Model Validation of Multirate Systems From Time-Domain Experimental Data
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Abstract—The model validation problem using time-domain experimental data is studied for multirate linear fractional uncertain models in this note. As a technical tool, the Carathéodory–Fejér (CF) interpolation problem with a nest operator constraint is first investigated. This problem is itself of interest mathematically and has potential applications in addressing other problems in control, signal processing, and circuit theory. A necessary and sufficient solvability condition for this interpolation problem is given. Then the validation tests are presented based on this condition and the lifting technique. Tractable convex optimization methods can be used to solve the validation problems.

Index Terms—Carathéodory–Fejér (CF) interpolation, model validation, multirate systems.

I. INTRODUCTION

Multirate systems, i.e., digital systems with signals having different sampling rates, have wide applications in control, communication, signal processing, econometrics and numerical mathematics. There are several reasons for this.

- In large scale multivariable digital systems, often it is unrealistic, or sometimes impossible, to sample all physical signals uniformly at one single rate. In such situations, one is forced to use multirate sampling.
- Multirate systems can often achieve objectives that cannot be achieved by single rate systems [1], [2].

The study of multirate systems goes back to late 1950s [3]. A renaissance of research in multirate systems has occurred since 1980 in control community, signal processing community and communication community. The driving force for studying multirate systems in signal processing comes from the need of sampling rate conversion, subband coding, and their ability to generate wavelets. Multirate signal processing is now one of the most vibrant areas of research in signal processing, see recent book [2] and references therein. In control community, two groups of research stand out: using multirate control to achieve what single rate control cannot as well as the limitation of doing this [1] and the optimal design of multirate controllers [4], [5]. In communication community, multirate sampling is used for blind system identification and equalization [6]. We also notice the cross discipline fertilization between signal processing and control in using $\mathcal{H}_\infty$ optimization to design filter banks [7], [8].

In this note, we will study the control-oriented model validation problems pertaining to the general multirate systems. There has recently been considerable research devoted to robust or control-oriented model validation [9], [10]. However, the research on model validation of multirate systems is almost nonexisting. Due to the wide applications of multirate systems, their model validation problem should receive comparable attention to those of single rate systems.

Model validation is a very important step in the process of control system modeling both in traditional stochastic setting and nonprobabilistic context. Generally, the model validation problem is to examine if the sets of experimental data are consistent with the model of the plant. A model is said to be invalidated when the validation test fails. Our confidence in the model set is increased if the model is consistent with the data. Since a model is either invalidated or not invalidated, it is actually more accurate to call the validation procedure as model invalidation.

More recently, motivated by the considerable research on control-oriented system identification, much attention has been paid on validation of uncertain models consisting of a nominal model and a norm bounded modeling uncertainty [11], [12], [10]. Such uncertainty models are the starting point for robust control. The first study of model validation for linear fractional transformation (LFT) model-sets was carried out by Smith and Doyle [10]. They show that the problem can be solved by a structured singular value type method. Chen [11] considered the general validation problems of linear fractional uncertain models in frequency domain and reduced it to the Nevanlinna-Pick interpolation problem, which can be solved by standard convex optimization methods. Based on the Carathéodory–Fejér (CF) interpolation problem, a purely time-domain formulation for models with an additive uncertainty is presented in [12]. It is shown that the problem can be solved as a convex program involving linear matrix inequalities (LMI). The time domain validation approach in a more general setup which is for LFT uncertain model-sets is studied in [13]. The similar setup is also used to consider the validation problem in a sampled-data framework [14], [15].

In this note, we extend the results in [13] and [12] to multirate systems. The setup is shown in Fig. 1, where $P_{mr}$ and $\Delta_{mr}$ are both multirate systems, and they together form a multirate uncertain system model with $P_{mr}$ fixed and $\Delta_{mr}$ unknown. The model validation problem considered in this note is as follows. Given $P_{nr}$, an uncertainty set which $\Delta_{mr}$ belongs to, a set of time domain experimental data on $u_i$ and $y_i$, and a set $\mathcal{E}$ of noise signals, find out if there exists a $\Delta_{mr}$ in the uncertainty set such that the experimental data can be reproduced with $P_{nr}$ and $\Delta_{mr}$ together with the noises $\mathcal{E}$. As a technical tool, we first propose and study a CF interpolation problem with a nest operator constraint. This problem is itself of interest mathematically and has potential applications in addressing other problems in control, signal processing, and circuit theory [16]. A necessary and sufficient solvability condition for this constrained CF interpolation problem is given. Then the validation tests are presented based on the above condition.

The note is organized as follows. The next section introduces some basic facts about the general multirate systems and shows how to convert a multirate system to an equivalent LTI system with a causality constraint. Section III addresses the tangential CF interpolation problem with nest operator constraint, which are the main tool to obtain the model validation test for general multirate systems.
II. GENERAL MULTIRATE SYSTEMS

The setup of a general MIMO multirate system is shown in Fig. 2. Here \( u_i \), \( i = 1, 2, \ldots, p \), are input signals whose sampling intervals are \( m_i h \), respectively, and \( y_j \), \( j = 1, 2, \ldots, q \) are output signals whose sampling intervals are \( n_j h \), respectively, where \( h \) is a real number called base sampling interval and \( m_i, n_j \) are natural numbers (positive integers). Such systems can result from discretizing continuous time systems using samplers of different rates or they can be found in their own right. We will assume that all signals in the system are synchronized at time 0, i.e., the time 0 instances of all signals occur at the same time. In this note, we will focus on those multirate systems that satisfy certain causal, linear, shift invariance properties which are to be defined below.

Since we need to deal with signals with different rates, it is more convenient and clearer to associate each signal explicitly with its sampling interval. Let \( \ell'(\tau) \) denote the space of \( \mathbb{R}^n \) valued sequences

\[
\ell'(\tau) = \{ \ldots, x(-\tau), x(0), x(\tau), \ldots \} : x(k\tau) \in \mathbb{R}^n \}
\]

The system in Fig. 2 is a map from \( \otimes_{i=1}^p \ell(m_i h) \otimes_{j=1}^q \ell(n_j h) \) to \( \otimes_{i=1}^p \ell(m_i h) \otimes_{j=1}^q \ell(n_j h) \). It is said to be linear if this map is a linear map.

Let \( l_i \in \mathbb{N} \) be a multiple of \( m_i \), and \( n_j, i = 1, 2, \ldots, p, j = 1, 2, \ldots, q \). Let \( \tilde{m}_i = l_i/m_i \) and \( \tilde{n}_j = l_i/n_j \). Denote the sets \( \{m_i\} \) and \( \{n_j\} \) by \( \tilde{M} \) and \( \tilde{N} \), respectively, and the sets \( \{\tilde{m}_i\} \) and \( \{\tilde{n}_j\} \) by \( \tilde{M} \) and \( \tilde{N} \) respectively. Let \( S : \ell'(\tau) \to \ell'(\tau) \) be the forward shift operator, i.e.,

\[
S \{x(-\tau), x(0), x(\tau), \ldots\} = \{x(-2\tau), x(-\tau), x(0), x(\tau), \ldots\}
\]

Define

\[
S_{\tilde{M}} = \text{diag} \{ S_{\tilde{m}_1}, \ldots, S_{\tilde{m}_p} \}
S_{\tilde{N}} = \text{diag} \{ S_{\tilde{n}_1}, \ldots, S_{\tilde{n}_q} \}
\]

Then the multirate system in Fig. 2 is said to be \( (\tilde{M}, \tilde{N}) \)-shift invariant or \( h \) periodic in real time if \( F_{\text{mult}} S_{\tilde{M}} \mathbb{X} = S_{\tilde{N}} F_{\text{mult}} \mathbb{X} \). Now let \( P : \ell'(\tau) \to \ell'(\tau) \) be the truncation operator, i.e.,

\[
P \{x((-k+1)\tau), x(k\tau), x((k+1)\tau), \ldots\} = \{x((k-1)\tau), x(k\tau), 0, \ldots\}
\]

if \( k\tau \leq t < (k+1)\tau \). Extend this definition to spaces \( \oplus_{\tilde{m}_i} \ell(m_i h) \) and \( \oplus_{\tilde{n}_j} \ell(n_j h) \) in an obvious way. Then the multirate system is said to be causal if

\[
P_{\text{mult}} u = P_{\text{mult}} v \Rightarrow P_{\text{mult}} F_{\text{mult}} u = P_{\text{mult}} F_{\text{mult}} v
\]

for all \( t \in \mathbb{R} \). In this note, we will concentrate on causal linear \( (\tilde{M}, \tilde{N}) \)-shift invariant systems. Such general multirate system covers many familiar classes of systems as special cases. If \( m_i, n_j \) are all the same, then this is an LTI single rate system. If \( m_i, n_j \) are all the same but \( l \) is a multiple of them, then it is a single rate \( l \)-periodic system [17]. If \( p = q = 1 \), this becomes the SISO dual rate system studied in [7]. If \( m_i \) are the same and \( n_j \) are the same, then this becomes the MIMO dual rate system studied in [18]. For systems resulted from discretizing LTI continuous time systems using multirate sample and hold schemes in [4], [5], \( l \) turns out to be the least common multiple of \( m_i \) and \( n_j \). The study of multirate systems in such a generality as indicated above, however, has never been done before.

A standard way for the analysis of such systems is to use lifting or blocking. Define a lifting operator \( L_r : \ell(\tau) \to \ell'(\tau) \) by the equation shown at the bottom of the page, and let

\[
L_{\tilde{M}} = \text{diag} \{ L_{\tilde{m}_1}, \ldots, L_{\tilde{m}_p} \}
L_{\tilde{N}} = \text{diag} \{ L_{\tilde{n}_1}, \ldots, L_{\tilde{n}_q} \}
\]

Then, the lifted system \( F = L_{\tilde{M}} F_{\text{mult}} L_{\tilde{N}}^{-1} \) is an LTI system in the sense that \( F \mathbb{S} = SF \). Hence, it has transfer function \( \tilde{F} \) in \( \lambda \)-transform. However, \( F \) is not an arbitrary LTI system, instead its direct feedthrough term \( \tilde{F}(0) \) is subject to a constraint that is resulted from the causality of \( F_{\text{mult}} \). This constraint is best described using the language of nests and nest operators [18], [5].

Let \( \mathcal{X} \) be a finite-dimensional vector space. A nest in \( \mathcal{X} \), denoted \( \{x_i\} \), is a chain of subspaces in \( \mathcal{X} \), including \( \{0\} \) and \( \mathcal{X} \), with the nonincreasing ordering

\[
\mathcal{X} = \{0\} \supseteq \mathcal{X}_1 \supseteq \cdots \supseteq \mathcal{X}_{l-1} \supseteq \mathcal{X}_l = \{0\}\]

Let \( \mathcal{U}, \mathcal{Y} \) be finite dimensional vector spaces. Denote by \( \mathcal{L}(\mathcal{U}, \mathcal{Y}) \) the set of linear operators \( \mathcal{U} \to \mathcal{Y} \). Assume that \( \mathcal{U} \) and \( \mathcal{Y} \) are equipped, respectively, with nest \( \{U_k\} \) and \( \{Y_k\} \) which have the same number of subspaces, say, \( l + 1 \) as above. A linear map \( T \in \mathcal{L}(\mathcal{U}, \mathcal{Y}) \) is said to be a nest operator if

\[
T U_k \subseteq Y_k, \quad k = 0, 1, \ldots, l
\]

Let \( \Pi_{U_k} : \mathcal{U} \to U_k \) and \( \Pi_{Y_k} : \mathcal{Y} \to Y_k \) be orthogonal projections. Then, (1) is equivalent to

\[
(I - \Pi_{Y_k}) T \Pi_{U_k} = 0, \quad k = 0, \ldots, l - 1
\]

The set of all nest operators (with given nests) is denoted \( \mathcal{N}(\{U_k\}, \{Y_k\}) \). If we decompose the spaces \( \mathcal{U} \) and \( \mathcal{Y} \) in the following way:

\[
\mathcal{U} = (U_0 \oplus U_1) \oplus (U_1 \oplus U_2) \oplus \cdots \oplus (U_{l-1} \oplus U_l)
\]

\[
\mathcal{Y} = (Y_0 \oplus Y_1) \oplus (Y_1 \oplus Y_2) \oplus \cdots \oplus (Y_{l-1} \oplus Y_l)
\]

Define

\[
L_r \{x(0), x(r\tau), \ldots\} \to \left\{ \begin{bmatrix} x(0) \\ \vdots \\ x((r-1)\tau) \\ x((2r-1)\tau) \end{bmatrix}, \begin{bmatrix} x(r\tau) \\ \vdots \\ x((2r-1)\tau) \end{bmatrix}, \ldots \right\}
\]
then a nest operator \( T \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\}) \) has the following block lower triangular form:

\[
T = \begin{bmatrix}
T_{11} & 0 & \cdots & 0 \\
T_{21} & T_{22} & \ddots & \vdots \\
\vdots & \vdots & \ddots & 0 \\
T_{l1} & T_{l2} & \cdots & T_{ll}
\end{bmatrix}.
\]  

(5)

Denote \( u = L_{\mathcal{Y}} u \), and \( y = L_{\mathcal{Y}} y \). Then

\[
\begin{aligned}
\underline{u}(0) &= [u_1(0) \cdots u_l(m_1 h) \cdots u_l(m_1 h)(\bar{m}_1 - 1)h] \ldots \\
u_l(0) \\
y_l(0) &= [y_l(0) \cdots y_l(m_1 h) \cdots y_l(m_1 h)(\bar{m}_1 - 1)h] \ldots \\
y_l(0)
\end{aligned}
\]

Define for \( k = 0, 1, \ldots, l \)

\[
\mathcal{U}_k = \{ u(0) : u_j(\bar{m}_j h) = 0 \text{ if } \bar{m}_j h < k h \}
\]

\[
\mathcal{Y}_k = \{ y(0) : y_j(\bar{m}_j h) = 0 \text{ if } \bar{m}_j h < k h \}.
\]

Here, \( \mathcal{U}_k \) and \( \mathcal{Y}_k \) represent respectively the input and output signals at or after time \( k h \) in the first period. Due to the causality of \( F_{mn} \), the direct through term of the lifted plant must satisfy

\[
\hat{F}(0) \mathcal{U}_k \subseteq \mathcal{Y}_k, \quad k = 0, 1, \ldots, l
\]

or, equivalently

\[
\hat{F}(0) \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\}).
\]  

(6)

Now we see that each multirate system has an equivalent single rate LTI system satisfying a causality constraint. This causality constraint is characterized by a nest operator constraint as in (6) on its transfer function.

We end this section by showing an example. Consider the system shown in Fig. 2. Let \( p = q = 2, m_1 = 2, m_2 = 6, n_1 = 4, n_2 = 3 \) and \( l = 12 \). Then \( \bar{m}_1 = 6, \bar{m}_2 = 2, \bar{n}_1 = 3 \) and \( \bar{n}_2 = 4 \). Let \( u \) and \( y \) be the lifted signals of \( u \) and \( y \) respectively. Then we have the equation shown at the bottom of the page. Denote the \( i \)th column of \( 8 \times 8 \) identity matrix by \( e_i \). Then

\[
\begin{align*}
\mathcal{U}_{l2} &= \mathcal{U}_{l1} = \{ 0 \} \\
\mathcal{U}_{l0} &= \mathcal{U}_0 = \text{span} \{ e_0 \} \\
\mathcal{U}_k &= \mathcal{U}_k = \text{span} \{ e_5, e_6 \} \\
\mathcal{U}_6 &= \mathcal{U}_6 = \text{span} \{ e_4, e_5, e_6, e_8 \} \\
\mathcal{U}_4 &= \mathcal{U}_4 = \text{span} \{ e_3, e_4, e_5, e_6, e_8 \} \\
\mathcal{U}_2 &= \mathcal{U}_1 = \text{span} \{ e_2, e_3, e_4, e_5, e_6, e_8 \} \\
\mathcal{U}_0 &= \mathbb{R}^8.
\end{align*}
\]

The nests \( \{\mathcal{Y}_k\} \) can be defined in a similar way. Then, \( \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\}) \) consists of matrices of the form

\[
\begin{bmatrix}
+ & 0 & 0 & 0 & 0 & + & 0 \\
+ & + & 0 & 0 & 0 & + & 0 \\
+ & + & + & 0 & 0 & + & 0 \\
+ & 0 & 0 & 0 & 0 & + & 0 \\
+ & + & 0 & 0 & 0 & + & 0 \\
+ & + & + & 0 & 0 & + & 0 \\
+ & + & + & + & 0 & + & 0
\end{bmatrix}
\]

where “+” represents an arbitrary number. Note that such matrices are not block lower triangular, but can be turned into block lower triangular matrices by permutations of rows and columns.

### III. MATHEMATICAL PREPARATIONS

Denote \( \mathcal{H}_\infty(\mathcal{U}, \mathcal{Y}) \) the Hardy class of all uniformly bounded analytic functions on \( \mathcal{D} \) with values in \( \mathcal{L}(\mathcal{U}, \mathcal{Y}) \), where \( \mathcal{D} \) denotes the open unit disc. Denote by \( \mathcal{H}_\infty(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\}) \) the set of functions \( G \in \mathcal{H}_\infty(\mathcal{U}, \mathcal{Y}) \) satisfying \( G(0) \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\}) \). The purpose of this section is to address the CF interpolation problem using functions in \( \mathcal{H}_\infty(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\}) \). Before going into this problem, we need to state a result on matrix positive completion.

#### A. Matrix-Positive Completion

The matrix-positive completion problem is as follows [19]: Given \( B_{ij}, |j - i| \leq q \), satisfying \( B_{ij} = B_{ji} \), find the remaining matrices \( B_{ij}, |j - i| > q \), such that the block matrix \( B = [B_{ij}]_{i,j=1}^{n} \) is positive definite. The matrix-positive problem was first proposed by Dym and Gohberg [19], who gave the following result.

**Lemma 1**: The matrix positive completion problem has a solution iff

\[
\begin{bmatrix}
B_{ii} & \cdots & B_{i,i+q} \\
\vdots & \ddots & \vdots \\
P_{i+q,i} & \cdots & B_{i+q,i+q}
\end{bmatrix} \succeq 0, \quad i = 1, \ldots, n-q.
\]  

(7)

Reference [20] gave a detailed discussion of such problem and presented an explicit description of the set of all solutions via a linear fractional map of which the coefficients are directly given in terms of the original data. However, Lemma 1 is enough for us.

#### B. CF Interpolation With Nest Operator Constraint

Let \( \mathcal{X}, \mathcal{U} \) and \( \mathcal{Y} \) be finite dimensional Hilbert spaces. The Hilbert space direct sum of \( n \) copies of \( \mathcal{X} \) will be denoted by \( \mathcal{X}^n \). Assume that \( \mathcal{U} \) and \( \mathcal{Y} \) are equipped respectively with nests \( \{\mathcal{U}_k\} \) and \( \{\mathcal{Y}_k\} \). Let \( U_i \) and \( Y_i, i = 0, 1, \ldots, n, \) be linear operators from \( \mathcal{X} \) to \( \mathcal{U} \) and from \( \mathcal{X} \) to \( \mathcal{Y} \) respectively. Denote

\[
U = \begin{bmatrix} U_0 \\ \vdots \\ U_n \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} Y_0 \\ \vdots \\ Y_n \end{bmatrix}.
\]  

(8)

\[
\begin{bmatrix}
u(0) \\
v(0)
\end{bmatrix} = \begin{bmatrix} u_1(0) & u_1(2h) & u_1(4h) & u_1(6h) & u_1(8h) & u_1(10h) & u_2(0) & u_2(2(6h)) \end{bmatrix}^T
\]

\[
\begin{bmatrix}
y(0) \\
y(0)
\end{bmatrix} = \begin{bmatrix} y_1(0) & y_1(4h) & y_1(8h) & y_2(0) & y_2(3h) & y_2(6h) & y_2(9h) \end{bmatrix}^T.
\]
The Toeplitz matrix generated by \( U \) is defined as

\[
T_U := \begin{bmatrix}
U_0 & 0 & \cdots & 0 \\
U_1 & U_0 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
U_n & U_{n-1} & \cdots & U_0
\end{bmatrix}.
\] (9)

The Toeplitz matrix \( T_Y \) generated by \( Y \) is defined in a similar way. The tangential CF interpolation problem with constraint \( \mathcal{N}(\{U_k\}, \{Y_k\}) \) for the data \( U_Y \) is to find (if possible) a function \( \tilde{G}(\lambda) = \sum_{i=0}^{\infty} G_i \lambda^i \) in \( \mathcal{H}_\infty(\{U_k\}, \{Y_k\}) \) such that \( \|\tilde{G}\|_\infty < 1 \) and

\[
Y = T_G U,
\]

where \( T_G \) is the Toeplitz matrix generated by \( \{G_0 \cdots G_n\} \). Note that the dimension of \( T_G \) depends on an integer \( n \), to simplify the notation, however, we choose to ignore this dependence. In fact, this does not cause any confusion if we always assume that all the matrix operations are compatible.

**Theorem 1:** There exists a solution to the CF interpolation problem with constraint \( \mathcal{N}(\{U_k\}, \{Y_k\}) \) for the data \( U, Y \) if and only if

\[
T_U^* \Pi_{\nu=\nu_k} U_T U - T_Y^* \Pi_{\nu=\nu_y} Y_T Y \geq 0
\] (10)

for all \( k = 1, \ldots, l \).

**Proof:** The nest operator constraint on the interpolation function \( \tilde{G} \) can be considered as an additional interpolation condition

\[
T_G \begin{bmatrix}
0 \\
\vdots \\
0 \\
I
\end{bmatrix} = \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}
\]

for some \( T \in \mathcal{N}(\{U_k\}, \{Y_k\}) \). By the solvability condition of the standard CF interpolation problem [21], the constrained CF interpolation problem has a solution iff

\[
\tilde{U}^* \tilde{U} - \tilde{Y}^* \tilde{Y} \geq 0
\] (11)

where

\[
\tilde{U} = \begin{bmatrix}
0 & U_0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & U_{n-1} & 0 & \cdots & 0 & U_0 \\
0 & 0 & U_n & \cdots & 0 & U_0 \\
0 & Y_0 & 0 & \cdots & 0 & 0 \\
0 & Y_{n-1} & 0 & \cdots & 0 & 0 \\
T & Y_n & 0 & \cdots & 0
\end{bmatrix}
\]

and

\[
\tilde{Y} = \begin{bmatrix}
0 & Y_0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & Y_{n-2} & \cdots & 0 \\
T & Y_n & Y_{n-2} & \cdots & 0
\end{bmatrix}
\]

Notice that the submatrices of \( \tilde{U} \) and \( \tilde{Y} \) formed by removing the first block column are block Toeplitz matrices and are equal to \( T_U \) and \( T_Y \) respectively. It follows from Schur complement that (12) is equivalent to:

\[
\begin{bmatrix}
I & U_0 & \cdots & 0 & 0 \\
T_U & T_Y & \vdots & \vdots & \vdots \\
\Pi_{\nu=\nu_k} U_k & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
T & Y & I
\end{bmatrix} \geq 0
\] (13)

for some \( T \in \mathcal{N}(\{U_k\}, \{Y_k\}) \). If we decompose the spaces \( \mathcal{H} \) and \( \mathcal{Y} \) as in (3)-(4), then a nest operator \( T \in \mathcal{N}(\{U_k\}, \{Y_k\}) \) has a block lower triangular form shown in (5). Therefore, the constrained CF interpolation problem has a solution if and only if (13) holds for a block lower triangular matrix \( T \). This is a matrix-positive completion problem. By Lemma 1, such a \( T \) exists iff

\[
\begin{bmatrix}
I & \Pi_{\nu=\nu_k} U_k & \cdots & \Pi_{\nu=\nu_k} U_0 & 0 \\
\Pi_{\nu=\nu_y} Y_k & T_u T_y & \vdots & \vdots & \vdots \\
\Pi_{\nu=\nu_y} Y_k & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\Pi_{\nu=\nu_y} Y_k & T_y & I
\end{bmatrix} \geq 0
\] (14)

for \( k = 0, 1, \ldots, l \). Here, \( \Pi_{\nu=\nu_k} \) and \( \Pi_{\nu=\nu_y} \) are operators from \( \mathcal{H} \) to \( \mathcal{U}_k \) and from \( \mathcal{Y}_k \) to \( \mathcal{Y}_k \) respectively. Using Schur complement twice, we see that (14) is equivalent to (10) for \( k = 0, 1, \ldots, l \). Finally, notice that (10) when \( k = 0 \) is implied by (10) when \( k = l \). This completes the proof.

The solvability condition for the standard CF interpolation problem without constraint is recovered when \( l = 1 \).

**IV. TIME-DOMAIN VALIDATION FOR MULTIRATE LFT UNCERTAIN MODEL**

In robust control theory, many problems can be treated in a unified framework using LFT machinery. In fact, additive, multiplicative and coprime factor uncertainty descriptions can all be represented as an LFT on the uncertainty, with a suitable choice of the coefficient matrix [22]. In this section, we will give the validation tests for multirate LFT uncertain models.

Suppose we have an uncertain multirate system shown in Fig. 1. Here, \( u_i, i = 1, \ldots, p \), are input signals whose sampling intervals are \( m_i \) and \( y_j, j = 1, \ldots, q \), are output signals whose sampling intervals are \( n_j \). Also, \( v, i = 1, \ldots, r \), and \( w, j = 1, \ldots, s \), are auxiliary signals whose sampling intervals are \( m_i \) and \( n_j \) respectively.

Assume that both \( P_{\text{ad}} \) and \( \Delta_{\text{ad}} \) are \( lb \) periodic in real time for some integer \( l \) as discussed in Section II. We can then convert the above multirate LFT uncertain system to an equivalent single rate LTI system with a causality constraint. Let \( \tilde{m}_i = l / m_i, \tilde{n}_j = l / n_j, \tilde{m}_i = l / m_i, \tilde{n}_j = l / n_j \). Also, let \( \tilde{y} = L_{\tilde{m}} u, \tilde{w} = L_{\tilde{n}} \tilde{w}, \tilde{w} = L_{\tilde{m}} v, \) and

\[
P = \begin{bmatrix}
L_{\tilde{m}} & 0 & \cdots & 0 \\
0 & L_{\tilde{m}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & L_{\tilde{m}}
\end{bmatrix}
\]

and

\[
\Delta = L_{\tilde{m}} \Delta_{\text{ad}} L_{\tilde{m}}^{-1}
\]
The model validation problem is to test whether the uncertain model is consistent with the experiments data, i.e., whether there exists a \( \Delta \in \mathcal{H}_\infty(\mathcal{N}(\{W_k\}, \{V_k\})) \) with \( \|\Delta\|_\infty \leq \gamma \) such that the following holds:

\[
\begin{align*}
Y &= T_{\gamma 11} U + T_{\gamma 21} V + E & (15) \\
W &= T_{\gamma 21} U + T_{\gamma 22} V & (16) \\
\Gamma &= T_{\Delta} W & (17)
\end{align*}
\]

for some \( F \in \mathcal{E} \), where \( \mathcal{E} \) is a compact convex set representing a bound on the error and

\[
\Gamma := \begin{bmatrix}
V_0 \\
\vdots \\
V_n
\end{bmatrix}, \quad W := \begin{bmatrix}
W_0 \\
\vdots \\
W_n
\end{bmatrix}.
\]

**Theorem 2**

For data \( U \) and \( Y \), define

\[
\Omega \in \{ Y : T_{\gamma 11} U + T_{\gamma 21} V + T_{\gamma 22} W \}, \quad E \in \mathcal{E} \}.
\]

The uncertain model (15)-(17) is not invalidated if and only if there exists a \( U \in \Omega \) such that \( H_k(Y) \geq 0 \) for \( k = 1, \ldots, l \), where

\[
H_k(Y) = \begin{bmatrix}
H_{k11}(Y) & H_{k12}(Y) \\
H_{k21}(Y) & I
\end{bmatrix}
\]

\[
H_{k11}(Y) = (T_{\gamma 21} T_{\gamma 11})^T (\Pi_{\gamma V_k V_k} T_{\gamma 11}) + (T_{\gamma 21} T_{\gamma 11})^T (\Pi_{\gamma W_k W_k}) (T_{\gamma 22} T_{\gamma 11})
\]

Define for \( k = 0, 1, \ldots, l \),

\[
\begin{align*}
V_k &= \{ u(0) : v_i(r_m h) = 0 \text{ if } r_m h < k \} \\
W_k &= \{ w(0) : w_i(r_n h) = 0 \text{ if } r_n h < k \} \\
U_k &= \{ u(0) : u_i(r_m h) = 0 \text{ if } r_m h < k \} \\
Y_k &= \{ y(0) : y_i(r_n h) = 0 \text{ if } r_n h < k \}.
\end{align*}
\]

Then \( \hat{P}(0) \in \mathcal{N}(\{U_k \cup V_k\}, \{Y_k \cup W_k\}) \) and \( \Delta \) satisfies \( \Delta \in \mathcal{N}(\{V_k\}, \{W_k\}) \). From now on, we will only consider the equivalent LTI system shown in Fig. 3 with such constraints.

Assume that an uncertain model of the lifted LTI equivalence of a multirate system is represented by the lower LFT \( F_l(\hat{P}, \hat{\Delta}) \), where the nominal model \( \hat{P} \in \mathcal{H}_\infty \) is given and satisfies \( \hat{P}_{22} \in \mathcal{H}_\infty(\{V_k\}, \{V_k\}) \) and \( \|\hat{P}_{22}\|_\infty \leq \frac{1}{\gamma} \), and \( \Delta \) is the uncertainty satisfying \( \|\Delta\|_\infty \leq \gamma \). Several time domain experiments are carried out so that several input–output pairs of the lifted system are collected

\[
\begin{align*}
U := \begin{bmatrix}
U_0 \\
\vdots \\
U_n
\end{bmatrix}, \quad Y := \begin{bmatrix}
Y_0 \\
\vdots \\
Y_n
\end{bmatrix}.
\end{align*}
\]

The model validation problem is to test whether the uncertain model is consistent with the experiments data, i.e., whether there exists a \( \hat{\Delta} \in \mathcal{H}_\infty(\mathcal{N}(\{W_k\}, \{V_k\})) \) with \( \|\hat{\Delta}\|_\infty \leq \gamma \) such that the following holds:

\[
\begin{align*}
\begin{bmatrix}
U \\
Y \\
W
\end{bmatrix} &= T_{\gamma 11} \begin{bmatrix}
U \\
V
\end{bmatrix} + T_{\gamma 21} \begin{bmatrix}
V \\
W
\end{bmatrix} + E \\
\begin{bmatrix}
V \\
W
\end{bmatrix} &= T_{\gamma 21} \begin{bmatrix}
U \\
V
\end{bmatrix} + T_{\gamma 22} \begin{bmatrix}
V \\
W
\end{bmatrix} \\
\begin{bmatrix}
V
\end{bmatrix} &= T_{\Delta} \begin{bmatrix}
W
\end{bmatrix}
\end{align*}
\]

Note that \( T_{\gamma 22} U = Y \), it follows from Theorem 1 that

\[
\Pi_{Y_k V_k} = \gamma^2 T_{T_{\gamma 22} \Pi_{Y_k V_k} T_{\gamma 22}} \geq 0
\]

for all \( k = 1, \ldots, l \). Therefore \( Q_k \geq 0 \) for all \( k = 1, \ldots, l \), and \( \Pi_{Y_k V_k} = \gamma^2 T_{T_{\gamma 22} \Pi_{Y_k V_k} T_{\gamma 22}} \geq 0 \) for all \( k = 1, \ldots, l \). Substituting (16) into (18) yields

\[
H_{k11} (Y) - T_{\gamma 22} Q_k T_{\gamma 11} \geq 0.
\]

Since \( Q_k \geq 0 \), it follows by Schur complement that (19) is equivalent to \( H_k(Y) \geq 0 \). Hence, the uncertain model is not invalidated if and only if there exists a \( Y \in \Omega \) such that \( H_k(Y) \geq 0 \) for \( k = 1, \ldots, l \).

The conditions in Theorem 2 are the well-known LMI feasibility conditions which is numerically feasible.

**V. Conclusion**

The model validation for general multirate systems is studied in this note. Based on the solutions to the constrained CF interpolation...
problem, the time domain validation test is presented for the general multirate LFT uncertain models. These tests can be carried out by solving feasibility problems involving LMIs.

On the Asymptotically Optimal Tuning of Robust Controllers for Systems in the CD-Algebra

Timo Hämäläinen and Seppo Pohjolainen

Abstract—In a previous paper, the authors have shown that a low-gain controller of the form $C_\varepsilon(s) = \sum_{k=0}^{\infty} \varepsilon K_k(s - i\omega_k)$ is able to track and reject constant and sinusoidal reference and disturbance signal for a stable plant in the Callier–Desoer (CD) algebra. In this note, we investigate the optimal tuning of the matrix gains $K_k$ of the controller $C_\varepsilon(s)$ as the scalar gain $\varepsilon \downarrow 0$. The cost function is the maximum error between the reference signal and the measured output signal over all frequencies and bounded reference and disturbance signal amplitudes. Closed forms for asymptotically globally optimal solutions are given. The optimal matrix gains $K_k$ are expressed in terms of the values of the plant transfer matrix at the reference and disturbance signal frequencies. Thus the matrices $K_k$ can be tuned with input-output measurements made from the open loop plant without knowledge of the plant model. Although the analysis is in the CD-algebra, to the authors’ knowledge the main results are new even for finite-dimensional systems.

Index Terms—Low-gain control, distributed parameter systems, Callier–Desoer (CD)-algebra, tracking, optimal control.

I. INTRODUCTION

In a previous paper [1], the authors solved the following robust regulation problem: Given a stable plant $P$ in the Callier–Desoer (CD)-algebra and reference and disturbance signals of the form

$$a_0 + \sum_{k=1}^{n} a_k \sin(\omega_k t + \phi_k), \quad a_k \in \mathbb{R}$$

find a low-order finite-dimensional controller so that the outputs asymptotically track the reference signals, asymptotically reject the disturbance signals, and the closed-loop system is stable with respect to a class of perturbations in the plant, see Fig. 1. In [1], it is shown that a low-gain controller $C_\varepsilon$ given by

$$C_\varepsilon(s) = \sum_{k=-\infty}^{\infty} \frac{\varepsilon K_k}{s - i\omega_k}$$

solves the robust regulation problem provided that the positive scalar gain $\varepsilon$ is small enough and the matrix gains $K_k$ satisfy the stability conditions

$$\sigma(P(i\omega_k)K_k) \subset \mathbb{C}^+, \quad k = -n, \ldots, n$$

where $\omega_0 = 0$ and $\omega_k = -\omega_k$ for $k = 1, \ldots, n$.

Conditions (3) give an easily verifiable condition for the stabilizing matrix gains $K_k$. Unfortunately there are no analogous conditions for the scalar gain $\varepsilon$. The closed-loop system will remain stable if $\varepsilon$ is below some bound $\varepsilon^*$, but there is no easy way to determine $\varepsilon^*$. The tuning of $\varepsilon$ has to be done more or less by trial and error. In the case of constant reference and disturbance signals Logemann and Townley have given an adaptive method of tuning $\varepsilon$ [2]. However it is a nontrivial problem to generalize their method to signals of the form (1). Therefore

REFERENCES