for every $1 \leq i \leq 3$, and $P_4 = \langle e, W, (e)(e)^3, (e)^4, R, z_4 \rangle$. Thus, $\{P_1, P_2, P_3, P_4\}$ form
the desired paths. See Figure 2 for an illustration.

3. MUTUALLY INDEPENDENT HAMILTONIAN PATHS

Theorem 4. $IHP_L(S_2) = 1$, $IHP_L(S_3) = 0$, and $IHP_L(S_n) = n - 2$ if $n \geq 4$.

Proof. Suppose that $n = 2$. Because $S_2$ is isomorphic to
$K_{1,1}$, it is easy to see that $IHP_L(S_2) = 1$. Suppose that $n = 3$. Obviously, $S_3$ is isomorphic to the cycle graph with six nodes.
It is easy to see that $S_3$ is not Hamiltonian laceable. Thus, $IHP_L(S_3) = 0$. Now, we assume that $n \geq 4$. Because $\delta(S_n) = n - 1$, $IHP_L(S_n) \leq n - 2$. To prove our theorem, we need to construct $(n - 2)$-mutually independent Hamiltonian paths of $S_n$ between any white node $u$ and any black node $v$. We prove this theorem by induction. Because $S_4$ is node transitive, we assume that $u = 1234$. The required Hamiltonian paths of $S_4$ are listed below:

$$
\begin{aligned}
&\begin{array}{c}
V = (1234) \\
\end{array}
\end{aligned}
$$

FIG. 2. Illustration of Theorem 3 on $S_5$. 

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Suppose that the theorem holds on $S_k$ for every $4 \leq k < n$. Without loss of generality, we assume that $(u)_n = n$ and $(v)_n = n - 1$. We let $C$ be the $(n - 2) \times (n - 2)$ matrix with

$$c_{ij} = \begin{cases} 
  i + j - 1 & \text{if } i + j - 1 \leq n - 2, \\
  i + j - n + 1 & \text{if } n - 2 < i + j - 1.
\end{cases}$$

More precisely,

$$C = \begin{bmatrix}
  1 & 2 & 3 & \ldots & n - 2 \\
  2 & 3 & 4 & \ldots & n - 1 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  n - 2 & 1 & \ldots & \ldots & n - 3
\end{bmatrix}$$

Obviously, $c_{i1}, c_{i2}, \ldots, c_{i,n-2}$ is a permutation of $\{1,2,\ldots,n-2\}$ for every $i$ with $1 \leq i \leq n - 2$. Moreover, $c_{i,j} \neq c_{i',j'}$ for any $1 \leq i < i' \leq n - 2$ and $1 \leq j < n - 2$. In other words, $C$ forms a latin square with entries in $\{1,2,\ldots,n-2\}$.

It follows from Theorem 3 that there exist hamiltonian paths $Q_1, Q_2, \ldots, Q_{n-2}$ of $S_n$ such that $Q_i$ is a path joining $u$ to a black node $s_i$ with (1) $(Q_1)_1 = c_{i1}$ for every $1 \leq i \leq n - 2$, and (2) $|\{Q_1(i), Q_2(i), \ldots, Q_{n-2}(i)\}| = n - 2$ for every $2 \leq i \leq (n - 1)!$. Again, there exist hamiltonian paths $R_1, R_2, \ldots, R_{n-2}$ of $S_n$ such that $R_i$ is a path joining $v$ to a white node $t_i$ with (1) $(R_1)_1 = c_{i(n-2)}$ for every $1 \leq i \leq n - 2$, and (2) $|\{R_1(i), R_2(i), \ldots, R_{n-2}(i)\}| = n - 2$ for every $2 \leq i \leq (n - 1)!$. For every $1 \leq i \leq n - 2$, by Theorem 2, there exists a path $W_i = (x_i^1, T_1, y_1^1, x_2^1, T_2, y_2^1, \ldots, x_{n-2}^1, t_{n-2}, y_{n-2}^1)$ joining the white node $x_i^1$ to the black node $y_{n-2}^1$ such that $x_i^1 = (s_i)^n$, $y_{n-2}^1 = (t_i)^n$, and $T_j$ is a hamiltonian path of $S_{n-j}$ joining $x_i^j$ to $y_i^j$ for every $1 \leq j \leq n - 1$. Hence, $\{P_1, P_2, \ldots, P_{n-2}\}$ form the desired paths. See Figure 3 for an illustration of the case $n = 5$.

Hence, the theorem is proved.

4. CONCLUSION

In this article, we introduce the concept of independent hamiltonian paths. We use $IHP(G)$ to denote the largest integer $k$ such that there exist $k$ mutually independent hamiltonian paths between any two different nodes of $G$. Moreover, we use $IHP_k(G)$ to denote the largest integer $k$ such that there exist $k$ mutually independent hamiltonian paths between any two nodes from different partite sets of $G$. We can apply mutually independent hamiltonian paths to the area of parallel processing. We proved that $IHP_1(S_2) = 1$, $IHP_2(S_3) = 0$, and $IHP_1(S_n) = n - 2$ if $n \geq 4$. Thus, there are $(n - 2)$ mutually independent hamiltonian paths between any two nodes $u$ and $v$ from different partite sets of $S_n$ if $n \geq 4$. We note that there are at most $(n - 2)$ mutually independent hamiltonian paths between $u$ and $v$. However, it is possible that there are $(n - 1)$ mutually independent hamiltonian paths between any two nonadjacent nodes $u$ and $v$.

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