On the generalization of the Euler polynomials with the real parameters

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Abstract

In this study, we give multiplication formula for generalized Euler polynomials of order $x$ and obtain some explicit recursive formulas. The multiple alternating sums with positive real parameters $a$ and $b$ are evaluated in terms of both generalized Euler and generalized Bernoulli polynomials of order $a$. Finally we obtained some interesting special cases.

1. Introduction

Throughout this paper $a$, $b$ and $c$ are positive real parameters and $x, x \in \mathbb{C}$, Srivastava et al. [12, p. 254, Eq. (20)] introduced and investigate the new type generalization of the Euler polynomials of order $x, E_n^x(x; \lambda; a, b, c)$, which are defined by means of the following generating function:

$$\left(\frac{2}{b^2 + at}\right)^x e^{xt} = \sum_{n=0}^{\infty} E_n^x(x; \lambda; a, b, c) \frac{t^n}{n!} \left(\left|t \ln \left(\frac{b}{a}\right) + \ln \lambda \right| < \pi; 1^x = 1\right).$$

(1.1)

If we set $\lambda = 1$ in (1.1), we arrive at the following generating function

$$\left(\frac{2}{b^2 + at}\right)^x e^{xt} = \sum_{k=0}^{\infty} E_k(x; a, b, c) \frac{t^k}{k!} \left(\left|t \ln \left(\frac{b}{a}\right) \right| < \pi; 1^x = 1\right).$$

(1.2)

If we set $x = 1$ in (1.2), we have

$$\frac{2e^{at}}{b^2 + at} = \sum_{k=0}^{\infty} E_k(x; a, b, c) \frac{t^k}{k!} \left|t \ln \left(\frac{b}{a}\right) \right| < \left|\frac{\pi}{\ln b - \ln a}\right|.$$

(1.3)

which have been studied by Luo et al. [7]. Details about these polynomials can be found in this paper.

Srivastava et al. [11, Eq. (23)] introduced the new type generalization of the Bernoulli polynomials of order $x, B_n^x(x; \lambda; a, b, c)$, which are given by following generating function:

$$\left(\frac{t}{b^2 - at}\right)^x e^{xt} = \sum_{n=0}^{\infty} B_n^x(x; \lambda; a, b, c) \frac{t^n}{n!} \left(\left|t \ln \left(\frac{b}{a}\right) + \ln \lambda \right| < 2\pi; a \neq b; 1^x = 1\right).$$

(1.4)
If we set \( z = 1 \) in (1.4), we have
\[
\sum_{n=0}^{\infty} B_n^\alpha(x; a, b, c) \frac{t^n}{n!} = \left( \frac{t}{b - d} \right)^z \left( t \ln \left( \frac{b}{a} \right) \right)^z < 2\pi; 1^z = 1.
\] (1.5)

If we set \( z = 1 \) in (1.5), we arrive at the following generating function
\[
\sum_{n=0}^{\infty} B_n^\alpha(x; a, b, c) \frac{t^n}{n!} = \frac{te^{zt}}{b - d} \left| t \right| < \frac{2\pi}{\ln b - \ln a},
\] (1.6)
which is defined by Luo et al. [6].

If we substitute \( a = 1 \) and \( b = c = e \), then classical Euler and Bernoulli polynomials can be deduced from (1.3) and (1.6), respectively.

Bernoulli and Euler polynomials and their numbers are used in number theory, complex analysis and approximation theory. Some theorems and relations on these polynomials are given [1–3,8–10]. Luo [5] gave the multiplication formula for the Apostol–Bernoulli and Apostol–Euler polynomials and their numbers by means of a suitable generating functions and give the basic properties of them. This work constitutes the main motivation of our paper. Methods used in this paper are similar to that of [5]. On the other hand Srivastava et al. [11,12] investigated a new generalization of the family of Bernoulli and Euler polynomials. They proved some interesting properties of these general polynomials and derived explicit representations for them in terms of a certain generalized Hurwitz–Lerch Zeta function and in series involving the familiar Gaussian hypergeometric function.

In this study, we define the generalized Euler polynomials and numbers of order \( x \) and prove multiplication formula for these polynomials. Moreover we give a formula for consecutive sums.

**Lemma 1** (Multinomial Identity [4, p. 28, Theorem B]). If \( x_1, x_2, \ldots, x_m \) are commuting elements of a ring (that is \( xx_j = x_j x_i \) for \( 1 \leq i < j \leq m \)) then we have for all integers \( n \geq 0 \):
\[
(x_1 + x_2 + \cdots + x_m)^n = \sum_{a_1 + a_2 + \cdots + a_m = n} \left( \begin{array}{c} n \\ a_1, a_2, \ldots, a_m \end{array} \right) x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m},
\] (1.7)
where summation takes place over all integers \( a_i \geq 0 \) and
\[
\left( \begin{array}{c} n \\ a_1, a_2, \ldots, a_m \end{array} \right) := \frac{n!}{a_1! a_2! \cdots a_m!}
\]
which are called the multinomial coefficients [4, p. 41, Definition B].

**Lemma 2** (Generalized Multinomial Identity [4, p. 41, Eq. (12 m)]). If \( x_1, x_2, \ldots, x_m \) are commuting elements of a ring (that is \( xx_j = x_j x_i \) for \( 1 \leq i < j \leq m \)) then we have for complex variable \( x \):
\[
(1 + x_1 + x_2 + \cdots + x_m)^2 = \sum_{\nu_1, \nu_2, \ldots, \nu_m \geq 0} \left( \begin{array}{c} 2 \\ \nu_1, \nu_2, \ldots, \nu_m \end{array} \right) x_1^{\nu_1} x_2^{\nu_2} \cdots x_m^{\nu_m},
\] (1.8)
where summation takes place over all integers \( \nu_i \geq 0 \), with
\[
\left( \begin{array}{c} 2 \\ \nu_1, \nu_2, \ldots, \nu_m \end{array} \right) := \frac{(x - 1)(x - 2) \cdots (x - (\nu_1 + 1) - (x - \nu_1 - 1) - \cdots - (x - \nu_m + 1))}{\nu_1! \nu_2! \cdots \nu_m!}
\]
which are called generalized multinomial coefficients defined by [4, p. 27, Eq. (10c’’)].

### 2. The multiplication formula for the generalized Euler polynomials

**Theorem 1.** For \( m \in \mathbb{N} \) and \( l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). The generalized Euler polynomials \( E_n^\alpha(mx; a, b, c) \) of order \( \alpha \) satisfy the following multiplication formulas
\[
E_n^\alpha(mx; a, b, c) = m^n \sum_{\nu_1, \nu_2, \ldots, \nu_{m-1} \geq 0} \left( \begin{array}{c} \alpha \\ \nu_1, \nu_2, \ldots, \nu_{m-1} \end{array} \right) (-1)^{\nu_1} \times E_n^{\alpha}\left(x + \frac{r(ln b - ln a) + \alpha(m - 1) ln a}{m ln c}, a, b, c\right), \ m \ odd,
\] (2.1)
and
\[
E_n^\alpha(mx; a, b, c) = \left( -\frac{2}{(n + 1)} \right)^m \sum_{0 \leq \nu_1, \nu_2, \ldots, \nu_{m-1} \leq l} \left( \begin{array}{c} l \\ \nu_1, \nu_2, \ldots, \nu_{m-1} \end{array} \right) \times (-1)^{\nu_1} E_n^{\alpha}\left(x + \frac{r(ln b - ln a) + l(m - 1) ln a}{m ln c}; a, b, c\right), \ m \ even.
\] (2.2)
where \( r = 2v_1 + 2v_2 + \cdots + (m - 1)v_m \).

**Proof.** It is easy to show that

\[
\frac{2}{b^* + a^*} = \frac{2e^{-\ln a x_{n+1}} e^{v x_{n+1}}}{1 + e^{v x_{n+1}}} \tag{2.3}
\]

for odd \( m \). By Eqs. (1.2), (1.8) and (2.3), we obtain assertion (2.1) of Theorem 1.

For even \( m \), we can similarly prove the assertion (2.2) of Theorem 1. Thus the proof is complete. \( \square \)

**Corollary 1.** Setting \( z = 1 \) in Eqs. (2.1) and (2.2), we have

\[
E_n(mx; a, b, c) = \begin{cases} 
  m^n \sum_{j=0}^{m-1} (-1)^j E_n \left( x + \frac{j}{m} \right), & m \text{ odd} \\
  (-2)^m m^n \sum_{j=0}^{m-1} (-1)^j B_{n+1} \left( x + \frac{j}{m} \right), & m \text{ even}
\end{cases} \tag{2.4}
\]

**Corollary 2 (Raabe’s multiplication theorem).** Setting \( a = 1 \) and \( b = c = e \) in Eq. (2.4), we have

\[
E_n(mx) = \begin{cases} 
  m^n \sum_{j=0}^{m-1} (-1)^j E_n \left( x + \frac{j}{m} \right), & m \text{ odd} \\
  (-2)^m m^n \sum_{j=0}^{m-1} (-1)^j B_{n+1} \left( x + \frac{j}{m} \right), & m \text{ even}
\end{cases}
\]

Next we define the multiple alternating sums as follows:

\[
\tilde{Z}_n^l(m; a, b) = (-1)^l \sum_{v_1, v_2, \ldots, v_m} \left( \begin{array}{c} l \\ v_1, v_2, \ldots, v_m \end{array} \right) \times (-1)^{v_1 + v_2 + \cdots + v_m} (\ln b - \ln a)^n (v_1 + 2v_2 + \cdots + mv_m)^n 
\]

and

\[
\tilde{Z}_n(m; a, b) = \sum_{j=0}^{m} (-1)^j (\ln b - \ln a)^n. 
\]

From this definition we prove the following theorem.

**Theorem 2.** For positive real numbers \( a, b \) and for \( m, l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), we have

\[
\tilde{Z}_n^l(m; a, b) = (-2)^l \sum_{k=0}^{l} \left( \begin{array}{c} l \\ k \end{array} \right) (-1)^{m+1} \times \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) E_j^l (mk + l; a, b, b) \tilde{E}_j^{l-k}(-mk; a, b, a). 
\] \tag{2.7}

**Proof.** By Eqs. (1.2) and (1.7) we obtain

\[
\sum_{n=0}^{\infty} \frac{\tilde{Z}_n^l(m; a, b) t^n}{n!} = \frac{e^{(\ln b - \ln a)} + (-1)^{m+1} e^i(\ln b - \ln a)}{1 + e^{(\ln b - \ln a)}} = \sum_{k=0}^{l} \left( \begin{array}{c} l \\ k \end{array} \right) \frac{e^{(\ln b - \ln a)} + (-1)^{m+1} e^i(\ln b - \ln a)}{1 + e^{(\ln b - \ln a)}} \tilde{Z}_n^{l-k}(mk + l; a, b, b) \tilde{E}_n^{l-k}(-mk; a, b, a) \frac{t^n}{n!} 
\]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation completes the proof. \( \square \)

**Corollary 3 (Luo [5, Eq. 53]).** Setting \( a = 1 \) and \( b = e \) in Eq. (2.7), we have

\[
\tilde{Z}_n^l(m; 1, e) = \tilde{Z}_n^l(m) = (-2)^l \sum_{k=0}^{l} \left( \begin{array}{c} l \\ k \end{array} \right) (-1)^{m+1} \times \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) E_j^l (mk + l; 1, b, b) \tilde{E}_j^{l-k}(0). 
\]
Corollary 4. Setting \( l = 1 \) in (2.7), we have

\[
\bar{Z}_n(m; a, b) = \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} \left( (-1)^{m+n-j-1} E_j(m+1; a, b, b)m^{n-j}(\ln a)^{n-j} + (\ln b)^j E_{n-j}(0; a, b, a) \right),
\]  

(2.8)

where \( E_j^0(1; a, b, b) = (\ln b)^j \) and \( E_{n-j}^0(-m; a, b, a) = (-1)^{n-j} m^{n-j}(\ln a)^{n-j} \).

Proposition 1. For \( n \geq 1 \), the generalized Euler polynomials \( E_k(x, a, b, c) \) satisfy the following equation

\[
\sum_{k=0}^{n} \binom{n}{k} E_k(0, a, b, c)(\ln b)^{n-k} + (\ln a)^{n-k} = 0.
\]

(2.9)

Proof. It is easy to see that

\[
\left( \frac{2}{b^2 + a^2} \right) \left( b^2 + a^2 \right) = 1.
\]

(2.10)

By using (1.3) and Taylor expansion in Eq. (2.10), the proof can be completed.

Corollary 5. If we set \( a = l = 1, \, b = e \) in Eq. (2.8) and use Proposition 1, then we have the following well-known formula for the alternating sum.

\[
\bar{Z}_n(m; 1, e) = \bar{Z}_n(m) = \frac{(-1)^{m+1} E_n(m+1) - E_n(0)}{2}.
\]

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References