Perfect Mendelsohn designs with block size six

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Abstract

Let v, k and λ be positive integers. A perfect Mendelsohn design with parameters v, k and λ, denoted by \((v, k, \lambda)-PMD\), is a decomposition of the complete directed multigraph \(\lambda k\) on \(v\) vertices into \(k\)-circuits such that for any \(r, 1 < r < k - 1\), and for any two distinct vertices \(x\) and \(y\) there are exactly \(\lambda\) circuits along which the (directed) distance from \(x\) to \(y\) is \(r\). It is known that a \((6, 6, 1)-PMD\) does not exist. In this paper we show that a \((v, 6, 1)-PMD\) exists for any \(v > 6\), where \(v \equiv 0\) or \(1\) (mod 6), with at most 150 possible exceptions of which 2604 is the largest.

1. Introduction

The concept of a perfect cyclic design was introduced by Mendelsohn [23]. This concept was further studied in a subsequent paper [8], where the notion of resolvability was discussed. A further development of the concept was made by Hsu and Keedwell [21], where the designs were called Mendelsohn designs and associated with complete mappings and near complete mappings. In what follows, we shall adapt the terminology and notation in [21] and present the following definitions involving the concept of Mendelsohn designs.

Definition 1.1. A set of \(k\) distinct elements \(\{a_1, a_2, \ldots, a_k\}\) is said to be cyclically ordered by \(a_1 < a_2 < \cdots < a_k < a_1\) and the pair \(a_i, a_{i+1}\) are said to be \(t\)-apart in a cyclic \(k\)-tuple \((a_1, a_2, \ldots, a_k)\), where \(i + t\) is taken modulo \(k\).

Definition 1.2. Let \(v, k\) and \(\lambda\) be positive integers. A \((v, k, \lambda)\)-Mendelsohn design (briefly \((v, k, \lambda)-MD\)) is a pair \((X, \mathcal{A})\), where \(X\) is a \(v\)-set (of points) and \(\mathcal{A}\) is a collection of...
cyclically ordered $k$-tuples of $X$ (called blocks) such that every ordered pair of points of $X$ are consecutive in exactly $\lambda$ of the blocks of $\mathcal{B}$. If for all $t = 1, 2, \ldots, k - 1$, every ordered pair of points of $X$ are $t$-apart in exactly $\lambda$ of the blocks of $\mathcal{B}$, then the $(v, k, \lambda)$-MD is called a perfect design and denoted briefly by $(v, k, \lambda)$-PMD.

We wish to remark that a $(v, k, \lambda)$-MD is equivalent to a decomposition of the complete directed multigraph $\mathcal{A}K^*$ on $v$ vertices into $k$-circuits and that a $(v, k, \lambda)$-PMD is equivalent to such a decomposition where for any $r$, $1 \leq r \leq k - 1$, and for any two distinct vertices $x$ and $y$ there are exactly $\lambda$ circuits along which the (directed) distance from $x$ to $y$ is $r$. It is easy to see that the number of blocks in a $(v, k, \lambda)$-MD is $\lambda v(v - 1)/k$. This leads to an obvious necessary condition for the existence of a $(v, k, \lambda)$-PMD, that is,

$$\lambda v(v - 1) \equiv 0 \pmod{k}. \quad (1)$$

This condition is known to be sufficient in many cases, but certainly not in all.

For $k = 3$, the existence question of a $(v, 3, \lambda)$-PMD has been solved in [5, 22], and an alternative proof can be found in [31]. The result can be stated as follows.

**Theorem 1.3.** A necessary and sufficient condition for the existence of a $(v, 3, \lambda)$-PMD is

$$\lambda v(v - 1) \equiv 0 \pmod{3},$$

except for the non-existing design $(6, 3, 1)$-PMD.

For $k = 4$, Mendelsohn started in [23] the investigation of the existence of $(v, 4, 1)$-PMD noticing that a $(v, 4, 1)$-PMD is equivalent to a quasigroup of order $v$ satisfying certain identities. A partial solution for $v \equiv 1 \pmod{4}$ was obtained by Bennett [4]. Zhang [29] discussed the remaining case $v \equiv 0 \pmod{4}$. An almost complete solution for the existence of a $(v, 4, \lambda)$-PMD was presented in [13], where $v = 12$ and $\lambda = 1$ is the only unsolved case. Bennett recently reported finding a construction for a $(12, 4, 1)$-PMD, so the possible exception $v = 12$ can now be removed. We state the result as follows.

**Theorem 1.4.** The necessary condition for the existence of a $(v, 4, \lambda)$-PMD, namely,

$$\lambda v(v - 1) \equiv 0 \pmod{4},$$

is also sufficient, except for $v = 4$ and $\lambda$ odd, $v = 8$ and $\lambda = 1$.

For $k = 5$, some new constructions by weighting and by $k$-difference sequence were introduced and an almost complete solution for the existence of a $(v, 5, \lambda)$-PMD was presented in [9, 10]. A $(110, 5, 1)$-PMD and a $(130, 5, 1)$-PMD were found recently in [1]. We state the result as follows.

**Theorem 1.5.** The necessary condition for the existence of a $(v, 5, \lambda)$-PMD, namely,

$$\lambda v(v - 1) \equiv 0 \pmod{5}$$

is also sufficient, except for $v = 6$ and $\lambda = 1$, and the possible exceptions of $(v, \lambda)$ where $\lambda = 1$ and $v \in \{10, 15, 20, 26, 30, 36, 46, 50, 56, 66, 86, 90, 126, 146, 186, 206, 246\}$, and $(v, \lambda) = (18, 5)$. 
For $k = 7$, a partial solution has been given in [7, 12].

**Theorem 1.6.** The necessary condition for the existence of a $(v, 7, \lambda)$-PMD, namely $\lambda v(v - 1) \equiv 0 \pmod{7}$, is also sufficient for all even $\lambda \geq 16$, with at most 29 possible exceptions for the pair $(v, \lambda)$, where $\lambda$ is even and $\lambda < 16$. The necessary condition $v \equiv 0$ or $1 \pmod{7}$ for the existence of a $(v, 7, 1)$-PMD is also sufficient for all $v \geq 421$ with at most 40 possible exceptions below this value.

Less work has been done for $k = 6$. In this case the necessary condition (1) becomes (i) $v \equiv 0$ or $1 \pmod{3}$ when $\lambda \not\equiv 0 \pmod{3}$, and (ii) all $v \geq 6$ when $\lambda \equiv 0 \pmod{3}$. It is clear that $\lambda = 1$ and $\lambda = 3$ are the basic cases. Bennett [3] briefly discussed the case $v \equiv 1 \pmod{6}$ and $\lambda = 1$. Yin [28] discussed the case $\lambda = 3$ and obtained the following result.

**Theorem 1.7.** There exists a $(v, 6, 3)$-PMD for every integer $v \geq 6$ with 27 possible exceptions $v \in \{6, 10, 12, 16, 18, 20, 22, 24, 26, 28, 30, 32, 33, 34, 38, 39, 40, 42, 44, 45, 48, 51, 52, 54, 55, 60, 62\}$.

In this paper we shall deal with the other basic case $\lambda = 1$, where $v \equiv 0, 1, 3,$ or $4 \pmod{6}$. Although the Wilson's theory on PBD-closure can be used to show that a $(v, 6, 1)$-PMD exists whenever $v$ is in these classes and $v$ is sufficiently large, neither a specific bound on $v$ nor a specific value of $v$ for $v \equiv 3$ or $4 \pmod{6}$ is known. We shall show that a $(v, 6, 1)$-PMD exists whenever $v > 6$ and $v \equiv 0$ or $1 \pmod{6}$, with at most 150 possible exceptions of which 2604 is the largest. It is known that a $(6, 6, 1)$-PMD does not exist.

For recent results on PMDs with some additional properties such as resolvability, incomplete PMDs, PMDs with holes, and perfect Mendelsohn covering designs, the reader is referred to [6, 11, 14, 30].

2. Constructions

In this section, we shall describe some constructions for PMDs which are either known or a generalization of known constructions. We first describe the concept of PMDs with holes.

Let $X$ be a $v$-set, let $\mathcal{H}$ be a set of subsets of $X$ and let $\mathcal{B}$ be a collection of cyclically ordered $k$-subsets of $X$ (called **blocks**). A hokey perfect Mendelsohn design (briefly HPMD) having hole set $\mathcal{H}$ is a triple $(X, \mathcal{H}, \mathcal{B})$, which satisfies the following properties:

(i) for any block $A$ (as a set) in $\mathcal{B}$ and any hole $H \in \mathcal{H}$, $|A \cap H| \leq 1$.

(ii) any ordered pair $(x, y) \in X^2 - (\bigcup_{H \in \mathcal{H}} H^2)$, $x \neq y$, appears $t$-apart in exactly $\lambda$ blocks in $\mathcal{B}$, where $t = 1, 2, \ldots, k - 1$. 
If $\mathcal{H} = \emptyset$, then an HPMD($\mathcal{H}$) is just a $(v, k, \lambda)$-PMD. Also, if $\mathcal{H} = \{Y\}$, then an HPMD($\mathcal{H}$) is called an incomplete PMD and denoted by $(v, |Y|, k, \lambda)$-IPMD.

If $\mathcal{H} = \{X_1, \ldots, X_n\}$ is a partition of $X$, then an HPMD($\mathcal{H}$) is called a frame PMD and denoted by $(v, k, \lambda)$-FPMD. The type of the FPMD is defined to be the multiset $\{|X_i|: 1 \leq i \leq n\}$. We usually use an “exponential” notation to describe types: a type $t_1^{u_1}t_2^{u_2} \cdots t_k^{u_k}$ denotes $u_i$ occurrences of $t_i$, $1 \leq i \leq k$. We wish to remark that the concept of an FPMD has been called holey PMD and denoted by HPMD in the literature (see e.g., [31]), but here we introduce the notation FPMD to save the notation HPMD for more general designs. It is clear that a $(v, k, \lambda)$-PMD is equivalent to a $(v, k, \lambda)$-FPMD of type $^p$.

If $\mathcal{H} = \{X_1, \ldots, X_n, H\}$, where $\{X_1, \ldots, X_n\}$ is a partition of $X$, then an HPMD($\mathcal{H}$) is called an incomplete frame PMD and denoted by $(v, |H|, k, \lambda)$-IFPMD. The type of the IFPMD is defined to be the multiset $\{(|X_i|, |X_i \cap H|): 1 \leq i \leq n\}$. We may also use an “exponential” notation to describe types of IFPMDs.

### 2.1. Direct constructions

Suppose $\zeta$ is a primitive element of the finite field $GF(q)$, where $q = ef + 1$. Let $t$ and $k$ be positive integers such that $ft = k$. Let $\mathcal{A}$ consist of the following blocks:

$$
\begin{align*}
&\zeta^i, \zeta^{i+1}, \ldots, \zeta^{i+t-1}, \\
&\zeta^{i+t+e}, \zeta^{i+e+1}, \ldots, \zeta^{i+e+t-1}, \\
&\ldots,
\end{align*}
$$

$$
\zeta^{i+(f-1)e}, \zeta^{i+(f-1)e+1}, \ldots, \\
\zeta^{i+(f-1)e+t-1}, \\
\ldots
$$

$i = 0, 1, \ldots, e - 1$.

In [28, Theorem 2.1], Yin showed that $(GF(q), dev \mathcal{A})$ is a $(q, k, t)$-PMD, where $dev \mathcal{A} = \bigcup_{b \in \mathcal{A}} dev B$ and $dev B = \{b_1 + g, b_2 + g, \ldots, b_k + g): g \in GF(q)B = (b_1, b_2, \ldots, b_k)\}$. We state the result as follows.

**Theorem 2.1.** If $q$ is a prime power and $f$ is the greatest common divisor of $q - 1$ and $k$, then there is a $(q, k, k/f)$-PMD.

If $f = k$, we obtain a corollary which was originally found by Mendelsohn [23].

**Corollary 2.2.** Let $v = p^r$ be any prime power and $k > 2$ be such that $k$ is a divisor of $v - 1$, then there exists a $(v, k, 1)$-PMD.

Mullin et al. [24] define a vector

$$
V(m, t) = (b_1, b_2, \ldots, b_{m+1})
$$

with elements from $GF(q)$, $q = mt + 1$ a prime power, satisfying the property that for each $k \in \{1, 2, \ldots, m + 1\}$, the set

$$
\{b_i - b_j: i \in \{1, 2, \ldots, m + 1\} \setminus \{m + 2 - k\}, j \equiv i + k (mod m + 2)\} \text{ and } 1 \leq j \leq m + 1
$$

is an $H$-module with $H$-basis $\{b_1, b_2, \ldots, b_{m+1}\}$. Theorem 2.1 follows immediately from the result in [24].
is a system of distinct representatives of the cyclotomic classes $C_0, C_1, \ldots, C_{m-1}$, where $C_i = \{ \zeta^i, \zeta^{m+i}, \ldots, \zeta^{(r-1)m+i} \}$, $0 \leq i \leq m - 1$ and $\zeta$ is a primitive element of $GF(q)$.

**Theorem 2.3.** If there is a vector $V(m, t)$ in $GF(q)$, where $q = mt + 1$ is a prime power, then there exists a $(q + t, t, m + 2, 1)$-IPMD.

**Proof.** Let $V(m, t)$ be $(b_1, \ldots, b_{m+1})$. Let $\Omega = \{ \infty; c \in C_0 \}$ consist of infinite elements such that $g + \infty = \infty$ for any $g \in GF(q)$. Denote

$$\mathcal{B} = \{ (cb_1, \ldots, cb_{m+1}, \infty; c \in C_0 \}.$$

Then it is readily checked that $(GF(q) \cup \Omega, \Omega, dev \mathcal{B})$ is the required IPMD. 

2.2. Recursive constructions

Let $K$ and $M$ be sets of positive integers. A *group divisible design* (GDD) $GD(K, \lambda, M; v)$ is a triple $(X, \mathscr{G}, \mathscr{B})$, where

(i) $X$ is a $v$-set (of points),
(ii) $\mathscr{G}$ is a collection of non-empty subsets of $X$ (called *groups*) with sizes in $M$ and which partition $X$,
(iii) $\mathscr{B}$ is a collection of subsets of $X$ (called *blocks*), each with size at least two in $K$,
(iv) no block meets a group in more than one point, and
(v) each pair set $\{x, y\}$ of points not contained in a group is contained in exactly $\lambda$ blocks.

The *group-type* (or *type*) of a GDD $(X, \mathscr{G}, \mathscr{B})$ is the multiset $\{ |G| : G \in \mathscr{G} \}$ and we shall use the "exponential" notation for its description: a group-type $1^i2^j3^k\ldots$, denotes $i$ occurrences of groups of size 1, $j$ occurrences of groups of size 2, and so on.

A *weighting* of a GDD $(X, \mathscr{G}, \mathscr{B})$ is any mapping $w : X \to \mathbb{Z}^* \cup \{0\}$.

A *transversal design* (TD) $T(k, m)$ is a GD$(\{ k \}, 1, \{ k \}; km)$. It is well known that a $T(k, m)$ is equivalent to $k - 2$ mutually orthogonal Latin squares (MOLS) of order $m$. If $(X, \mathscr{G}, \mathscr{B})$ is a $T(k, m)$ and $(Y, \mathscr{H}, \mathscr{D})$ is a $T(k, n)$, where $Y \subseteq X$ and $\mathscr{H} = \{ Y \cap G : G \in \mathscr{G} \}$, then the latter is called a sub-TD of the former, and $(X, \{ \mathscr{G}, \mathscr{Y} \}, \mathscr{D} - \mathscr{A})$ is called an incomplete TD (ITD), denoted by $T(k, m) - T(k, n)$.

A *pairwise balanced design* (PBD) $B(K, \lambda; v)$ is a GD$(K, \lambda, \{ 1 \}; v)$. A $B(\{ k \}, \lambda; v)$ is called a balanced incomplete block design (BIBD) and denoted by $B(k, \lambda; v)$.

The following weighting construction is an analogue of Wilson's fundamental construction for GDDs [27].

**Theorem 2.4.** Suppose there is a "master" GD$(K, \lambda, M; v)$ $(X, \mathscr{G}, \mathscr{B})$. Suppose $w : X \to \mathbb{Z}^* \cup \{0\}$ is a weighting such that for any block $B \in \mathscr{B}$, there exists an "input" $(\sum_{x \in B} w(x), k, \mu)$-FPMD of type $\{ w(x) : x \in B \}$. Then there exists a $(\sum_{x \in X} w(x), k, \lambda \mu)$-FPMD of type $\{ \sum_{x \in G} w(x) : G \in \mathscr{G} \}$.
Proof. Let $S_x$ denote $w(x)$ copies of $x$ and $S_T = \bigcup_{x \in T} S_x$. Suppose the input FPMD for block $B$ is $(S_x, \{S_x : x \in B\}, \mathcal{A}_B)$. Denote $\mathcal{B}^* = \bigcup_{B \in \mathcal{A}} \mathcal{A}_B$. Then $(S_x, \{S_G : G \in \mathcal{G}\}, \mathcal{B}^*)$ is the required FPMD. □

As a special case of Theorem 2.4, we have the following construction.

**Theorem 2.5.** Let $v$, $k$, $\lambda$ and $\mu$ be positive integers. Suppose there exists a PBD $B(K, \lambda; v)$ and for each $k' \in K$ there exists a $(k', k, \mu)$-PMD. Then there exists a $(v, k, \lambda\mu)$-PMD.

To construct PMDs from FPMDs and IFPMDs we need the "filling in holes" constructions (see [31]), which we describe below.

**Theorem 2.6.** If there exist a $(v, w, k, \lambda)$-IFPMD of type $\{(u_i, n_i) : 1 \leq i \leq h\}$ and a $(u_i + d, n_i + d, k, \lambda)$-IPMD for $1 \leq i \leq h$, then there exists a $(v + d, w + d, k, \lambda)$-IPMD. Moreover, if there exists a $(w + d, k, \lambda)$-PMD, then there exists a $(v + d, k, \lambda)$-PMD.

**Theorem 2.7.** Suppose there exists a $(v, k, \lambda)$-FPMD of type $\{u_i : 1 \leq i \leq h\}$. If a $(u_i + d, n_i + d, k, \lambda)$-IPMD exists for $1 \leq i \leq h$ (or $1 \leq i \leq h$), then there exists a $(v + d, k, \lambda)$-IPMD. Moreover, if there exists an $(x, k, \lambda)$-PMD for $x = u_1 + d$ (or $x = d$), then there exists a $(v + d, k, \lambda)$-PMD.

**Theorem 2.8.** If there exists a $(v, w, k, \lambda)$-IPMD and a $(w, k, \lambda)$-PMD, then there exists a $(v, k, \lambda)$-PMD.

For the convenience of later use we combine weighting and filling in holes constructions to state some recursive constructions for PMDs, which are analogues to the singular indirect product (SIP), singular direct product (SDP), and direct product (DP) constructions for BIBDs.

**Theorem 2.9.** (SIP). If there exist a $(v, k, \lambda)$-PMD, a $T(k, m) - T(k, n)$ and an $(m + d, n + d, k, \lambda)$-IPMD, then there exists a $(vm + d, vn + d, k, \lambda)$-IPMD. Further, if a $(vm + d, k, \lambda)$-PMD exists, then there exists a $(vm + d, k, \lambda)$-PMD.

**Proof.** Let $(X, \mathcal{B})$ be the given $(v, k, \lambda)$-PMD. Denote $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$. Let $X^* = X \times M$, $H^* = X \times N$ and $\mathcal{H}^* = \{\{x\} \times M : x \in X\} \cup \{H^*\}$. For any $B \in \mathcal{B}$, construct on $B \times M$ a $T(k, m) - T(k, n)$ with the empty $T(k, n)$ on $B \times N$. Consider each block in the ITD as a cyclically ordered $k$-subset according to the ordering in $B$. Denote all these blocks by $\mathcal{A}_B$ and let $\mathcal{B}^* = \bigcup_{B \in \mathcal{A}} \mathcal{A}_B$. It is readily checked that $(X^*, \mathcal{H}^*, \mathcal{B}^*)$ is an $(mv, nv, k, \lambda)$-IFPMD of type $(m, n)^\ast$. The conclusion then follows from Theorem 2.6. □
Theorem 2.10 (SDR). If there exist a \((v, k, \lambda)\)-PMD, a \(T(k, m)\) and a \((m + w, w, K, \lambda)\)-PMD, then there exist a \((v, k, \lambda)\)-IPMD for \(x = w\) or \(m + w\). Further, if an \((x, k, \lambda)\)-PMD exists, then there exist a \((vm + w, k, \lambda)\)-PMD and a \((vm + w, v, k, \lambda)\)-IPMD.

Proof. Let \(n = 0\) in the proof of Theorem 2.9. The resultant \((mv, m, k, \lambda)\)-IFPMD is just a \((mv, k, \lambda)\)-FPMD of type \(m^*\). Then the conclusion follows from Theorem 2.7. Notice that the \(T(k, n)\) on \(B \times M\) contains, without loss of generality, a block \(B \times \{1\}\) and \((X \times \{1\}, \{B \times \{1\}; B \in \mathcal{B}\}\) is a sub-\((v, k, \lambda)\)-PMD. Its deletion leads to the required \((vm + w, v, k, \lambda)\)-IPMD. □

Theorem 2.11. (DP) If there exist a \((v, k, \lambda)\)-PMD, a \((u, k, \lambda)\)-PMD and a \(T(k, u)\), then there exist a \((vu, k, \lambda)\)-PMD and a \((vu, x, k, \lambda)\)-IPMD for \(x = u\) or \(v\).

Proof. Let \(u = m + w\) and \(w = 0\) in Theorem 2.10. □

We need some constructions to obtain incomplete transversal designs. The following lemmas are special cases of the working corollaries in [18]. For more about TDs see [15, 17].

Lemma 2.12. If \(T(7, t)\) and \(T(6, m + m_j)\) \((j = 1, 2, \ldots, t)\) all exist, then also a \(T(6, mt + \sum_{j=1}^t m_j) - T(6, \sum_{j=1}^t m_j)\) exists.

Lemma 2.13. If \(T(8, t)\) and \(T(6, \sum_{j=1}^t m_j)\) and \(T(6, m + m_1j + m_2j) - T(6, m_1j) - T(6, m_2j)\) \((j = 1, 2, \ldots, t)\) all exist, then also a \(T(6, mt + \sum_{j=1}^t m_j) - T(6, \sum_{j=1}^t m_j)\) exists.

Lemma 2.14. If \(T(6 + d, t), T(6, m)\) and \(T(6, m + w_i) - T(6, w_i)\) \((i = 1, 2, \ldots, d)\) all exist, then also a \(T(6, mt + w) - T(6, m + w)\) exists, where \(w = \sum_{i=1}^d w_i\).

Lemma 2.15. If \(T(i + d, t), T(6, m), T(6, m + m_j) - T(6, m_j)\) and \(T(6, m + w_i + m_j) - T(6, w_i) - T(6, m_j)\) \((i = 1, \ldots, d\) and \(j = 2, \ldots, t)\) all exist, then also a \(T(6, mt + w + \sum_{j=2}^t m_j) - T(6, m + w) - T(6, \sum_{j=2}^t m_j)\) exists, where \(w = \sum_{i=1}^d w_i\).

Lemma 2.16 [26]. A \(T(6, m)\) exists if \(m \geq 5, m \neq 6, 10, 14, 18, 22, 26, 30, 34, \text{ or } 42\).

3. \(v \equiv 1 \pmod{6}\)

Let \(P_{1,6}\) denote the set of prime powers congruent to 1 modulo 6. By Corollary 2.2, a \((q, 6, 1)\)-PMD exists whenever \(q \in P_{1,6}\). If a PBD \(B(P_{1,6}, 1; \nu)\) exists, then by
Theorem 2.8 we have a \((v, 6, 1)\)-PMD. Denote \(Q = E \cup F\), where
\[
\]
\[
F = \{205, 253, 295, 391, 445, 655, 685, 745, 781, 805, 1243, 1255, 1585, 1795, 1819, 1921\}.
\]

**Theorem 3.1** [25]. If \(v \equiv 1 \pmod{6}\), \(v \geq 1\), and \(v \notin Q\), then there is a PBD \(B(P_{1,6}, 1; v)\).

**Corollary 3.2.** If \(v \equiv 1 \pmod{6}\), \(v \geq 1\), and \(v \notin Q\), then there exists a \((v, 6, 1)\)-PMD.

In the remaining part of this section we shall show that for any \(v \in F\) there exists a \((v, 6, 1)\)-PMD.

**Lemma 3.3.** There exists a \((v, 6, 1)\)-PMD for \(v \in \{295, 655\}\).

**Proof.** Greig [19, Theorem 111] showed that a resolvable BIBD \(B(6, 1; 30t + 6)\) exists if \(t\) is even, \(4 \leq t \leq 832\), and \(6t + 1\) is a prime power. Add one new point to each parallel class to form a PBD \(B(\{7, 6t + 1\}, 1; 36t + 7)\). Construct a \((7, 6, 1)\)-PMD on each block of size 7, to obtain a \((36t + 7, 6t + 1, 6, 1)\)-PMD. For \(t = 8, 18\), a \((49, 6, 1)\)-PMD and a \((109, 6, 1)\)-PMD exist. We may apply Theorem 2.8 to obtain a \((36t + 7, 6, 1)\)-PMD for \(t = 8, 18\).

**Lemma 3.4.** A \((v, 6, 1)\)-PMD exists for \(v \in \{1795, 1819, 1921\}\).

**Proof.** First, we write \(246 + a - 7.31 + 29 + a\) and apply Lemma 2.13 with \(t = 31\), \(m = 7\), \(m_{1j} \in \{0, 1\}\) and \(m_{2j} \in \{0, 1\}\), such that \(\sum_{j=1}^{t} m_{1j} = 29\) and \(\sum_{j=1}^{t} m_{2j} = a\), \(0 \leq a \leq 31\). We obtain a \(T(6, 246 + a) - T(6, a)\). Next we apply Theorem 2.9 to obtain a \((7.246 + (49 + 6a), 6, 1)\)-PMD since a \((295, 49, 6, 1)\)-PMD exists from the proof of Lemma 3.3 and a \((49 + 6a, 6, 1)\)-PMD exists for \(a = 4, 8\) or 25 from Corollary 3.2. The conclusion then follows.

**Lemma 3.5.** Suppose there are a \((m, 6, 1)\)-PMD and a \((v, 6, 1)\)-PMD. Then

(i) a \((vm, 6, 1)\)-PMD exists if there is a \(T(6, m)\); and

(ii) a \((v(m - 1) + 1, 6, 1)\)-PMD exists if there is a \(T(6, m - 1)\).

**Proof.** (i) and (ii) are special cases of Theorem 2.10, where \(w = 0\) and 1, respectively.

**Lemma 3.6.** A \((v, 6, 1)\)-PMD exists for \(v \in \{253, 445, 685, 745, 781, 805, 1255\}\).

**Proof.** Apply Lemma 3.5(ii) with the following parameters: \(253 = 7.36 + 1\), \(445 = 37.12 + 1\), \(685 = 19.36 + 1\), \(745 = 31.24 + 1\), \(781 = 13.60 + 1\), and \(805 = 18.36 + 1\).
805 = 67.12 + 1, 1255 = 19.66 + 1. The required TDs come from Lemma 2.16, and the PMDs from Corollary 2.2.

**Lemma 3.7.** If \( q = 4t + 1 \) is a prime integer and \( 2 < t < 500 \), where \( t \) is odd, then there exists a \((q + t, t, 6,1)\)-PMD.

**Proof.** From [18, Appendix] there exists a vector \( V(4, t) \) in GF(\( q \)). Then the conclusion follows from Theorem 2.3.

**Lemma 3.8.** A \((q + t, 6,1)\)-PMD exists if \( t \) is not underlined in Table 1.

**Proof.** When \( t \) is not underlined in Table 1, \( t \equiv 1 \pmod{6} \) and \( t \notin E \). By Theorem 3.12, a \((t, 6,1)\)-PMD exists. We may fill in the size \( t \) hole in the \((q + t, t, 6,1)\)-IPMD given in Lemma 3.7 to obtain a \((q + t, 6,1)\)-PMD.

**Lemma 3.9.** Suppose there exist a \( T(14, u) \) and a \((u + w, w, 6,1)\)-IPMD. Then there exists a \((13u + 3t + w, 3t + w, 6,1)\)-IPMD, \( 0 \leq t \leq u \). Moreover, if there exists a \((3t + w, 6,1)\)-PMD, then there exists a \((13u + 3t + w, 6,1)\)-PMD.

**Proof.** Apply Theorem 2.4 with a \( T(14, u) \) as the master GDD. Give \( t \) points weight 3 and other points weight 0 in the first group. Give weight one to other points of the GDD. We need a \((13,6,1)\)-FPMD of type \( l^{13} \) and a \((16,6,1)\)-FPMD of type \( l^{13}3^1 \) as input designs. The former is equivalent to a \((13,6,1)\)-PMD and the latter to a \((16,3,6,1)\)-IPMD, which come from Corollary 3.2 and Lemma 3.7 with \( t = 3 \), respectively. We obtain a \((13u + 3t, 6,1)\)-FPMD of type \( u^{13}(3t)^1 \), where \( 0 \leq t \leq u \). The conclusion then follows from filling in holes construction.

<table>
<thead>
<tr>
<th>Table 1</th>
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<tbody>
<tr>
<td>( q )</td>
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<tr>
<td>29</td>
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<td>461</td>
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<td>509</td>
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<td>557</td>
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</table>
Lemma 3.10. A \((v, 6, 1)\)-PMD exists for \(v \in \{205, 391, 1243\}\).

Proof. In Lemma 3.9, take \(w = 0\) and \((u, t) = (13, 12), (25, 22)\) or \((79, 72)\). By Lemma 3.8 and Corollary 2.2 we obtain the required PMDs. □

Lemma 3.11. A \((1585, 6, 1)\)-PMD exists.

Proof. Apply Lemma 3.5(ii) with \(u = 66\) and \(m = 25\). A \(T(6, 24)\) exists from Lemma 2.16. We obtain the required PMD. □

Combining Corollary 3.2, Lemmas 3.3, 3.4, 3.6, 3.10 and 3.11, we obtain the main result of this section.

Theorem 3.12. If \(v \equiv 1 \pmod{6}\), \(v \geq 1\) and \(v \not\equiv 1 \pmod{6}\), then there is a \((v, 6, 1)\)-PMD.

4. \(v \equiv 0 \pmod{6}\)

To ease the notation we define

\[
\text{PMD}(6) = \{v: \text{a \((v, 6, 1)\)-PMD exists}\},
\]

Denote by \(\langle a, b \rangle\) the set of integers \(v \equiv 0 \pmod{6}\) such that \(a \leq v \leq b\).

Lemma 4.1 [31, Corollary 2.3]. There does not exist a \((6, 6, 1)\)-PMD.

In this section we shall show that for any integer \(v \geq 2610\) and \(v \equiv 0 \pmod{6}\) there exists a \((v, 6, 1)\)-PMD. We shall also discuss the small orders \(v \equiv 0 \pmod{6}\) and \(v < 2610\). We first give a preliminary bound \(v \geq 49512\). Denote

\[
O_r = \max\{v: v \text{ odd and } T(r + 2, v) \text{ does not exist}\}.
\]

It is well known [18] that \(O_{12} \leq 3565\). That is, a \(T(14, v)\) exists whenever \(v\) is odd and \(v \geq 3567\).

Lemma 4.2. Suppose that a \(T(14, 36u - 1)\) exists, where \(u \equiv 1 \pmod{6}\) and \(u \in \text{PMD}(6)\). If \(0 \leq t \leq 36u - 1\) and \(3t + 1 \in \text{PMD}(6)\), then \(468u + 3t - 12 \in \text{PMD}(6)\).

Proof. In Lemma 3.9 take \(w = 1\). We obtain \(13(36u - 1) + 3t + 1 = 468u + 3t - 12 \in \text{PMD}(6)\). We need a \((36u, 1, 6, 1)\)-IPMD, which comes from Lemma 3.5(i) since a \(T(6, 36)\) exists from Lemma 2.1. □

Proposition 4.3. If \(v \equiv 0 \pmod{6}\) and \(v \geq 49512\), then \(v \in \text{PMD}(6)\).
Proof. When $t$ is even and $t \geq 440$, we have $3t + 1 \equiv 1 \pmod{6}$, $3t + 1 \geq 1321$, and so $3t + 1 \in \text{PMD}(6)$. If $u \geq 103$, then $36u - 1 \geq 3567$ and a $T(14, 36u - 1)$ exists. By Lemma 4.2, $468u + 3t - 12 \in \text{PMD}(6)$ if $u \geq 103$, $u \equiv 1 \pmod{6}$ and $u \in \text{PMD}(6)$, where $440 \leq t \leq 36u - 1$. Therefore, $\langle 468u + 1308, 576u - 18 \rangle \subseteq \text{PMD}(6)$. We may take all these $u$'s as $u_0 = 103, u_1 = 109, u_2 = 121, \ldots$, missing those $u$'s for which a $(u, 6, 1)$-PMD is unknown. Since $0 < u_{i+1} - u_i \leq 12$, it is easy to see that $468u_i + 1308 \leq 468(u_i + 12) + 1308 \leq 576u_i - 18$, where $u_i \geq u_0 = 103$. Thus the intervals $\langle 468u_i + 1308, 576u_i - 18 \rangle$ for $i = 0, 1, \ldots$ are consecutively overlapped, where $468u_0 + 1308 = 49512$. The proof is complete. \hfill \Box

Next, we discuss the small orders.

**Proposition 4.4.** For $v \in \langle 12, 378 \rangle$, $v \in \text{PMD}(6)$ if $v = 36, 66, 126, 186, 210, 216, 246, 252, 336$ or 366.

**Proof.** Apply SDP construction, we know $210 = 7.29 + 7 \in \text{PMD}(6)$. The DP construction guarantees that $252 = 7.36 \in \text{PMD}(6)$. The conclusion then follows from Lemma 3.8. \hfill \Box

**Lemma 4.5.** $\langle 13q + t, 13q + 13t \rangle - \{13q + v: v \in E \text{ and } t \leq v \leq 13t\} \subseteq \text{PMD}(6)$, where $t$ and $q = 4t + 1$ are taken from Table 1.

**Proof.** Apply Lemma 3.9 with $u = q$ and $w = t$. \hfill \Box

**Proposition 4.6.** $\langle 384, 468 \rangle \subseteq \text{PMD}(6)$.

**Proof.** In Lemma 2.12, take $t = 8$, $m = 7$ and $m_j = \{0, 1\}$. We obtain a $T(6, 61) - T(6, 5)$ and so a $T(6, 61)$. The $T(6, 61)$ contains a sub-$T(6, 8)$. Thus we have a $T(6, 61) - T(6, 8)$. Apply SIP construction with $v = 7$ and $\lambda = 1$, we have a $(432, 61, 6, 1)$-IPMD since a $(66, 13, 6, 1)$-IPMD exists from Lemma 3.7. We have $432 \in \text{PMD}(6)$ since $61 \in \text{PMD}(6)$. The conclusion then follows from Lemma 4.5 where $t = 7$. \hfill \Box

**Lemma 4.7.** $\{528, 1008, 1680\} \subseteq \text{PMD}(6)$.

**Proof.** Delete one point from a $T(7, 7)$, we have a $GDD(7, 1, 6; 48)$ of type $6^6$. Give weight $m$ to each point, where $m = 11, 21$ or 35, and fill in holes in the resultant FPMD, we obtain the required PMDs since $66, 126, 210 \in \text{PMD}(6)$ from Proposition 4.4. \hfill \Box

**Lemma 4.8.** $540 \in \text{PMD}(6)$.

**Proof.** A $GDD(7, 1, 3; 45)$ of type $3^{15}$. Further, give weight 12 to each point of the
FPMD, we obtain a \((540, 6, 1)\)-FPMD of type \(36^15\) since a \(T(6, 12)\) exists. Filling in size 36 holes gives a \((540, 6, 1)\)-PMD.

**Lemma 4.9.** \(666, 684 \in PMD(6)\).

**Proof.** Since a \(T(6, 35)\) exists, apply Lemma 3.5 with \(684 = 19.36\) and \(666 = 19.35 + 1\).

**Proposition 4.10.** \(v \in \langle 474, 696 \rangle, v \in PMD(6)\) if \(v = 486, 528, 540, 558, 636, 666, 684\) or \(696\).

**Proof.** We may apply SDP construction with \(558 = 19.29 + 7\) to obtain a \((558, 6, 1)\)-PMD. Then the conclusion follows from Lemma 3.8 and Lemmas 4.7-4.9.

**Proposition 4.11.** \(\langle 702, 858 \rangle - \{744, 804, 834\} \subset PMD(6)\).

**Proof.** Apply Lemma 4.5 with \(t = 13\).

**Lemma 4.12.** \(936 \in PMD(6)\).

**Proof.** Apply Theorem 2.4 with a \(T(14, 13)\) as the master GDD. Give weight three to six points in the first group and weight zero to the remaining points of the group. Give weight one to other points of the GDD. We have a \((187, 6, 1)\)-FPMD of type \(13^{13}18^{1}\). Give weight five to each point of the FPMD. A similar construction in the proof of Theorem 2.9 gives a \((935, 6, 1)\)-FPMD of type \(65^{13}90^{1}\). Adding one new point and filling in the holes with a \((v, 6, 1)\)-PMD, where \(v = 66\) and \(91\) leads to a \((936, 6, 1)\)-PMD.

**Lemma 4.13.** \(990 \in PMD(6)\).

**Proof.** Delete one point from a BIBD \(B(7, 1; v)\) for \(v = 91\), see [20] for its existence. We obtain a GDD of type \(6^{12}\). Give weight 11 to each point and apply Wilson’s fundamental construction, we have a GDD of type \(66^{15}\). Since the GDD has block size 7, we obtain a \((990, 6, 1)\)-FPMD of type \(66^{15}\) and then a \((990, 6, 1)\)-PMD.

**Proposition 4.14.** \(v \in \langle 864, 1074 \rangle, v \in PMD(6)\) if \(v = 876, 882, 900, 936, 966, 990, 996, 1008\) or \(1026\).

**Proof.** Apply DP construction with \(882 = 7.126\) and \(900 = 25.36\), we have \(882, 900 \in PMD(6)\). The conclusion then follows from Lemmas 3.8, 4.7 and 4.12-4.13.

**Proposition 4.15.** \(\langle 1080, 1194 \rangle - \{1098, 1158, 1188\} \in PMD(6)\).
Proof. Apply SIP construction with \( v = 7 \) and \( \lambda = 1 \). A \((186,37,6,1)\)-IPMD exists from Lemma 3.7. Write 149 = 7.19 + 16 and apply Lemma 2.13 to obtain a \( T(6, 149 + a) - T(6, a) \) for \( 0 \leq a \leq 19 \). Thus, a \((1043 + (37 + 6a), 6,1)\)-PMD exists if \( 37 + 6a \notin E \). □

**Proposition 4.16.** For \( v \in \langle 1200, 1392 \rangle \), \( v \in \text{PMD}(6) \) if \( v \in \langle 1254, 1392 \rangle - \{1326\} \).

Proof. Using \((216,43,6,1)\)-IPMD in Table 1 and a \( T(6, 173 + a) - T(6, a) \) for \( 0 \leq a \leq 23 \), we may apply SIP construction to obtain \( \langle 1254, 1392 \rangle - \{1266, 1326, 1356\} \subseteq \text{PMD}(6) \), where the ITD comes from the expression 173 = 7.23 + 12. 1266 \( \in \text{PMD}(6) \) by Lemma 3.7. Write 1356 = 13.101 + 43. We may apply Lemma 4.5 with \( t = 25 \) to obtain 1356 \( \in \text{PMD}(6) \). □

**Proposition 4.17.** \( \langle 1398, 1998 \rangle - \{1740, 1770, 1860, 1890, 1944\} \subseteq \text{PMD}(6) \).

Proof. Apply Lemma 4.5 with \( t = 25 \) we have \( \langle 1398, 1638 \rangle - \{1428, 1458, 1548, 1578, 1632\} \subseteq \text{PMD}(6) \).

Write 197 = 7.27 + 8, a \( T(6, 197 + a) - T(6, a) \) for \( 0 \leq a \leq 27 \) exists from Lemma 2.13. Since a \((246,49,6,1)\)-IPMD exists from Lemma 3.7, we may apply SIP construction to obtain \( \{1428, 1458, 1548, 1578\} \subseteq \text{PMD}(6) \).

Apply Lemma 3.9 with \( u = 125 \) and \( w = 1 \), we have \( \langle 1626, 1998 \rangle - \{1680, 1740, 1770, 1860, 1890, 1944, 1980\} \subseteq \text{PMD}(6) \). We need only to show that 1680, 1980 \( \in \text{PMD}(6) \), where the first one comes from Lemma 4.7. Write 1980 = 13.149 + 43 and apply Lemma 4.5 with \( t = 37 \), we have 1980 \( \in \text{PMD}(6) \). □

**Lemma 4.18.** 2052 \( \in \text{PMD}(6) \).

Proof. From [16] there is a PBD \( B(\{t, q + t\}, 1; t(q^2 + q + 1)) \) for a prime power \( q \) and \( 0 < t < q^2 - q + 1 \). Take \( q = 7 \) and \( t = 36 \) to obtain a PBD \( B(\{36, 43\}, 1; 2052) \). The conclusion follows from Theorem 2.5 with \( \lambda = \mu = 1 \). □

**Lemma 4.19.** \( \{2082, 2172\} \subseteq \text{PMD}(6) \).

Proof. Start with a \( T(38, 53) \) and give \( t \) points weight 9 and other points weight zero in one group, and give weight one to other points of the TD. Since we have an input \((37,6,1)\)-FPMD of type \( 1^{37} \) and an input \((46,6,1)\)-FPMD of type \( 1^{46} \), which is equivalent to a known \((46,9,6,1)\)-IPMD from Lemma 3.7. We obtain from Theorem 2.4, a FPMD of type \( 53^{37}(9t)^1 \). Adding 13 new points and filling in holes with a \((66,13,6,1)\)-IPMD and a \((9t + 13,6,1)\)-PMD, where \( t = 12, 22 \), we have \( \{2082, 2172\} \subseteq \text{PMD}(6) \). □

**Proposition 4.20.** \( \langle 2004, 2604 \rangle - \{2202, 2256, 2484, 2568, 2604\} \subseteq \text{PMD}(6) \).
Proof. Apply Lemma 4.5 with $t = 37$, we have $\langle 2004, 2418 \rangle - \{2052, 2082, 2172, 2202, 2256, 2292, 2352, 2388 \} \subseteq \text{PMD}(6)$.

Apply Lemma 4.5 with $t = 43$, we have $\langle 2292, 2604 \rangle - \{2304, 2364, 2394, 2484, 2514, 2568, 2604 \} \subseteq \text{PMD}(6)$. We need to show that $\{2052, 2082, 2172, 2514 \} \subseteq \text{PMD}(6)$.

Write $2514 = 7.317 + 295$ and $317 = 7.43 + 16$. A $T(6, 317 + 36) - T(6, 36)$ exists from Lemma 2.13 and a $(396, 79, 6, 1)$-IPMD from Lemma 3.7. Applying SIP construction produces a $(2514, 6, 1)$-PMD. The conclusion then follows from Lemmas 4.18 and 4.19. □

Combining the previous propositions in this section we have proved the following theorem.

**Theorem 4.21.** $\langle 12, 2604 \rangle - E_0 \subseteq \text{PMD}(6)$, where $E_0$ is shown in Table 2.

In what follows, we shall show that $\langle 2610, 49506 \rangle \subseteq \text{PMD}(6)$.

**Proposition 4.22.** $\langle 2610, 3198 \rangle \subseteq \text{PMD}(6)$.

Proof. Apply Lemma 4.5 with $t = 49$, we have the conclusion except for $v \in \{2616, 2676, 2706, 2796, 2826, 2880, 2916, 2976, 3012, 3054 \}$. Apply Lemma 4.5 with $t = 43$, we know that the first four integers are in PMD(6).

Apply Lemma 4.5 with $t = 7$, we have a $(438, 6, 1)$-PMD. In fact, the construction gives a $(438, 13, 6, 1)$-IPMD. Write $425 = 7.59 + 12$, we may apply Lemma 2.13 to obtain a $T(6, 425 + 11) - T(6, 11)$. Further apply SIP construction with $v = 7$ and $\lambda = 1$. Since a $(79, 6, 1)$-PMD exists, we know that $3054 = 7.425 + 79 \in \text{PMD}(6)$.

For the remaining five integers, we use DP construction with $377 + 91 = 13(29 + 7)$ to obtain a $(377 + 91, 91, 6, 1)$-IPMD. Write $377 = 7.53 + 6$

<table>
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<tr>
<th>Orders of $v \equiv 0 \pmod{6}$, where a $(v, 6, 1)$-PMD is unknown</th>
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<tbody>
<tr>
<td>12  18  24  30  42  48  54  60  72  78  84  90  96</td>
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<tr>
<td>102 108 114 120 132 138 144 150 156 162 168 174 180</td>
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<tr>
<td>192 198 204 222 228 234 240 258 264 270 276 282 288</td>
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<tr>
<td>294 300 306 312 318 324 330 342 348 354 360 372 378</td>
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<tr>
<td>474 480 492 498 504 510 516 522 534 546 552 564 570</td>
</tr>
<tr>
<td>576 582 588 594 600 606 612 618 624 630 642 648 654</td>
</tr>
<tr>
<td>660 672 678 690 744 804 834 864 870 888 894 906 912</td>
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<tr>
<td>918 924 930 942 948 954 960 972 978 984 1002 1014 1020</td>
</tr>
<tr>
<td>1032 1038 1044 1050 1056 1062 1068 1074 1098 1158 1188 1200 1206</td>
</tr>
<tr>
<td>1212 1218 1224 1230 1326 1326 1242 1248 1326 1326 1740 1770 1860 1890 1944</td>
</tr>
<tr>
<td>2202 2256 2484 2568 2604</td>
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</tbody>
</table>
and apply Lemma 2.15 with \( t = 53, m = 7, w_i, m_j \in \{0, 1\}, \) and \( m + w = 13, \) we obtain a \( T(6, 377 + a) - T(6, a) \) for \( 0 \leq a \leq 52. \) Applying SIP construction with \( v = 7 \) and \( \lambda = 1 \) produces a \( (7.377 + (91 + 6a), 6, 1)\)-PMD, where \( 91 + 6a = 187, 241, 277, 337, 373 \in PMD(6). \) The proof is complete. □

Proposition 4.23. \( \langle 3204, 3828 \rangle \subseteq PMD(6). \)

Proof. Filling in size 49 hole with a \((49, 7, 6, 1)\)-IPMD we obtain a \((239 + 7, 7, 6, 1)\)-IPMD from a \((197 + 49, 49, 6, 1)\)-IPMD. Since a \( T(14, 239) \) exists, applying Lemma 3.9 gives \( \langle 3204, 3828 \rangle - \{3222, 3252, 3342, 3372, 3426, 3462, 3522, 3558, 3600, 3756, 3774, 3804 \} \subseteq PMD(6). \)

Since \( 252 = 7.36 \in PMD(6) \) and a \( T(14, 251) \) exists, we may apply Lemma 3.9 to delete the last 10 numbers except for 3756, which can be solved by applying Lemma 4.5 with \( t = 67. \)

Since \( 468 = 13.36, \) the DP construction gives a \((468, 13, 6, 1)\)-IPMD. A \( T(7, 65) \) exists from [17], we may apply Lemma 2.12 to obtain a \( T(6, 455 + a) - T(6, a) \) for \( 0 \leq a \leq 65. \) The SIP construction gives a \((v, 6, 1)\)-PMD for \( v \in \langle 3198, 3276 \rangle - \{3240\}. \) This guarantees that \( 3222, 3252 \in PMD(6). \) The proof is complete. □

Proposition 4.24. \( \langle 3834, 4368 \rangle \subseteq PMD(6). \)

Proof. Apply Lemma 4.5 with \( t = 67, \) we have \( \langle 3834, 4368 \rangle - \{3852, 3912, 3948, 3990, 4146, 4164, 4194, 4296 \} \subseteq PMD(6). \)

Since \( 35 \leq a \leq 73 \) exists, we obtain a \((19.35, 6, 1)\)-IFPMD with type \((35, 7)\). Adding a new point to the IFPMD and construct a \((666, 133, 6, 1)\)-IPMD on each hole, we obtain a \((666, 133, 6, 1)\)-IPMD. Write \( 533 = 7.71 + 36, \) we have a \( T(6, 533 + a) - T(6, a) \) for \( 0 \leq a \leq 71. \) Applying SIP construction with \( v = 7, \lambda = 1 \) and \( a = 50 \) gives a \((4164, 6, 1)\)-PMD since \( 133 + 6.50 = 433 \subseteq PMD(6). \)

Apply Lemma 4.2 with \( u = 7 \) and \( t = 196, \) we have \( 3852 \in PMD(6). \)

Lemma 4.5 with \( t = 73 \) takes care of all the remaining cases. □

Proposition 4.25. \( \langle 4374, 4758 \rangle \subseteq PMD(6). \)

Proof. Apply Lemma 4.5 with \( t = 73, \) we have \( \langle 4368, 4758 \rangle - \{4458, 4476, 4506, 4608 \} \subseteq PMD(6). \)

Write \( 557 = 7.78 + 11, \) a \( T(6, 557 + a) - T(6, a) \) exists from Lemma 2.13, where \( 0 \leq a \leq 78. \) Using a \((696, 139, 6, 1)\)-IPMD from Lemma 3.7 and applying SIP construction gives \( \langle 4458, 4476, 4506 \rangle \subseteq PMD(6). \) Since \( 666 = 19.35 + 1 \) and a \((666, 19, 6, 1)\)-IPMD exists, we may apply SIP construction to obtain \( 4608 = 7.647 + 79, \) where a \( T(6, 647 + 10) - T(6, 10) \) comes from Lemma 2.13 with \( 647 = 7.88 + 31. \) □

Proposition 4.26. \( \langle 4734, 5280 \rangle \subseteq PMD(6). \)
Proof. Write $653 = 7.91 + 16$, a $T(6, 653 + a) - T(6, a)$ exists from Lemma 2.13, where $0 \leq a \leq 91$. A $(816, 163, 6, 1)$-IPMD exists from Lemma 3.7. We apply SIP construction with $v = 7$ and $\lambda = 1$ to obtain $\langle 4734, 5280 \rangle - \{4806, 4836, 4890, 4926, 4986, 5022, 5064, 5220, 5238, 5268 \} \subseteq \text{PMD}(6)$.

Write $677 = 7.88 + 61$, a $T(6, 677 + a) - T(6, a)$ exists from Lemma 2.13, where $0 \leq a \leq 88$. A $(846, 169, 6, 1)$-IPMD exists from Lemma 3.7. Applying SIP construction solves the last seven cases.

Write $384 = 7.53 + 13$, the SDP construction gives a $(384, 13, 6, 1)$-IPMD. Since a $T(14, 371)$ exists from [17], we may apply Lemma 3.9 with $w = 13$ to obtain $\{4836, 4890 \} \subseteq \text{PMD}(6)$.

In a $T(38, 125)$, give weight 9 to $t$ points and zero to other points in a group and give weight one to other points of the TD. Adding a new point to the resultant FPMD of type $125^{37}(9t)$ we obtain $\{4806, 4836, 4890, 4926, 4986, 5022, 5064, 5220, 5238, 5268 \} \subseteq \text{PMD}(6)$.

Proposition 4.27. $\langle 5286, 6318 \rangle \subseteq \text{PMD}(6)$.

Proof. Apply Lemma 4.5 with $t = 97$, we have $\langle 5286, 6318 \rangle - \{5292, 5322, 5376, 5412, 5472, 5508, 5550, 5706, 5724, 5754, 5856 \} \subseteq \text{PMD}(6)$.

Since $402 \in \text{PMD}(6)$ and a $T(14, 40)$ exists, by Lemma 3.9 we know that all the remaining integers except 5706 are in PMD(6).

The same construction from 408 $\in \text{PMD}(6)$ gives 5706 $\in \text{PMD}(6)$. □

Lemma 4.28. Let $u$ and $t$ be integers, $0 \leq t \leq u - 1$. Suppose $\{u, 3t + 1\} \subseteq \text{PMD}(6)$ and a $T(14, u - 1)$ exists, then there exists a $(13(u - 1) + 3t + 1, 6, 1)$-PMD.

Proof. Take $w = 1$ in Lemma 3.9. □

Proposition 4.29. $\langle 6324, 8622 \rangle \subseteq \text{PMD}(6)$.

Proof. Take $u = 456, 462, 468, 528, 540$ in Lemma 4.28. The only number left is 7500. Write 1013 = 7.143 + 12, a $T(6, 1013 + 26) - T(6, 26)$ then follows from Lemma 2.13. Since a $(1266, 253, 6, 1)$-IPMD exists from Lemma 3.7, we may apply SIP construction to obtain $7500 = 7.1013 + 409 \in \text{PMD}(6)$, where 409 = 253 + 6.26 $\in \text{PMD}(6)$. □

Proposition 4.30. $\langle 8628, 10926 \rangle \subseteq \text{PMD}(6)$.

Proof. Take $u = 666$ and 684 in Lemma 4.28. The only numbers left are 8628, 8634, 8640, 8700, 8760, 8790 and 9294. Write 1181 = 7.167 + 12, we have a $T(6, 1181 + a) - T(6, a)$ for $0 \leq a \leq 167$ by Lemma 2.13. Since a $(1476, 295, 6, 1)$-IPMD exists from Lemma 3.7, we may apply SIP construction to obtain $7.1181 + (295 + 6a) \in \text{PMD}(6)$, where $a \in \{11, 12, 13, 23, 38, 122\}$. This takes care of all these numbers but 8760.
Write $1109 = 7.154 + 31$ and $8760 = 7.1109 + (277 + 6.120)$. Since a $(1386,277,6,1)$-IPMD exists from Lemma 3.7 and $997 \in \text{PMD}(6)$, a similar construction gives $8760 \in \text{PMD}(6)$. \hfill $\square$

**Proposition 4.31.** $\langle10926,49512\rangle \subseteq \text{PMD}(6)$.

**Proof.** Apply Lemma 4.28 with $3t + 1 \geq 1321$, we have $(a, b) \subseteq \text{PMD}(6)$, where $a = 13(u - 1) + 1321$ and $b = 13(u - 1) + 3(u - 2) + 1$. The parameters $u$, $a$ and $b$ are shown in Table 3. A $T(14, u - 1)$ exists from [17]. The conclusion then follows. \hfill $\square$

Summarizing the results in this section we have the following theorem.

**Theorem 4.32.** Suppose $v \equiv 0 \pmod{6}$. If $v \geq 2610$, then there exists a $(v, 6, 1)$-PMD. Below this value, a $(6,6,1)$-PMD does not exist and there are at most 135 integers $v$ shown in Table 2 for which the existence of a $(v, 6, 1)$-PMD is undecided.

**Proof.** For $v < 2610$, see Theorem 4.21. For $v \geq 2610$, see Propositions 4.22–4.27, 4.29–4.31 and 4.3. \hfill $\square$

### 5. Concluding remarks

We have surveyed the recent existence results on $(v,k,\lambda)$-PMDs in Section 1 and described various constructions in Section 2. For $k = 6$ the basic cases are $\lambda = 1$ and 3. Since the case $\lambda = 3$ has been studied already, we have discussed the case $\lambda = 1$ in this paper. Combining the results in Sections 3 and 4 we have the following theorem.

**Theorem 5.1.** Suppose $v \equiv 0$ or $1 \pmod{6}$. If $v \geq 2605$, there exists a $(v, 6, 1)$-PMD. Below this value, a $(6,6,1)$-PMD does not exist and there are at most 150 integers

Table 3

<table>
<thead>
<tr>
<th>$u$</th>
<th>$a = 13u + 1308$</th>
<th>$b = 16u - 18$</th>
<th>$u$</th>
<th>$a = 13u + 1308$</th>
<th>$b = 16u - 18$</th>
</tr>
</thead>
<tbody>
<tr>
<td>702</td>
<td>10434</td>
<td>11214</td>
<td>756</td>
<td>11136</td>
<td>12078</td>
</tr>
<tr>
<td>810</td>
<td>11838</td>
<td>12942</td>
<td>876</td>
<td>12696</td>
<td>13998</td>
</tr>
<tr>
<td>966</td>
<td>13866</td>
<td>15438</td>
<td>1080</td>
<td>15348</td>
<td>17262</td>
</tr>
<tr>
<td>1194</td>
<td>16830</td>
<td>19086</td>
<td>1332</td>
<td>18624</td>
<td>21294</td>
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<tr>
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<td>21276</td>
<td>24558</td>
<td>1782</td>
<td>24474</td>
<td>29664</td>
</tr>
<tr>
<td>2178</td>
<td>29622</td>
<td>34830</td>
<td>2574</td>
<td>34770</td>
<td>41166</td>
</tr>
<tr>
<td>3060</td>
<td>41088</td>
<td>48942</td>
<td>3660</td>
<td>48888</td>
<td>58547</td>
</tr>
</tbody>
</table>
\( v \in E \cup E_0 \), where \( E \) and \( E_0 \) are shown in Section 3 and Table 2, respectively, for which the existence of a \((v, 6, 1)\)-PMD is undecided.

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References