Stochastic sensitivity of 3D-cycles

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Abstract

The limit cycles of nonlinear systems under the small stochastic disturbances are considered. The random trajectories of forced system leave the deterministic cycle and form some stochastic bundle around it. The probabilistic description of this bundle near cycle based on stochastic sensitivity function (SSF) is suggested. The SSF is a covariance matrix for periodic solution of linear stochastic first approximation system. This matrix is a solution of the boundary problem for linear matrix differential equation. For 3D-cycles this matrix differential equation on the basis of singular expansion is reduced to the system of three scalar equations only. The possibilities of SSF to describe some peculiarities of stochastically forced Roessler model are demonstrated.

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1. Introduction

Analysis of limit cycles under the stochastic disturbances was started by Pontryagin et al. [17] and continued by many researchers, see e.g. Stratonovich [23], Ibrahim [12], Soong and Grigoriu [22], Baras [4]. The random trajectories of forced system leave the closed curve of deterministic cycle and due to cycle stability form some bundle around it. Stochastic cycles were considered both near and far from Hopf bifurcation point. A qualitative effect of external fluctuations on the Hopf scenario was found and investigated by Arnold et al. [3] and Moss and McClintock [15]. A nonuniformity of stochastic bundles along cycles for nonlinear parameters far from bifurcation points attracts attention of researchers, see e.g. Kurrer and Schulten [13], Deissler and Farmer [9], Ali and Menzinger [1]. A small external noises acting on limit cycles may give rise of local phase-dependent response of the oscillations. Local instability of cycle is a reason of its significant sensitive dependence and can cause of noise-induced transition to chaos [6]. A variance of stochastic bundles perpendicular to the deterministic orbit is a natural measure of

limit cycles sensitivity. Kolmogorov–Fokker–Planck (KFP) equation gives the most detailed probabilistic description. However, the direct using of this equation is very difficult even for the simplest situations. It is important to note that when stochastic disturbances are small, they lead to famous problems of analysis of equations with small coefficients near higher derivatives. Under these circumstances asymptotics and approximations are actively used. Van Kampen approximation of KFP solution near deterministic cycle was used by Kurver and Schulten [13]. Asymptotic analysis of distribution density for small noises based on quasipotential function [11] is actively developed, see e.g. Day [8], Dembo and Zeitouni [10], Naeh et al. [16], Roy [19], Smelyanskiy et al. [21]. The probabilistic distribution for the bundle of random trajectories localized near cycle has Gaussian approximation. Quasipotential gives exponential asymptotics for stationary probability density. In the vicinity of cycle the first approximation of quasipotential is an orbital quadratic form [14,20]. Matrix of this quadratic form defines a covariance of the normal deviations of random trajectories for any point on a cycle. This matrix function plays a role of stochastic sensitivity function (SSF) of a cycle. SSF is a natural probabilistic measure of stochastic cycles response to small random disturbances.

For the case of cycle on a plane (2D-cycle) this matrix SSF due to its singularity has analytical representation using scalar function [14]. The possibilities of this scalar SSF to predict some peculiarities of 2D-cycles for stochastic forced Van-der-Pol oscillator and Brusselator are presented in [5,6]. A new critical values of Brusselator parameters with the help of SSF have been found. For these values a very small noise transfers Brusselator to chaotic regime.

The main attention of this paper is concentrated on the sensitivity analysis of 3D-cycles. In Section 1, the construction of SSF is introduced on the basis of linear stochastic first approximation systems. In Theorem 1, the convergence of the orthogonal to cycle projections for any solutions of the first approximation systems to some stochastic periodic process is proved. The covariance matrix of this periodic process is SSF mentioned above. This matrix is a solution of some boundary periodic value problem for linear matrix differential equation. A common iterative method for solution of this problem for n-dimensional cycles was presented in [7]. In Section 2, for the 3D-cycles a new effective method of the SSF construction is suggested. On the basis of singular expansion the matrix differential equation is reduced by Theorem 2 to the system of three scalar equations only. According to Theorem 3, a periodic solution of this simple system can be found by stabilization method. In Section 3, an application of suggested method to the sensitivity analysis of stochastically forced Roessler model cycle is demonstrated.

2. Stochastic forced cycles and projective dynamics of the first approximation systems

Consider the deterministic system of non-linear differential equations

\[ \dot{x} = f(x) \] (1)

Suppose the system (1) has T-periodic solution \( x = \xi(t) \) with exponentially stable phase curve \( \gamma \). It means that for small neighbourhood \( \Gamma \) of cycle \( \gamma \) there exist constants \( K > 0, \ell > 0 \) such that for any solution \( x(t) \) of system (1) with \( x(0) = x_0 \in \Gamma \) the following inequality holds

\[ ||\Delta(x(t))|| \leq Ke^{-\ell t}||\Delta(x_0)||. \]

Here \( \Delta(x) = x - \gamma(x) \) is a deviation of a point \( x \) from a cycle \( \gamma \), \( \gamma(x) \) is the point on cycle \( \gamma \) that is nearest to \( x \).
Let $\Pi_t$ be a hyperplane orthogonal to cycle at the point $\xi(t)(0 \leq t < T)$. By $\Gamma_t$ denote a neighbourhood 
$\Gamma_t = \Pi_t \cap \Pi_t$. It is supposed that $\Gamma_t \cap \Gamma_s = \emptyset$ for $t \neq s$.
Consider the first approximation system
\[
\dot{z} = F(t)z, \quad F(t) = \frac{\partial f}{\partial x}(\xi(t))
\]
(2)
for small deviations $z = x - \xi(t)$ of the system (1) solutions from cycle $\gamma$.
In the following analysis of this deviation dynamics the projector $Pr = I - rr^\top/r^\top r$ onto subspace 
orthogonal to the vector $r \neq 0$ will be used. For $r(t) = f(\xi(t))$ a matrix $P(t) = P_{Pr}$ is a projective matrix 
onto hyperplane $\Pi_t$. Stability of cycle means that the projections $P(t)z(t)$ of the solutions $z(t)$ of system 
(2) go to zero as $t \to +\infty$.

**Definition** (see [20]). The system (2) is called $P$-stable if there exist constants $K > 0, l > 0$ such that 
for any solution $z(t)$ of system (2) with $z(0) = z_0$ the following inequality holds
\[
\|P(t)z(t)\| \leq Ke^{-lt}\|P(0)z_0\|
\]
A $P$-stability of linear system (2) is necessary and sufficient condition [20] for exponential stability of 
cycle $\gamma$ of non-linear system (1).
In the sensitivity analysis of deterministic cycle $\gamma$ forced by random noises a stochastic system
\[
\dot{x} = f(x) + \varepsilon\sigma(x)w,
\]
(3)
in Ito’s sense [2] is used. Here $w(t)$ is a $n$-dimensional Wiener process, $\sigma(x)$ is $n \times n$-matrix function
defining the disturbances dependence on state of a system, $\varepsilon$ is a parameter of its intensity.
The random trajectories of system (3) form some bundle around limit cycle $\gamma$. Suppose the neigh-
bourhood $\Gamma$ is an invariant for a system (3). Therefore bundle of random trajectories is contained in
$\Gamma$ completely.
Let us describe the probabilistic characteristics of this bundle with the help of vector random variable
$X_t$. Values of $X_t$ are the intersection points of the system (3) random trajectories with $\Gamma_t$ (see Fig. 1,
where vectors $u_1, u_2$ and $v_1, v_2$ along with angle $\phi$ will be introduced in Section 3). The probabilistic
distribution of a bundle is stabilized as time increases. Therefore, a random variable $X_t$ in neighbourhood $\Gamma$
is an invariant for a system (3). Therefore bundle of random trajectories is contained in $\Gamma$ completely.
For small noises the function $\rho_t(x, \varepsilon)$ in vicinity of cycle $\gamma$ has an exponential Gaussian asymptotics
$\rho_t^0(x, \varepsilon)$
\[
\rho_t(x, \varepsilon) = \rho_t^0(x, \varepsilon) = K \exp \left( -\frac{(x - \xi(t))^T W(t)(x - \xi(t))}{2\varepsilon^2} \right)
\]
with mean value $m_t = \xi(t)$ and covariance matrix $\text{cov}_t = D(t, \varepsilon) = \varepsilon^2 W(t)$. This distribution concentrated in hyperplane $\Pi_t$ is singular: rank $D(t, \varepsilon) \leq n - 1$. For nondegenerated noises ($\det(\sigma(x)) \neq 0$) it follows that
rank $D(t, \varepsilon) = n - 1$.
The covariance matrix $D(t, \varepsilon)$ characterizes a dispersion of intersection points of random trajectories 
with hyperplane $\Pi_t$. The matrix $W(t) = 1/\varepsilon^2 D(t, \varepsilon)$ plays a role of stochastic sensitivity function (SSF) 
of cycle for random input. Let us connect a matrix $W(t)$ with the solutions of some linear stochastic systems.
For small noises the system of the first approximation for (3) looks like
\[ \dot{z} = F(t)z + \varepsilon G(t)\dot{w}, \quad G(t) = \sigma(\xi(t)). \] (4)

Sensitivity of system (4) solution \( z \) to noise with intensity \( \varepsilon \) is characterized by a variable \( u = (1/\varepsilon)z \) governed by following equation
\[ \dot{u} = F(t)u + G(t)\dot{w}. \] (5)

As will be shown by Theorem 1, the dynamics of projections \( P(t)u(t) \) for this system solutions is connected with system
\[ \dot{v} = F(t)v + P(t)G(t)\dot{w}. \] (6)

The additive noises of system (6) are the projections of system (5) noises. Covariance matrix \( U(t) = \text{cov}(u(t), u(t)) \) of the arbitrary solution \( u(t) \) of system (5) satisfies to equation
\[ \dot{U} = F(t)U + U F^T(t) + S(t), \quad S(t) = G(t)G^T(t). \] (7)

For covariance matrix \( V(t) = \text{cov}(v(t), v(t)) \) of the arbitrary solution \( v(t) \) of system (6), the following equation holds
\[ \dot{V} = L V, \quad L(V) = FV + VF^T + PSP. \] (8)

**Theorem 1.** Let system (2) be \( P \)-stable. Then
(a) Eq. (8) with conditions
\[ V(t)v(t) = 0 \] (9)
\[ V(0) = V(T) \] (10)
has on the interval \([0, +\infty)\) matrix \( W(t) \) as a unique solution.

![Diagram](image)
(b) The system (6) has the solution $\bar{v}(t)$ with T-periodic covariance matrix $\text{cov}(\bar{v}(t), \bar{v}(t)) = W(t)$.

(c) For any solution $v(t)$ of system (6) the projection $P(t)V(t)P(t)$ of covariance matrix $V(t) = \text{cov}(v(t), v(t))$ converges to T-periodic matrix $W(t)$:

$$\lim_{t \to +\infty} (P(t)V(t)P(t) - W(t)) = 0.$$  

(d) For any solution $v(t)$ of system (6) the projection $P(t)v(t)$ converges in mean square to $\bar{v}(t)$:

$$\lim_{t \to +\infty} E\|P(t)v(t) - \bar{v}(t)\|^2 = 0.$$  

(e) For any solution $u(t)$ of system (5) the projection $P(t)U(t)P(t)$ of covariance matrix $U(t) = \text{cov}(u(t), u(t))$ converges to T-periodic matrix $W(t)$:

$$\lim_{t \to +\infty} (P(t)U(t)P(t) - W(t)) = 0.$$  

(f) For any solution $u(t)$ of system (5) the projection $P(t)u(t)$ converges in mean square to $\bar{v}(t)$:

$$\lim_{t \to +\infty} E\|P(t)u(t) - \bar{v}(t)\|^2 = 0.$$  

Proof. The assertion (a) is proved in [14]. The assertion (b) follows easily if to take $\bar{v}(0)$ such that $\text{cov}(\bar{v}(0), \bar{v}(0)) = W(0)$. Let us prove the assertion (c).

From Eq. (8) and equality $W(t) = P(t)W(t)P(t)$ we get $P(t)V(t)P(t) - W(t) = P(t)\Delta(t)P(t)$, where $\Delta(t) = V(t) - W(t)$ is a solution of a homogeneous equation

$$\Delta = F\Delta + \Delta F^\top.$$

Matrix $\Delta(t)$ has an explicit representation $\Delta(t) = Z(t)\Delta(0)Z^\top(t)$, where $Z(t)$ is a fundamental matrix of system (2). Under the condition of $P$-stability of system (2), we have $P(t)Z(t) \to 0$ as $t \to +\infty$. Thus the projection $P(t)V(t)P(t)$ converges to $T$-periodic matrix $W(t)$ as $t \to +\infty$.

The assertion (d) follows from assertion (c) and relations

$$E[\|P(t)v(t) - \bar{v}(t)\|^2] \leq 2E[\|P(t)v(t) - P(t)\bar{v}(t)\|^2] + 2E[\|P(t)\bar{v}(t) - \bar{v}(t)\|^2],$$

$$E[\|P(t)v(t) - P(t)\bar{v}(t)\|^2] = \text{tr}(P(t)\Delta(t)P(t)),$$

$$E[\|P(t)\bar{v}(t) - \bar{v}(t)\|^2] = \text{tr}((P(t) - I)W(t)(P(t) - I)).$$

For the proof of the assertions (e) and (f) the following Lemma is required. □

Lemma 1. For any solution $u(t)$ of system (5) there exists a solution $v(t)$ of system (6) such that

$$P(t)u(t) = P(t)v(t).$$  

Proof. Let the scalar function $u(t)$ be a solution of stochastic equation

$$\dot{u} = \frac{1}{t^r}t^r G\dot{w}$$

where $\dot{w}$ is a solution of equation

$$\dot{w} = G\dot{w}$$

and $G$ is a positive definite matrix.
Consider vector function $v(t) = u(t) - \alpha(t)r(t)$. It follows from equalities

$$
\dot{v} = \dot{u} - \dot{\alpha}r - \alpha \dot{r} = Fu + G\dot{w} - \frac{1}{r}r^T Gw - aFr = Fv + PG\dot{w},
$$

that $v(t)$ is a solution of (6). The identity (11) follows from the condition $P(t)r(t) \equiv 0$. Lemma 1 is proved.

Let us prove the assertion (e). From an identity (11) it follows that

$$
P(t)U(t)P(t) \equiv P(t)V(t)P(t).
$$

(12)

Now assertion (e) follows from (12) and (c).

The proof of the assertion (f) follows from Lemma 1 and (d). Theorem 1 is proved.

Remark 1. Due to equality

$$
P(t)V(t)P(t) - W(t) = P(t)Z(t)(V(0) - W(0))Z^T(t)P(t),
$$

projection $P(t)V(t)P(t)$ tends to matrix $W(t)$ as fast as projection $P(t)Z(t)$ of the system (2) fundamental matrix $Z(t)$ tends to zero.

Dynamics of $Z(t)$ is defined by eigenvalues (multipliers) $|\rho_1| \geq |\rho_2| \geq \cdots \geq |\rho_n|$ of a monodromy matrix $B = Z(T)$. First multiplier $\rho_1$ always is equal to unit ($\rho_1 = 1$), and $\rho_2$ determines the convergence character. An inequality $|\rho_2| < 1$ (criterion of system (2) $P$-stability) is a necessary and sufficient condition of the projection $P(t)V(t)P(t)$ convergence to the matrix $W(t)$.

As we see, analysis of normal deviations of random trajectories from deterministic cycle requires to consider the system (6) along with traditional first approximation system (5).

Due to exponential stability of cycle the deterministic system (2) is $P$-stable. This provides for stochastic system (6) an existence of a stable periodic regime defined by the process $\tilde{v}(t)$ with covariance matrix $W(t)$. Consider small normal deviations of random trajectories from cycle. These deviations defined by projections of the system (5) solutions converge in mean square to $\tilde{v}(t)$ as $t \to \infty$.

So, in steady-state regime of stochastic auto-oscillations the matrix $W(t)$ is a dispersion measure for the points of random trajectories intersection with ortogonal hyperplanes. Here $W(t)$ plays a role of a matrix sensitivity function.

3. Singular decomposition of matrix sensitivity function for 3D-cycle

Due to Theorem 1 the covariance matrix $W(t)$ of steady-state stochastic auto-oscillations is the unique solution of system (8)–(10). Let us find a solution of this system on the basis of singular decomposition.

In the three-dimensional case ($n = 3$) studied here, a singular decomposition of symmetrical nonnegative defined $3 \times 3$-matrix $V(t)$ looks like

$$
V(t) = \lambda_1(t)v_1(t)v_1^T(t) + \lambda_2(t)v_2(t)v_2^T(t) + \lambda_3(t)v_3(t)v_3^T(t),
$$

where $\lambda_1(t) \geq \lambda_2(t) \geq \lambda_3(t)$ are eigenvalues, and $v_1(t), v_2(t), v_3(t)$ are eigenvectors of the matrix $V(t)$.

From the condition (9) it follows that for any $t$ a matrix $V(t)$ is degenerated (distribution of intersection points is concentrated in hyperplane $\Pi_t$). It means that $\lambda_3(t) \equiv 0$, and appropriate eigenvector $v_3(t) = r(t)/\|r(t)\|$ is tangent to cycle $\gamma$. It allows to write the matrix $V(t)$ decomposition as follows

$$
V(t) = \lambda_1(t)v_1(t)v_1^T(t) + \lambda_2(t)v_2(t)v_2^T(t).
$$

(13)
Here $V(t)$ is determined by scalar functions $\lambda_1(t)$, $\lambda_2(t)$ and vectors $v_1(t)$, $v_2(t)$. In a case of non-degenerate noises the functions $\lambda_1(t)$, $\lambda_2(t)$ are strictly positive and determine for any $t$ a dispersion of random trajectories around cycle along vectors $v_1(t)$, $v_2(t)$. Values $\lambda_1(t)$, $\lambda_2(t)$ determine the size and $v_1(t)$, $v_2(t)$ determine the directions of a dispersion ellipse axes. The equation of this ellipse in plane $\Pi_i$ looks like

$$(x - \xi(t))^\top W^\ast(t)(x - \xi(t)) = 2k^2,$$

where the parameter $\delta$ determines a fiducial probability $P = 1 - e^{-\delta}$.

Denote by $u_1(t), u_2(t)$ some orthonormal basis of the plane $\Pi_i$. One can easily find this basis if $T$-periodic solution $\xi(t)$ is known (see Remark 2). The eigenvectors $v_1(t), v_2(t)$ can be represented by rotation of basis $u_1(t), u_2(t)$ with some angle $\phi(t)$ (see Fig. 1).

$$v_1(t) = u_1(t) \cos \phi(t) + u_2(t) \sin \phi(t),$$

$$v_2(t) = -u_1(t) \sin \phi(t) + u_2(t) \cos \phi(t).$$

Thus, the decomposition (13) and (14) allows to express an unknown solution of a system (8)--(10) by means of three scalar functions $\lambda_1(t)$, $\lambda_2(t)$, $\phi(t)$.

Denote

$$P_1(t) = v_1(t) \cdot v_1(t), \quad P_2(t) = v_2(t) \cdot v_2(t).$$

Remark that $P_i(t)(i = 1, 2)$ are the projective matrices

$$P_i v_j = \delta_{ij}, \quad P_i v_j = 0(i \neq j), \quad P = P_1 + P_2.$$  \hfill (15)

Rewrite the decomposition (13) as follow

$$V(t) = \lambda_1(t) - P_1(t) + \lambda_2(t) - P_2(t).$$

**Lemma 2.** For the vector functions $v_i(t)$ and matrix functions $P_i(t)$ ($i = 1, 2$) the following identities hold

$$v_i^\top(t) P_i(t) v_1(t) = 0,$$  \hfill (16)

$$v_i^\top(t) P_i(t) v_2(t) = 0.$$  \hfill (17)

$$v_i^\top(t) P_i(t) v_2(t) = 0,$$  \hfill (18)

$$v_i^\top(t) P_i(t) v_2(t) = 0.$$  \hfill (19)

$$v_i^\top(t) P_i(t) v_2(t) = -\psi(t) + \bar{u}_i^\top(t) u_2(t),$$  \hfill (20)

$$v_i^\top(t) P_i(t) v_2(t) = -\psi(t) + \bar{u}_1^\top(t) u_2(t).$$  \hfill (21)

**Proof.** The identity (16) follows directly from

$$v_i^\top(t) P_1 v_1 = v_i^\top(t) v_1 v_1 = v_i^\top(t) v_1 v_1 = v_i^\top(t) v_1 v_1 = v_i^\top(t) v_1 = [v_i^\top(t) v_1] = 0.$$

The identity (19) is proved similarly. The identity (17) follows from

$$v_i^\top(t) P_2 v_1 = v_i^\top(t) v_2 v_1 + v_i^\top(t) v_1 v_1 = v_i^\top(t) v_2 v_1 + v_i^\top(t) v_1 v_1 = 0.$$

The identity (18) is proved similarly.
Using identities
\[
\begin{align*}
\dot{v}_1 &= u_1 \cos \psi + u_2 \sin \psi + v_2 \\
\dot{u}_2 &= 0, \quad \dot{u}_2 = 0, \quad \dot{u}_2 = -\dot{u}_2 \dot{u}_2
\end{align*}
\]
we obtain
\[
\begin{align*}
v_1^T P_1 v_2 &= v_1^T [v_1^T v_1 + v_1^T v_2] v_2 = v_1^T v_2 = (u_1 \cos \psi + u_2 \sin \psi + v_2) v_2 \\
&= (u_1^T \cos \psi + u_2^T \sin \psi)v_2 + \psi \\
&= -\dot{u}_1^T v_1 \cos \psi + \dot{u}_2^T u_2 \cos^2 \psi - \dot{u}_2^T u_1 \sin \dot{u}_2 u_2 \sin \dot{u}_2 u_2 \sin \psi + \psi = \dot{u}_1^T v_2,
\end{align*}
\]
implying (20). The identity (21) is proved similarly.

\section*{Theorem 2}
The matrix \(V(t)\) is the solution of a system (8) and (9) if and only if scalar functions \(\lambda_1(t), \lambda_2(t), \phi(t)\) of decomposition (13) and (14) satisfy to a system
\[
\begin{align*}
\lambda_1 &= \lambda_1 v_1^T [F + F^T] v_1 + v_1^T S v_1 \\
\lambda_2 &= \lambda_2 v_2^T [F + F^T] v_2 + v_2^T S v_2 \\
(\lambda_1 - \lambda_2) \psi &= \lambda_2 v_1^T F v_2 + \lambda_1 v_2^T F^T v_2 + v_1^T S v_2 - (\lambda_1 - \lambda_2) \dot{u}_1^T u_2.
\end{align*}
\]

\section*{Proof}
Let \(V(t)\) be a solution of (8) and (9). Substituting decomposition
\[
V = \lambda_1 P_1 + \lambda_2 P_2
\]
in differential Eq. (8) we obtain
\[
V = \lambda_1 P_1 + \lambda_1 P_2 = \lambda_1 F P_1 + \lambda_2 F P_2 + \lambda_1 P_1 F^T + \lambda_2 P_2 F^T + (P_1 + P_2) S (P_1 + P_2).
\]
Multiplying this equality by \(v_i^T\) from the left and by \(v_j\) from the right and using the projective matrices properties (15) and Lemma 2 we get
\[
\begin{align*}
v_1^T V v_1 &= \lambda_1 v_1^T F v_1 + \lambda_1 v_1^T F^T v_1 + v_1^T S v_1 \\
v_2^T V v_2 &= \lambda_2 v_2^T F v_2 + \lambda_2 v_2^T F^T v_2 + v_2^T S v_2 \\
v_1^T V v_2 &= \lambda_1 (\psi + u_2 u_2) + \lambda_2 (-\psi + u_2 u_2) = \lambda_2 v_2^T F v_2 + \lambda_1 v_2^T F^T v_2 + v_2^T S v_2.
\end{align*}
\]
Thus, \(\lambda_1(t), \lambda_2(t), \phi(t)\) satisfy the system (22).

Let us prove a converse. Let \(\lambda_1(t), \lambda_2(t), \phi(t)\) be solutions of system (22). Consider the matrix \(V = \lambda_1 P_1 + \lambda_2 P_2\). This matrix \(V\) satisfies (9). Due to (23) for \(i, j = 1, 2\) the following identities hold
\[
v_i^T (V - L(V)) v_j = 0.
\]
After differentiating of identity \(r^T V r = 0\) we get
\[
\begin{align*}
[r^T V r] &= r^T V r + r^T V r + r^T V r = r^T V r = 0 \\
r^T (V - L(V)) r &= 0.
\end{align*}
\]
Therefore, (24) is true and for \(i, j = 1, 2\). From (24) the equality (8) follows directly. Theorem 2 is proved.
As we see, construction of the solution \( V(t) \) of system (8) and (9) on the basis of singular decomposition is reduced to the solution of a system (22) for three scalar functions. The matrix \( W(t) \) (required stochastic sensitivity function of cycle) of system (8)–(10) solution can be obtained by the following limit procedure.

**Theorem 3.** Let \( T \)-periodic matrix \( W(t) \) be a solution of system (22) on the interval \([0, +\infty)\). Put \( V(t) = \lambda_1(t) \cdot P_1(t) + \lambda_2(t) \cdot P_2(t) \), where \( P_i(t) = v_i(t)v_i^\top(t) \) with vector functions \( v_i(t) \) obtained from (14). Then matrix \( V(t) \) tends to matrix \( W(t) \) as \( t \to +\infty \)

\[
\lim_{t \to +\infty} (V(t) - W(t)) = 0 \tag{25}
\]

**Proof.** Due to Theorem 2 matrix \( V(t) \) is a solution of system (8) and (9) and therefore \( V(t) = P(t)V(t)P(t) \). Now convergence (25) follows from the assertion (c) of Theorem 1.

**Remark 2.** For computation of system (22) right sides it is necessary to know vector functions \( u_1(t), u_2(t) \) and \( \dot{u}_1(t) \). To find it, the following method is offered. Rewrite an initial system \( \dot{x}(t) = f(x(t), \phi(t)) \) by coordinates

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, x_3), \\
\dot{x}_2 &= f_2(x_1, x_2, x_3), \\
\dot{x}_3 &= f_3(x_1, x_2, x_3), \\
\xi(t) &= (\xi_1(t), \xi_2(t), \xi_3(t))^\top
\end{align*}
\]

Tangent vector \( r(t) = (f_1(t), f_2(t), f_3(t))^\top \) has coordinates

\[
fi(t) = fi(\xi(t)), i = 1, 2, 3.
\]

One can choose the vectors \( u_1(t), u_2(t) \) of plane \( \Pi_t \) orthonormal basis in following form

\[
\begin{align*}
u_1 &= g_1 \cdot \begin{pmatrix}
-f_2 \\
f_1 \\
0
\end{pmatrix}, \\
u_2 &= g_2 \cdot \begin{pmatrix}
-f_1f_3 \\
f_2 \\
f_1^2 + f_2^2
\end{pmatrix},
\end{align*}
\]

where

\[
g_1 = (f_1^2 + f_2^2)^{-1/2}, \quad g_2 = (f_3^2(f_1^2 + f_2^2) + (f_1^2 + f_2^2))^{-1/2}
\]

As a result of such choice the following formula holds

\[
\begin{align*}
\dot{u}_1 &= \dot{g}_1 \cdot \begin{pmatrix}
-f_2 \\
f_1 \\
0
\end{pmatrix} + g_1 \cdot \begin{pmatrix}
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\
0 & 0 & 0
\end{pmatrix} \cdot \begin{pmatrix}
f_1 \\
f_2 \\
f_3
\end{pmatrix},
\end{align*}
\]

where

\[
\dot{g}_1 = -(f_1^2 + f_2^2)^{-3/2} \cdot \left( f_1 \left( \frac{\partial f_2}{\partial x_1} f_1 + \frac{\partial f_2}{\partial x_2} f_2 + \frac{\partial f_2}{\partial x_3} f_3 \right) + f_2 \left( \frac{\partial f_1}{\partial x_1} f_1 + \frac{\partial f_1}{\partial x_2} f_2 + \frac{\partial f_1}{\partial x_3} f_3 \right) \right)
\]
4. Example: stochastically forced Roessler model

Consider the stochastic system

\[
\begin{align*}
\dot{x} &= -y - z + \epsilon \dot{w}_1 \\
\dot{y} &= x + \alpha y + \epsilon \dot{w}_2 \\
\dot{z} &= \alpha + z(x - \mu) + \epsilon \dot{w}_3
\end{align*}
\]

obtained by the addition of small additive random disturbances to classic deterministic Roessler model [18]. Here \(w_i(t) (i = 1, 2, 3)\) are independent standard Wiener processes. Deterministic \((\epsilon = 0)\) Roessler system with \(\alpha = 0.2, \mu = 1.2\) has stable limit cycle. Projection of this cycle is plotted in Fig. 2. Under the random disturbances the trajectory of forced system leaves the deterministic cycle and form some bundle around it. For a numerical simulation of random trajectories of stochastic Roessler model an appropriate stochastic component of random disturbances was introduced in the deterministic fourth-order Runge–Kutta scheme on each step \(h = 10^{-4}\). We calculate \(x_{m+1} = RK(x_m, h) + \epsilon \Delta w_m\). Here \(x_m\) is numerical approximation for value \(x(t_m)\) of system (3) random solution \(x(t), RK(x_m, h)\) is one-step function of Runge–Kutta scheme, \(\Delta w_m = w(t_{m+1}) - w(t_m)\) is the Wiener process increment.

The projection of a random trajectories bundle for noise intensity \(\epsilon = 0.04\) is shown in Fig. 2b. Here an arrangement of the orthogonal hyperplanes is also marked. In this projection the random bundle looks uniformly. More detailed analysis of random trajectories intersection points with hyperplanes demonstrates more complicated behavior.

Fig. 3 shows the arrangement of intersection points (asterisks) for marked hyperplanes. One can see in Fig. 3 that the random trajectories are distributed around of cycle non-uniformly.

We shall characterize a dispersion of random trajectories bundle in the cut by plane \(\Pi\) for noise intensity \(\epsilon\) by empirical covariance matrix \(\hat{D}(t, \epsilon)\). A matrix \(\hat{W}(t, \epsilon) = 1/\epsilon^2 \hat{D}(t, \epsilon)\) plays a role of empirical sensitivity function of a cycle at the point \(\xi(t)\). The positive eigenvalues \(\hat{\lambda}_1(t) > \hat{\lambda}_2(t) > 0\) of matrix \(\hat{W}\) can be used as the natural scalar characteristics of this sensitivity. In Fig. 4 the values \(\hat{\lambda}_1, \hat{\lambda}_2\) calculated for 100 rotations around cycle with noise \(\epsilon = 10^{-4}\) are plotted by asterisks.
Fig. 3. Intersection points (asterisks) of random trajectories with normal hyperplanes and fiducial ellipses.

Fig. 4. Stochastic sensitivity of Roessler cycle for $\alpha = 0.2$, $\mu = 1.2$, $\epsilon = 0.0001$: (a) curve $\lambda_1$ and values $\bar{\lambda}_1$; (b) curve $\lambda_2$ and values $\bar{\lambda}_2$. 
Compare these empirical characteristics obtained by direct numerical simulation with outcomes based on theoretical sensitivity function \( W(t) \). Due to Theorem 1 this matrix is the solution of system (8)–(10). Singular decomposition \( W(t) \) allows to express the matrix \( W(t) \) by three scalar functions \( \lambda_1(t), \lambda_2(t), \varphi(t) \). These functions can be found from the system (22) by stabilization method (see Theorem 3).

The eigenvalues \( \lambda_1(t) > \lambda_2(t) \) of matrix \( W(t) \) are the convenient scalar characteristics of cycle sensitivity. For stochastic cycle of Roessler model the plots of functions \( \lambda_1(t), \lambda_2(t) \) (solid lines) are shown in Fig. 4.

From Fig. 4, one can see that theoretical curves \( \lambda_1(t), \lambda_2(t) \) are arranged near the values \( \bar{\lambda}_1, \bar{\lambda}_2 \) of empirical sensitivity function and reflect the main features (sharp peaks, monotonicity intervals) of this function clearly. Fig. 4 shows peculiarities of random bundle behaviour such that non-uniformity of bundle width along the cycle and large dispersion overfall in normal directions.

The functions \( \lambda_1(t), \lambda_2(t), \varphi(t) \) allow to construct a concentration ellipse for any normal plane. The values \( \lambda_1(t), \lambda_2(t) \) determine a size and configuration of this ellipse. The angle \( \varphi(t) \) sets orientation of axes of this ellipse in a plane \( \Pi_t \). As one can see in Fig. 3 the constructed ellipses reflect the dispersion of intersection points precisely. For ellipses plotted in Fig. 3 the value of fiducial probability equals 0.85.

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