A STONE-WEIERSTRASS THEOREM FOR MV-ALGEBRAS AND UNITAL \(\ell\)-GROUPS

LEONARDO MANUEL CABRER AND DANIELE MUNDICI

Abstract. Working jointly in the equivalent categories of MV-algebras and lattice-ordered abelian groups with strong order unit (for short, unital \(\ell\)-groups), we prove that isomorphism is a sufficient condition for a separating subalgebra \(A\) of a finitely presented algebra \(F\) to coincide with \(F\). The separation and isomorphism conditions do not individually imply \(A = F\). Various related problems, like the separation property of \(A\), or \(A \cong F\) (for \(A\) a separating subalgebra of \(F\)), are shown to be (Turing-)decidable. We use tools from algebraic topology, category theory, polyhedral geometry and computational algebraic logic.

1. Introduction

A unital \(\ell\)-group, [2, 11] group equipped with a translation invariant lattice structure, and \(u \geq 0\) is an element whose positive integer multiples eventually dominate every element of \(G\). An MV-algebra \(A = (A, 0, \oplus, \neg)\) is an abelian monoid \((A, 0, \oplus)\) equipped with an operation \(\neg\) such that \(\neg \neg x = x\), \(x \oplus \neg 0 = \neg 0\) and \(\neg (\neg x \oplus y) \oplus y = \neg (\neg y \oplus x) \oplus x\). Recently, MV-algebras, and especially finitely presented MV-algebras, [15], [20, §6], have found applications to lattice-ordered abelian groups [4, 5, 6, 14, 18], the Farey-Stern-Brocot AF C\(^*\)-algebra of [17], (see [3, 9, 19]), probability and measure theory, [10, 21], multisets [7], and vector lattices [22]. The versatility of MV-algebras stems from a number of factors, including:

(i) The categorical equivalence \(\Gamma\) between unital \(\ell\)-groups and MV-algebras [16], which endows unital \(\ell\)-groups with the equational machinery of free algebras, finite presentability, and word problems, despite the archimedean property of the unit is not definable by equations.

(ii) The duality between finitely presented MV-algebras and rational polyhedra, [4, 15, 20].

(iii) The one-to-one correspondence, via \(\Gamma\) and Grothendieck’s \(K_0\), between countable MV-algebras and AF C\(^*\)-algebras whose Murray-von Neumann order of projections is a lattice, [16].

(iv) The deductive-algorithmic machinery of the infinite-valued Lukasiewicz calculus \(L_\infty\) is immediately applicable to MV-algebras, [8, 20].

This paper deals with finitely generated subalgebras of finitely presented MV-algebras and unital \(\ell\)-groups. We will preferably focus on the equational class of MV-algebras, where freeness and finite presentations are immediately definable. Finitely presented MV-algebras are the Lindenbaum algebras of finitely axiomatizable theories in \(L_\infty\). Finitely generated projective MV-algebras, in particular, are a key tool for the proof-theory of \(L_\infty\) (see [4, 13, 15] for an algebraic analysis of admissibility, exactness and unification in \(L_\infty\)). Remarkably enough, the characterization of projective MV-algebras and unital \(\ell\)-groups is a deep open problem in algebraic topology, showing that unital \(\ell\) -groups have a greater complexity than \(\ell\)-groups, (see [5, 6]). As a matter of fact, while the well-known Baker-Beynon duality ([1] and references therein), shows that finitely presented \(\ell\)-groups coincide with finitely generated projective \(\ell\)-groups, the class of finitely generated projective unital \(\ell\)-groups (resp., finitely generated projective MV-algebras) is strictly contained in the class of finitely presented unital \(\ell\)-groups (resp., finitely presented MV-algebras).

Key words and phrases. MV-algebra, lattice ordered abelian group, strong order unit, unital \(\ell\)-group, projective MV-algebra, infinite-valued Lukasiewicz calculus Stone-Weierstrass theorem, retraction, McNaughton function, piecewise linear function, basis, Schauder hat, regular triangulation, unimodular triangulation, duality, polyhedron, finite presentation, isomorphism problem, Markov unrecognizability theorem.

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Let $A$ be a finitely generated subalgebra of a finitely presented MV-algebra, (or unital $\ell$-group) $F$. In Theorem 3.1 we prove: it is decidable whether $A$ is separating. If $A$ is separating and $F$ is a finitely generated free MV-algebra then $A$ is projective. This is proved in Theorem 3.2. Further, $A = F$ iff $A \cong F$: this is our MV-algebraic Stone Weierstrass theorem (4.4). For separating subalgebras $A$ of free $n$-generator MV-algebras or unital $\ell$-groups, the isomorphism problem $A \cong F$ is decidable. See Theorem 4.8.

As is well known, [8, 9.1], the free $n$-generator MV-algebra $M([0,1]^n)$ consists of all $n$-variable McNaughton functions defined on $[0,1]^n$. For any MV-term $\tau = \tau(X_1, \ldots, X_n)$ we let $\hat{\tau} \in M([0,1]^n)$ be obtained by evaluating $\tau$ in $M([0,1]^n)$. In Section 5 we introduce a method to write down a list of MV-terms $\tau_1, \ldots, \tau_k$ in the variables $X_1, \ldots, X_n$ in such a way that the subalgebra of $M([0,1]^n)$ generated by $\hat{\tau}_1, \ldots, \hat{\tau}_k$ is separating and distinct from $M([0,1]^n)$. In Theorem 5.4 we prove that every finitely generated separating proper subalgebra $A$ of $M([0,1]^n)$ is obtainable by this method. In the light of Theorem 3.2, a large class of projective non-free MV-algebras can be effectively introduced. In Section 6, we connect our presentation of finitely generated separating subalgebra $A$ of $M([0,1]^n)$ with the notion of basis [20, §6]. In Theorem 6.3, for any finitely generated separating subalgebra $A$ of $M([0,1]^n)$ we provide an effective method to transform every generating set of $A$ into a basis of $A$. As another application, in Section 7, we prove the decidability of the problem of recognizing whether two different sets of terms generate the same separating subalgebra of a free MV-algebra.

Finally, in Section 8 we discuss the mutual relations between presentations of MV-algebras as finitely generated subalgebras of free MV-algebras, and the traditional finite presentations in terms of principal quotients of free MV-algebras. Many results proved in this paper for separating finitely generated unital $\ell$-subgroups of free unital $\ell$-groups, fail for finitely presented $\ell$-groups, and are open problems for finitely presented unital $\ell$-groups. The separation hypothesis plays a crucial role in most decidability results of the earlier sections. Actually, the final two results of this paper (Theorems 8.1 and 8.2) show that without this hypothesis, decision problems for a subalgebra $A$ of $M([0,1]^n)$ generated by $\hat{\tau}_1, \ldots, \hat{\tau}_k$ become as difficult as their classical counterparts where $A$ is presented as a principal quotient of $M([0,1]^n)$.

2. MV-algebras, rational polyhedra, regular triangulations

MV-algebras, [8, 20]. We assume familiarity with the categorical equivalence $\Gamma$ between MV-algebras and unital $\ell$-groups, [16, 8]. For any closed set $Y \subseteq [0,1]^n$ we let $M(Y)$ denote the MV-algebra of restrictions to $Y$ of all McNaughton functions defined on $[0,1]^n$. A set $S \subseteq M(Y)$ is said to be separating (or, $S$ separates points of $Y$) if for all $x, y \in Y$ such that $x \neq y$ there is $f \in S$ with $f(x) \neq f(y)$. Unless otherwise specified, $Y$ will be nonempty, whence the MV-algebra $M(Y)$ will be nontrivial.

By an MV-term $\tau = \tau(X_1, \ldots, X_n)$ we mean a string of symbols obtained from the variable symbols $X_i$ and the constant symbol $0$ by a finite number of applications of the MV-algebraic connectives $\land, \lor$. The map $\hat{\tau}$ sending $X_i$ to the $i$th coordinate function $\pi_i: [0,1]^n \to [0,1]$ canonically extends to a map interpreting each MV-term $\tau(X_1, \ldots, X_n)$ as a McNaughton function $\hat{\tau} \in M([0,1]^n)$. McNaughton theorem [8, 9.1.5] states that this map is onto the MV-algebra $M([0,1]^n)$. The set $\pi_1, \ldots, \pi_n$ freely generates the free MV-algebra $M([0,1]^n)$.

Rational polyhedra and their regular triangulations, [20, 24]. Let $n = 1, 2, \ldots$ be a fixed integer. A point $x$ lying in the $n$-cube $[0,1]^n$ is said to be rational if so are its coordinates. In this case, there are uniquely determined integers $0 \leq c_i \leq d_i$ such that $x$ can be written as $x = (c_1/d_1, \ldots, c_n/d_n)$, with $c_i$ and $d_i$ relatively prime for each $i = 1, \ldots, n$. By definition, the homogeneous correspondent of $x$ is the integer vector $\hat{x} = (dc_1/d_1, \ldots, dc_n/d_n, d) \in \mathbb{Z}^{n+1}$, where $d > 0$ is the least common multiple of $d_1, \ldots, d_n$. The integer $d$ is said to be the denominator of $x$, denoted $\text{den}(x)$. A rational polyhedron $P$ in $[0,1]^n$ is a subset of $[0,1]^n$ which is a finite union of simplexes in $[0,1]^n$ with rational vertices. A polyhedral complex $\Pi$ is said to be rational if the vertices of each polyhedron in $\Pi$ are rational.

A rational $m$-dimensional simplex $T = \text{conv}(v_0, \ldots, v_m) \subseteq [0,1]^n$ is regular if the set $\{\hat{v}_0, \ldots, \hat{v}_m\}$ of homogeneous correspondents of its vertices can be extended to a basis of the free abelian group
$\mathbb{Z}^{n+1}$. An (always finite) simplicial complex $\Delta$ is \textit{regular} if each one of its simplexes is \textit{regular}. The \textit{support} $|\Delta|$ of $\Delta$, i.e., the point-set union of all simplexes of $\Delta$, is the most general possible rational polyhedron in $\mathbb{R}^n$ (see [20, 2.10]). We also say that $\Delta$ is a regular \textit{triangulation} of the rational polyhedron $|\Delta|$.

Regular simplexes and complexes are called “unimodular” in [8, 14, 15, 18], and “(Farey) regular” in [21]. In the literature on polyhedral topology, notably in [24], the adjective “linear” has a different meaning. Throughout this paper, the adjective “linear” is to be understood in the affine sense.

\textit{Finitely presented MV-algebras are dual to rational polyhedra, [4, 15, 20].} An MV-algebra $A$ is said to be \textit{finitely presented} if it is isomorphic to the quotient MV-algebra $M([0,1]^n)/I$, for some $n = 1, 2, \ldots$, and some principal ideal $I$ of $M([0,1]^n)$. If $I$ is generated by $g \in M([0,1]^n)$ then $M([0,1]^n)/I \cong M(g^{-1}(0))$, and $g^{-1}(0)$ is a (possibly empty) rational polyhedron in $[0,1]^n$.

Given two rational polyhedra $P \subseteq [0,1]^n$ and $Q \subseteq [0,1]^k$ a \textit{Z-map} is a continuous piecewise linear map $\eta: P \to Q$ such that each linear piece of $\eta$ has integer coefficients. Following [20, 3.2], given rational polyhedra $P \subseteq [0,1]^n$ and $Q \subseteq [0,1]^k$, we write $P \cong Q$ (and say that $P$ and $Q$ are \textit{Z-homeomorphic}) if there is a homeomorphism $h$ of $P$ onto $Q$ such that both $h$ and $h^{-1}$ are Z-maps. We also say that $h$ is a \textit{Z-homeomorphism}.

The functor $M$ sending each polyhedron $P$ to the MV-algebra $M(P)$, and each Z-map $\eta: P \to Q$ to the map $M(\eta): M(Q) \to M(P)$ defined by

$$M(\eta): f \mapsto f \circ \eta,$$

for any $f \in M(Q)$, where $\circ$ denotes composition, determines a categorical equivalence between rational polyhedra with Z-maps, and the opposite of the category of finitely presented MV-algebras with homomorphisms. For short, $M$ is a \textit{duality} between these categories. See [4, 15, 20] for further details.

3. “\textit{Presenting}” MV-algebras by a finite list of MV-terms

Every finite set of MV-terms $\tau_1, \ldots, \tau_k$ in the variables $X_1, \ldots, X_n$ determines the subalgebra $A$ of $M([0,1]^n)$ generated by the McNaughton functions $\hat{\tau}_1, \ldots, \hat{\tau}_k$. Then $A$ is finitely presented [20, 6.6]. As we will show throughout this paper, when $A$ is presented via generators $\hat{\tau}_1, \ldots, \hat{\tau}_k$, several decision problems turn out to be solvable.

Our first example is as follows:

**Theorem 3.1.** The following separation problem is decidable:

**INSTANCE:** A list of MV-terms $\tau_1, \ldots, \tau_k$ in the variables $X_1, \ldots, X_n$.

**QUESTION:** Does the set of functions $\{\hat{\tau}_1, \ldots, \hat{\tau}_k\}$ separate points of $[0,1]^n$?

**Proof.** Let $g = (\hat{\tau}_1, \ldots, \hat{\tau}_k): [0,1]^n \to [0,1]^k$ be defined by $g(x) = (\hat{\tau}_1(x), \ldots, \hat{\tau}_k(x))$ for all $x \in [0,1]^n$.

By [20, 3.4], the range $R$ of $g$ is a rational polyhedron in $[0,1]^k$. The separation problem equivalently asks if $g$ is one-to-one. Equivalently, is $g$ is a piecewise linear homeomorphism of $[0,1]^n$ onto $R$? Fix $i \in \{1, \ldots, k\}$. By induction on the number of connectives in the subterms of $\tau_i$ one can effectively list the linear pieces $l_{i1}, \ldots, l_{it}$, of the piecewise linear function $\hat{\tau}_i$. Since all these linear pieces have integer coefficients, the routine stratification argument of [20, 2.1] yields (as the list of the sets of vertices of its simplexes) a rational polyhedral complex $\Pi_i$ over $[0,1]^n$ such that $\hat{\tau}_i$ is linear on each simplex of $\Pi_i$. As in [24, 1.4], we now subdivide $\Pi_i$ into a rational triangulation $\Delta_i$, without adding new vertices. By induction on $k$, we compute a rational triangulation $\Delta$ of $[0,1]^n$ which is a joint subdivision of $\Delta_1, \ldots, \Delta_k$. Thus $g$ is linear over $\Delta$. (Using the effective desingularization procedure of [20, 2.9–2.10] we can even insist that $\Delta$ is regular.) We next write down the set $\Delta' = \{g(T) \subseteq [0,1]^k \mid T \in \Delta\}$ as the list of the set of vertices of each convex polyhedron $g(T)$. The following two conditions are now routinely checked:

(I) $\Delta'$ is a rational triangulation (of $R$), and

(II) $g$ maps the set of vertices of $\Delta$ one-to-one into the set of vertices of $\Delta'$. (By definition of $\Delta'$, $g$ maps vertices of $\Delta$ onto vertices of $\Delta'$.)
Indeed, the separation problem has a positive answer iff the both (I) and (II) are satisfied: This completes the proof of the decidability of the separation problem. □

An MV-algebra $D$ is said to be projective if whenever $\psi: A \to B$ is a surjective homomorphism and $\phi: D \to B$ is a homomorphism, there is a homomorphism $\theta: D \to A$ such that $\phi = \psi \circ \theta$. As is well known, $D$ is projective iff it is a retract of a free MV-algebra $F$: in other words, there is a homomorphism $\omega$ of $F$ onto $D$ and a one-to-one homomorphism $\iota$ of $D$ into $F$ such that the composite function $\omega \circ \iota$ is the identity function on $D$.

A large class of finitely generated projective MV-algebras, and automatically, of finitely generated projective unital $\ell$-groups, can be effectively presented by their generators, combining the following theorem with the techniques of Section 5 below:

**Theorem 3.2.** Let $A$ be a finitely generated subalgebra of the free MV-algebra $\mathcal{M}([0, 1]^n)$. Suppose $A$ is separating (a decidable property, by Theorem 3.1). Then $A$ is projective.

*Proof.* Let $\{g_1, \ldots, g_k\}$ be a generating set of $A$. As already noted, the separation hypothesis means that the map $g: [0, 1]^n \to [0, 1]^k$ defined by $g(x) = (g_1(x), \ldots, g_k(x))$, $(x \in [0, 1]^n)$ is one-to-one. Since $g$ is continuous, $g$ is a homeomorphism of $[0, 1]^n$ onto its range $R \subseteq [0, 1]^k$. Further, $g$ is piecewise linear and each linear piece of $g$ is a polynomial with integer coefficients. Thus, $g$ is a $\mathbb{Z}$-map. By [20, 3.4], $R$ is a rational polyhedron. Further:

1. The piecewise linearity of $g$ yields a triangulation $\Delta_g$ of $[0, 1]^n$ such that $g$ is linear over every simplex of $\Delta_g$. Since $g$ is a homeomorphism, the set $g(\Delta_g) = \{g(T) \mid T \in \Delta_g\}$ is a rational triangulation of $R$, making $R$ into what is known as an $n$-dimensional PL-ball.
   A classical result of Whitehead [25] shows that $R$ has a collapsible triangulation $\nabla$.

2. By [4, 4.10], the polyhedron $R = g([0, 1]^n)$ has a point of denominator 1, and has a strongly regular triangulation $\Delta$, in the sense that $\Delta$ is regular and for every maximal simplex $M$ of $\Delta$ the greatest common divisor of the denominators of the vertices of $M$ is equal to 1.

By [6, 6.1(III)], the MV-algebra $\mathcal{M}(R)$ is projective. By [20, 3.6], $A$ is isomorphic to $\mathcal{M}(R)$, whence the desired conclusion follows. □

We refer to [16] and [8] for background on unital $\ell$-groups and their categorical equivalence $\Gamma$ with MV-algebras. In particular, [16, 4.16] deals with the freeness properties of the unital $\ell$-group $\mathcal{M}_{\text{group}}([0, 1]^n)$ of all (continuous) piecewise linear functions $f: [0, 1]^n \to \mathbb{R}$ where each linear piece of $f$ has integer coefficients, and with the constant function 1 as the distinguished order unit. An equivalent definition of $\mathcal{M}_{\text{group}}([0, 1]^n)$ is given by

$$
\Gamma(\mathcal{M}_{\text{group}}([0, 1]^n)) = \mathcal{M}([0, 1]^n).
$$

(2)

Projective unital $\ell$-groups are the main concern of [5, 6]. From the foregoing theorem we immediately have:

**Corollary 3.3.** If $(G, u)$ is a finitely generated unital $\ell$-subgroup of $\mathcal{M}_{\text{group}}([0, 1]^n)$, and is separating, in the sense that for each $x \neq y \in [0, 1]^n$ there exists $f \in G$ such that $f(x) \neq f(y)$, then $(G, u)$ is projective. □

4. An MV-algebraic Stone-Weierstrass theorem

Let $P \subseteq [0, 1]^n$ be a rational polyhedron and $\{g_1, \ldots, g_k\}$ a generating set of a subalgebra $A$ of $\mathcal{M}(P)$. Under which conditions does $A$ coincide with $\mathcal{M}(P)$?

One obvious necessary condition is that $A$ be isomorphic to $\mathcal{M}(P)$—but this condition alone is not sufficient: for instance, by [20, 3.6], the subalgebra of $\mathcal{M}([0, 1])$ generated by $x \oplus x$ is isomorphic to $\mathcal{M}([0, 1])$ but does not coincide with it, because the points 1/2 and 1 are not separated by $A$, but are separated by the identity function $x \in \mathcal{M}([0, 1])$.

Another necessary condition is given by observing that $A$ must separate points of $P$. Again, this condition alone is not sufficient for $A$ to coincide with $\mathcal{M}(P)$, as the following example shows:
Example 4.1. Let $A$ be the subalgebra of the free one-generator MV-algebra $\mathcal{M}([0,1])$ generated by the two elements $x \oplus x$ and $(x \oplus x)$. See the picture below. It is easy to see that $A$ is separating. However, $A$ does not coincide with $\mathcal{M}([0,1])$: no function $f \in A$ satisfies $f(1/2) = 1/2$, while the McNaughton function $x \in \mathcal{M}([0,1])$ does.

While individually taken, separation and isomorphism are necessary but not sufficient conditions for a subalgebra $A$ of $\mathcal{M}(P)$ to coincide with $\mathcal{M}(P)$, putting these two conditions together, we will obtain in Theorem 4.4 an MV-algebraic variant of the Stone-Weierstrass theorem.

To this purpose, let us agree to say that a subalgebra $A$ of an MV-algebra $B$ is an epi-subalgebra if the inclusion map is an epi-homomorphism. Stated otherwise, for any two homomorphisms $h, g: B \to C$, if $h \vert A = g \vert A$ then $h = g$. We then have:

Lemma 4.2. Let $P \subseteq [0,1]^n$ be a rational polyhedron and $A$ a finitely generated subalgebra of $\mathcal{M}(P)$. Then the following conditions are equivalent:

(i) $A$ is a separating subalgebra of $\mathcal{M}(P)$;

(ii) $A$ is an epi-subalgebra of $\mathcal{M}(P)$.

Proof. Let $g_1, \ldots, g_n \in \mathcal{M}(P)$ be a set of generators for $A$. Then $A$ is separating iff the map $g = (g_1, \ldots, g_n): P \to [0,1]^n$ is one-to-one. By [4, Theorem 3.2], $g$ is one-to-one iff $g$ is a mono $\mathbb{Z}$-map. Recalling (1), this latter condition is equivalent to stating that the map $\mathcal{M}(g): \mathcal{M}([0,1]^n) \to \mathcal{M}(P)$ is an epi-homomorphism. Equivalently, the range $A$ of $\mathcal{M}(g)$ is an epi-subalgebra of $\mathcal{M}(P)$.

Lemma 4.3. Let $P$ and $Q$ be rational polyhedra in $[0,1]^n$ and $\eta: P \to Q$ a one-to-one $\mathbb{Z}$-map. Then the following are equivalent:

(i) $P$ is $\mathbb{Z}$-homeomorphic to $\eta(P)$;

(ii) $\eta$ is a $\mathbb{Z}$-homeomorphism of $P$ onto $\eta(P)$.

Proof. For the nontrivial direction, let $\gamma: \eta(P) \to P$ be a $\mathbb{Z}$-homeomorphism. Then $\gamma \circ \eta$ is a one-to-one $\mathbb{Z}$-map from $P$ into $P$. By [4, Theorem 3.6], $\gamma \circ \eta$ is a $\mathbb{Z}$-homeomorphism. It follows that $\eta = \gamma^{-1} \circ (\gamma \circ \eta)$ is a $\mathbb{Z}$-homeomorphism. $\Box$

We are now ready to prove the main result of this section:

Theorem 4.4. Let $P \subseteq [0,1]^n$ be a rational polyhedron and $A$ a subalgebra of $\mathcal{M}(P)$. Then the following conditions are equivalent:

(i) $A = \mathcal{M}(P)$;

(ii) $A$ is isomorphic to $\mathcal{M}(P)$ and is a separating subalgebra of $\mathcal{M}(P)$;

(iii) $A$ is isomorphic to $\mathcal{M}(P)$ and is an epi-subalgebra of $\mathcal{M}(P)$.

Proof. (ii)$\iff$(iii) follows directly from Lemma 4.2. (i)$\Rightarrow$(ii) is trivial. To prove (ii)$\Rightarrow$(i), let $e: \mathcal{M}(P) \to A$ be an isomorphism, and $i: A \to \mathcal{M}(P)$ be the inclusion map. Since $e$ is bijective and $i$ is epi, then $i \circ e$ is epi. Similarly, since $e$ and $i$ are one-to-one, then so is $i \circ e$. Let $\eta: P \to P$ be the unique $\mathbb{Z}$-map such that $i \circ e(f) = f \circ \eta$ for each $f \in \mathcal{M}(P)$. By [4, Theorem 3.2], $\eta$ is one-to-one and onto. By [4, Theorem 3.6], $\eta$ is a $\mathbb{Z}$-homeomorphism, that is, $\eta^{-1}$ is a $\mathbb{Z}$-map. Again with reference to (1), the map $\mathcal{M}(\eta^{-1}): \mathcal{M}(P) \to A$ is the inverse of $i \circ e$. As a consequence, $i$ is surjective and $A = i(\mathcal{M}(P)) = \mathcal{M}(P)$. $\Box$
Recalling (2) we immediately have:

**Corollary 4.5.** A finitely generated separating unital $t$-subgroup of $M_{\text{group}}([0,1]^n)$ isomorphic to $M_{\text{group}}([0,1]^n)$ coincides with $M_{\text{group}}([0,1]^n)$.

When $P = [0,1]^n$, Theorem 4.4 has the following stronger form:

**Theorem 4.6.** Any $n$-generator separating subalgebra $A$ of $M([0,1]^n)$ (equivalently, any $n$-generator epi-subalgebra of $M([0,1]^n)$) coincides with $M([0,1]^n)$.

Proof. Let $\{g_1,\ldots,g_n\}$ be a generating set of $A$ in $M([0,1]^n)$. The map $g = (g_1,\ldots,g_n) : [0,1]^n \to [0,1]^n$ is one-to-one. From [4, Theorem 3.6] it follows that $g$ is a $Z$-homeomorphism. With reference to (1), the map $\mathcal{M}(g)$ yields an isomorphism from $\mathcal{M}([0,1]^n)$ onto $\mathcal{M}([0,1]^n)$, whence $\mathcal{M}([0,1]^n) = A$.

**Problem 4.7.** Prove or disprove: Any epi-subalgebra $B$ of a semisimple MV-algebra $C$ isomorphic to $C$ coincides with $C$.

**Theorem 4.8.** The following isomorphism problem is decidable:

**INSTANCE:** MV-terms $\tau_1,\ldots,\tau_k$ in the variables $X_1,\ldots,X_n$ such that the subalgebra $A$ of $\mathcal{M}([0,1]^n)$ generated by $\hat{\tau}_1,\ldots,\hat{\tau}_k$ is separating (a decidable condition, by Theorem 3.1).

**QUESTION:** Is $A$ isomorphic to $\mathcal{M}([0,1]^n)$?

Proof. Let us write $g = (\hat{\tau}_1,\ldots,\hat{\tau}_k) : [0,1]^n \to [0,1]^k$. By hypothesis $g$ is a homeomorphism. Let $R$ be the range of $g$. As in the proof of Theorem 3.1, let $\Delta'$ be a rational triangulation of $[0,1]^n$ such that $g$ is linear on each simplex of $\Delta'$. Using the desingularization procedure of [20, 2.8] (which is also found in [8, Theorem 9.1.2]), we compute a regular subdivision $\Delta$ of $\Delta'$, by listing the sets of vertices of its simplexes. We have the following equivalent conditions:

- $\Delta \cong \Delta'$
- $\Delta' \cong \mathcal{M}([0,1]^n)$, (because $\Delta \cong \mathcal{M}(R)$, by [20, 3.6])
- $\Delta \cong \mathcal{M}(R)$, (by duality, [20, 3.10])
- $g$ is a $Z$-homeomorphism of $[0,1]^n$ onto $R$, (by Lemma 4.3)
- $g(S)$ is regular for each $S \in \Delta$, and $\text{den}(g(r)) = \text{den}(r)$ for each vertex $v$ of $\Delta$. This last equivalence follows because the homeomorphism $g$ is a $Z$-map of $[0,1]^n$ onto $R$. (see [20, 3.15(i)→(iii)]).

**Proposition 4.9.** The following problem is decidable:

**INSTANCE:** MV-terms $\tau_1,\ldots,\tau_k$ in the variables $X_1,\ldots,X_n$.

**QUESTION:** Is the subalgebra $A$ of $\mathcal{M}([0,1]^n)$ generated by $\hat{\tau}_1,\ldots,\hat{\tau}_k$ free and separating?

Proof. Again, let us write $g = (\hat{\tau}_1,\ldots,\hat{\tau}_k) : [0,1]^n \to [0,1]^k$. Let $R$ be the range of $g$. We first claim that $A$ is separating and free iff $A$ is separating and isomorphic to $\mathcal{M}([0,1]^n)$.

For the nontrivial direction, as repeatedly noted, since $A$ is separating and $A \cong \mathcal{M}(R)$ then $g$ is a homeomorphism of $[0,1]^n$ onto $R$. Since by [20, 4.18], $R$ is homeomorphic to the maximal spectral space $\mu(A)$ of $A$, then $\mu(A)$ is homeomorphic to $[0,1]^n$. Further, for each $m = 1,2,\ldots$, the maximal spectral space of the free $m$-generator MV-algebra $\mathcal{M}([0,1]^m)$ is homeomorphic to $[0,1]^m$. As is well known, whenever $m \neq n$ the $m$-cube $[0,1]^m$ is not homeomorphic to $[0,1]^n$. Since by hypothesis $A$ is free and finitely generated, the only possibility for $A$ to be isomorphic to some free MV-algebra $\mathcal{M}([0,1]^m)$ is for $m = n$, which settles our claim.

To conclude the proof, by Theorem 4.4, $A$ is free and separating iff $A$ is (separating and) equal to $\mathcal{M}([0,1]^n)$. This is decidable, by Theorems 3.1 and 4.8.
Problem 4.10. Prove or disprove the decidability of the following problems:
(a) INSTANCE: MV-terms \(\tau_1, \ldots, \tau_k\) in the variables \(X_1, \ldots, X_n\).
   QUESTION: Is the subalgebra \(A\) of \(M([0, 1]^n)\) generated by \(\hat{\tau}_1, \ldots, \hat{\tau}_k\) free?
(b) INSTANCE: MV-terms \(\tau_1, \ldots, \tau_k\) in the variables \(X_1, \ldots, X_n\).
   QUESTION: Is the subalgebra \(A\) of \(M([0, 1]^n)\) generated by \(\hat{\tau}_1, \ldots, \hat{\tau}_k\) isomorphic to \(M([0, 1]^n)\)?

By a quirk of fate, replacing isomorphism by equality in Problem (b) we have:

Proposition 4.11. The following problem is decidable:
INSTANCE: MV-terms \(\tau_1, \ldots, \tau_k\) in the variables \(X_1, \ldots, X_n\).
QUESTION: Does the subalgebra \(A\) of \(M([0, 1]^n)\) generated by \(\hat{\tau}_1, \ldots, \hat{\tau}_k\) coincide with \(M([0, 1]^n)\)?

Proof. By Theorem 4.4, \(A = M([0, 1]^n)\) iff \(A\) is separating and isomorphic to \(M([0, 1]^n)\) iff \(A\) is separating and free, by the claim in the proof of Proposition 4.9. The latter conjunction of properties is decidable, by the same proposition. \(\square\)

5. Subalgebras of \(M([0, 1]^n)\) and Rational Triangulations of \([0, 1]^n\)

In this section a method is introduced to write down a list of MV-terms \(\tau_1, \ldots, \tau_k\) in the variables \(X_1, \ldots, X_n\) in such a way that the subalgebra of \(M([0, 1]^n)\) generated by the McNaughton functions \(\hat{\tau}_1, \ldots, \hat{\tau}_k\) is simultaneously separating and distinct from \(M([0, 1]^n)\). Conversely, every finitely generated separating proper subalgebra of \(M([0, 1]^n)\) is obtainable by this method.

In combination with Theorem 3.2, a large class of projective MV-algebras can be effectively introduced by this method.

The procedure starts with a rational triangulation \(\Delta\) of \([0, 1]^n\) equipped with a set \(H\) of functions \(f \in M([0, 1]^n)\), called “hats”. Each hat of \(H\) is pyramid-shaped and linear on each simplex of \(\Delta\). One then lets \(A\) be the algebra generated by \(\mathcal{H}\). In more detail:

Definition 5.1. A weighted triangulation of \([0, 1]^n\) is a pair \((\Delta, (a_1, \ldots, a_u))\), where \(\Delta\) is a triangulation of \([0, 1]^n\) with rational vertices \(v_1, \ldots, v_u\) and their associated positive integers \(a_1, \ldots, a_u\), where for each \(i = 1, \ldots, u\) \(a_i\) is a divisor of \(\text{den}(v_i)\). We write \((\Delta, a)\) as an abbreviation of \((\Delta, (a_1, \ldots, a_u))\).

The function \(h_i : [0, 1]^n \rightarrow [0, 1]\) which is linear on every simplex of \(\Delta\) and satisfies \(h_i(v_i) = a_i/\text{den}(v_i)\) and \(h_i(v_j) = 0\) for each \(j \neq i\) is called the \(i\)th hat of \((\Delta, a)\).

Given a weighted triangulation \((\Delta, a)\) of \([0, 1]^n\), the set of its hats is denoted \(\mathcal{H}_{\Delta,a}\). If each linear piece of every hat \(h_i \in \mathcal{H}_{\Delta,a}\) has integer coefficients, (i.e., \(h_i \in M([0, 1]^n)\)), we say that the set \(\mathcal{H}_{\Delta,a}\) is basic.

Lemma 5.2. (i) Let \(T\) be an \(n\)-simplex with rational vertices \(w_0, \ldots, w_n \in [0, 1]^n\). For each \(i = 0, \ldots, n\) let \(l_i : T \rightarrow [0, 1]\) be the linear function satisfying \(l_i(w_i) = 1/\text{den}(w_i)\) and \(l_i(w_j) = 0\) for \(j \neq i\). Then \(T\) is regular iff \(l_i\) has integer coefficients, for each \(i = 0, \ldots, n\).

(ii) Let \(\Delta\) be a regular triangulation of \([0, 1]^n\) with vertices \(v_1, \ldots, v_u\). For each \(i = 1, \ldots, u\) let \(a_i \geq 1\) be a divisor of \(\text{den}(v_i)\). Let \(a = (a_1, \ldots, a_u)\). Then \((\Delta, a)\) is a weighted triangulation and \(\mathcal{H}_{\Delta,a}\) is a basic set.

(iii) There is an effective (= Turing-computable) procedure to test if a weighted triangulation \((\Delta, a)\) of \([0, 1]^n\) determines a basic set \(\mathcal{H}_{\Delta,a}\).

Proof. (i) Let \(M\) be the \((n+1) \times (n+1)\) matrix whose \(i\)th row consists of the integer coordinates of the homogeneous correspondent \(\hat{w}_i\) of \(w_i\). Assume \(T\) is not regular. By definition, \(|\text{det}(M)| \geq 2\).

The absolute value of the determinant of the inverse matrix \(M^{-1}\) is a rational number lying in the open interval \((0, 1)\). So \(M^{-1}\) is not an integer matrix. Since, the \(i\)th column of \(M^{-1}\) yields the coefficients of the linear function \(l_i\), not all these functions can have integer coefficients. Conversely, if \(T\) is regular then \(M^{-1}\) is an integer matrix, whose columns yield the coefficients of the linear functions \(l_i, i = 1, \ldots, n\).
(ii) Evidently, $(\Delta, a)$ is a weighted triangulation. Fix an $n$-simplex $T$ of $\Delta$ with its vertices $w_0, \ldots, w_n$ and corresponding linear functions $l_0, \ldots, l_n$. By (i), the coefficients of each $l_i$ are integers, and so are the coefficients of $a_i l_i$. Thus, $H_{\Delta, a}$ is a basic set.

(iii) For every $n$-simplex $T$ of $\Delta$ let $M_T$ be the $(n+1) \times (n+1)$ integer-valued matrices whose rows are the homogeneous correspondents of the vertices of $T$. Let $D_T$ be the $(n+1) \times (n+1)$ diagonal matrix whose diagonal entries are given by the subsequence of $a$ associated to the vertices of $T$. The rational matrix $M_T^{-1} D$ is effectively computable from the input data $(\Delta, a)$. Arguing as in (i), it is easy to see that the set $H_{\Delta, a}$ is basic iff $M_T^{-1} D$ is an integer matrix for each $T$. $\square$

The following is an example of a basic set $H_{\Delta, a}$ where the triangulation $\Delta$ is not regular.

**Example 5.3.** Fix an integer $u \geq 3$, and let $V = \{k/u \mid k = 0, 1, \ldots, u\}$. Let $\Delta$ be the rational triangulation of $[0,1]$ whose vertices are precisely those in $V$. Assume each vertex $v_k$ of $\Delta$ is associated to the integer $a_k = \text{den}(v_k)$. We then have a weighted triangulation $(\Delta, a)$ of $[0,1]$ and $H_{\Delta, a}$ is a basic set. For $k = 1, \ldots, u - 1$, the hat $h_k$ of $H_{\Delta, a}$ is a piecewise linear function with four linear pieces, connecting the five points of the unit square $(0,0), ((k-1)/u,0), (k/u,1), ((k+1)/u,0), (1,0)$. Each hat $h_k$ of $H_{\Delta, a}$ has value 1 at $v_k$. In detail:

$$h_k(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{k-1}{u} \\ ux - (k - 1) & \text{if } \frac{k-1}{u} \leq x < \frac{k}{u} \\ -ux + k + 1 & \text{if } \frac{k}{u} \leq x < \frac{k+1}{u} \\ 0 & \text{if } \frac{k+1}{u} \leq x \leq 1. \end{cases}$$

Further,

$$h_0(x) = \begin{cases} -ux + 1 & \text{if } 0 \leq x < \frac{1}{u} \\ 0 & \text{if } \frac{1}{u} \leq x \leq 1 \end{cases}, \quad h_u(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{u-1}{u} \\ ux - (u - 1) & \text{if } \frac{u-1}{u} \leq x \leq 1. \end{cases}$$

A moment’s reflection shows that the subalgebra of $M([0,1])$ generated by $H_{\Delta, a}$ is separating and differs from $M([0,1])$.

The next two results show that basic sets $H_{\Delta, a}$ generate all possible separating proper subalgebras of free MV-algebras and unital $\ell$-groups:

**Theorem 5.4.** Let $A$ be a subalgebra of $M([0,1]^n)$.

(i) $A$ is finitely generated, separating, and distinct from $M([0,1]^n)$ iff $A$ is generated by a basic set $H_{\Delta, a}$, for some weighted triangulation $(\Delta, a) = (\Delta, (a_1, \ldots, a_n))$ of $[0,1]^n$ such that $a_j \neq 1$ for some $j = 1, \ldots, u$.

(ii) If $A$ is finitely generated, separating, and distinct from $M([0,1]^n)$ then for every weighted triangulation $(\Delta, a) = (\Delta, (a_1, \ldots, a_n))$ of $[0,1]^n$ such that $H_{\Delta, a}$ is a basic generating set of $A$, it follows that $a_j \neq 1$ for some $j = 1, \ldots, u$.

**Proof.** (i) $(\Rightarrow)$ We have only to check $A \neq M([0,1]^n)$. Let $v_1, \ldots, v_u$ be the vertices of $\Delta$. For each $f \in A$ the value $f(v_j)$ is an integer multiple of $a_j / \text{den}(v_j)$, whence $f(v_j) \neq 1 / \text{den}(v_j)$. We claim that some function in $M([0,1]^n)$ attains the value $1 / \text{den}(v_j)$ at $v_j$. As a matter of fact, desingularization as in [17] or [20, 5.2] yields a regular triangulation $\Sigma$ of $[0,1]^n$ such that $v_j$ is one of the vertices of $\Sigma$. Let $h_j : [0,1]^n \rightarrow [0,1]$ be the Schauder hat of $\Sigma$ at $v_j$, as in [8, 9.1.3–9.1.5]. By definition, $h_j$ is linear on every simplex of $\Sigma$, $h_j = 1 / \text{den}(v_j)$ and $h_j(v_i) = 0$ for all other vertices of $\Sigma$. Our claim is settled.

By [8, 9.1.4], $h_j$ belongs to $M([0,1]^n)$. So $A$ is strictly contained in $M([0,1]^n)$.

$(\Leftarrow)$ Let $\{g_1, \ldots, g_k\}$ be a generating set of $A$. Let $g = (g_1, \ldots, g_k) : [0,1]^n \rightarrow [0,1]^k$, and $R$ be the range of $g$. Since $A$ is separating then $g^{-1}$ is a piecewise linear homeomorphism of $R$ onto $[0,1]^n$ and each linear piece of $g^{-1}$ has rational coefficients—just because each linear piece of $g$ has integer coefficients. Let $\nabla$ be a regular triangulation of $M(R)$ such that $g^{-1}$ is linear over every simplex of $\nabla$. The computability of $\nabla$ follows by direct inspection of the proof of [20, 2.9].
Let \( w_1, \ldots, w_u \) be the vertices of \( \nabla \). For each \( i = 1, \ldots, u \) let \( v_i = g^{-1}(w_i) \). Since \( g \) has integer coefficients there is an integer \( 1 \leq a_i \) such that \( \text{den}(v_i) = a_i \cdot \text{den}(w_i) \). The set of simplexes \( \Delta = \{ g^{-1}(T) \subseteq [0,1]^n \mid T \in \nabla \} \) is a rational triangulation of \([0,1]^n\). Since \( \mathcal{M}([0,1]^n) \) strictly contains \( A \), by Theorem 4.4 it is impossible for \( A \) to be isomorphic to \( \mathcal{M}([0,1]^n) \). Since by \([20, 3.6]\) \( A \cong \mathcal{M}(R) \), then \( \mathcal{M}(R) \) is not isomorphic to \( \mathcal{M}([0,1]^n) \). By duality \([20, 3.10]\), \([0,1]^n \) is not \( \mathbb{Z} \)-homeomorphic to \( R \). As observed in proof of Theorem 4.8, \( g \) is not a \( \mathbb{Z} \)-homeomorphism of \([0,1]^n \) onto \( R \). By \([20, 3.15]\) there is a rational point \( r \in [0,1]^n \) such that \( \text{den}(g(r)) \) is a divisor of \( \text{den}(r) \) different from \( \text{den}(r) \). Stated otherwise, \( \text{den}(r) = m \cdot \text{den}(g(r)) \) with \( m \neq 1 \). By \([20, 5.2]\), it is no loss of generality to assume that \( \nabla \) has \( g(r) \) among its vertices. Thus, for some \( j \) we can assume that \( r \) is the \( j \)th vertex of \( \Delta \) and write

\[
r = v_j, \quad g(r) = w_j, \quad 1 \neq a_j = \frac{\text{den}(v_j)}{\text{den}(w_j)}.
\]

As in the proof of (i) above, let \( \mathcal{H}_\nabla = h_1, \ldots, h_u \) be the set of Schauder hats of \( \nabla \). By Lemma 5.2(i), for every \( i = 1, \ldots, u \) each linear piece of \( h_i \) has integer coefficients. By \([20, 5.8]\), \( \mathcal{H}_\nabla \) generates \( \mathcal{M}(R) \). By construction of \( \nabla \), the composite function \( h_i \circ g \) belongs to \( \mathcal{M}([0,1]^n) \), has value \( a_i / \text{den}(v_i) \geq 1/\text{den}(v_i) \) at \( v_i \), has value zero at any other vertex of \( \Delta \), and is linear over every simplex of \( \Delta \). Therefore, the weighted triangulation \( (\Delta, (a_1, \ldots, a_u)) = (\Delta, a) \) determines the basic set

\[
\mathcal{H}_{\Delta, a} = \mathcal{H}_\nabla \circ g = \{ h_i \circ g \mid h_i \in \mathcal{H}_\nabla, \ i = 1, \ldots, u \},
\]

which generates \( A \), just as \( \mathcal{H}_\nabla \) generates \( \mathcal{M}(R) \cong A \).

(ii) We argue by cases:

In case \( \Delta \) is not regular, let \( T = \text{conv}(w_0, \ldots, w_n) \) be an \( n \)-simplex in \([0,1]^n \) that fails to be regular. By hypothesis, the linear pieces of each hat of \( \mathcal{H}_{\Delta, a} \) have integer coefficients, and so do, in particular, the linear functions \( l_0, \ldots, l_n : \mathbb{R}^n \to \mathbb{R} \) given by the following stipulations, for each \( t = 0, \ldots, n : \)

- \( l_t \) is linear over \( T \),
- \( l_t(w_t) = a_t / \text{den}(v_t) \), and
- \( l_t(w_s) = 0 \) for each \( s \neq t \).

By Lemma 5.2(i), not all \( a_t \) can be equal to 1.

In case \( \Delta \) is regular, suppose \( a_i = 1 \) for each \( i = 1, \ldots, u \) (absurdum hypothesis). Then \( \mathcal{H}_{\Delta, a} \) is precisely the set of Schauder hats of \( \Delta \). By \([20, 5.8]\), \( \mathcal{H}_{\Delta, a} \) generates \( \mathcal{M}([0,1]^n) \), which contradicts the hypothesis \( A \neq \mathcal{M}([0,1]^n) \).

\( \square \)

**Corollary 5.5.** Let \( (G, u) \) be a unital \( \ell \)-subgroup of \( \mathcal{M}_{\text{group}}([0,1]^n) \).

(i) \( (G, u) \) is finitely generated, separating, and distinct from \( \mathcal{M}_{\text{group}}([0,1]^n) \) iff \( (G, u) \) is generated by a basic set \( \mathcal{H}_{\Delta, a} \) for some weighted triangulation \( (\Delta, a) \) of \([0,1]^n \) such that \( a_j \neq 1 \) for some \( j \).

(ii) If \( (G, u) \) is finitely generated, separating, and distinct from \( \mathcal{M}_{\text{group}}([0,1]^n) \) then for every weighted triangulation \( (\Delta, a) \) of \([0,1]^n \) such that \( \mathcal{H}_{\Delta, a} \) is a basic generating set of \( (G, u) \), it follows that \( a_j \neq 1 \) for some \( j \).

\( \square \)

6. **Computing a Basis of a Finitely Generated Subalgebra of \( \mathcal{M}([0,1]^n) \)**

By \([20, 6.6]\), every finitely generated subalgebra \( A \) of \( \mathcal{M}([0,1]^n) \) is finitely presented, i.e., \( A \) is a principal quotient of a free MV-algebra. Equivalently, \([20, 6.1, 6.3]\); \( A \) has a basis, i.e., a set of nonzero elements \( B = \{ b_1, \ldots, b_z \} \), together with integers \( 1 \leq m_1, \ldots, m_z \) (called “multipliers”) such that

(a) \( B \) generates \( A \).

(b) \( m_1 b_1 + \cdots + m_z b_z = 1 \) where the sum is computed in the unital \( \ell \)-group \( (G, u) \) of \( A \) given by \( \Gamma(G, u) = A \). See \([20, 6.1(iii)]\).
(c) For each k-element subset \( C = b_{i_1}, \ldots, b_{i_k} \) of \( B \) with \( b_{i_1} \wedge \ldots \wedge b_{i_k} \neq 0 \), the set of maximal ideals of \( A \) containing \( B \setminus C \) is homeomorphic to a \((k-1)\)-simplex, \((k = 1, 2, \ldots)\).

We now prove that every basic set is a basis of the MV-algebra it generates.

**Proposition 6.1.** Suppose the weighted triangulation \((\Delta, (a_1, \ldots, a_n)) = (\Delta, \mathbf{a})\) of \([0,1]^n\) determines the basic set \(\mathcal{H}_{\Delta, \mathbf{a}}\). Let \(v_1, \ldots, v_n\) be the vertices of \(\Delta\). Then the MV-subalgebra \(A\) of \(\mathcal{M}(\{0,1\}^n)\) generated by \(\mathcal{H}_{\Delta, \mathbf{a}}\) is separating, and \(\mathcal{H}_{\Delta, \mathbf{a}}\) is a basis of \(A\), whose multipliers \(m_i\) coincide with \(\text{den}(v_i)/a_i\) for each \(i = 1, \ldots, n\).

**Proof.** \(A\) is a separating subalgebra of \(\mathcal{M}([0,1]^n)\) because it is generated by the hats of a triangulation of \([0,1]^n\). From the definition of \(\mathcal{H}_{\Delta, \mathbf{a}}\) together with \([20, 4.18]\) and \([20, 6.1(\text{ii}')]\), it follows that the conclusion of condition (c) above is equivalent to saying that the set of points \(x \in [0,1]^n\) such that \(m_{i_1}b_{i_1}(x) + \cdots + m_{i_k}b_{i_k}(x) = 1\) is homeomorphic to a \((k-1)\)-simplex. It is now easy to see that \(\mathcal{H}_{\Delta, \mathbf{a}}\) satisfies condition (c). (See the proof of \([20, 5.8(\text{ii})]\)). Condition (b) is trivially satisfied. Therefore, \(\mathcal{H}_{\Delta, \mathbf{a}}\) is a basis of \(A\). \(\square\)

**Remark 6.2.** By Lemma 5.2(iii), one can decide whether a weighted triangulation \(\Delta\) with multiplicities \(\mathbf{a}\) determines a basic set \(\mathcal{H}_{\Delta, \mathbf{a}}\). By contrast, the decidability of the problem whether \(\{\hat{\tau}_1, \ldots, \hat{\tau}_k\}\) is a basis of the MV-algebra it generates is open. See \([20, \text{p.213}]\). Interestingly enough, perusal of \([20, \S 6.5]\) shows that every basis of \(A \subseteq \mathcal{M}([0,1]^n)\) becomes a basic generating set of \(A\) after finitely many binary algebraic blowups.

In the light of the foregoing proposition, the following theorem provides an effective method to transform every generating set of \(A \subseteq \mathcal{M}([0,1]^n)\) into a basis of \(A\):

**Theorem 6.3** (Effective basis generation). Every list of terms \(\tau_1, \ldots, \tau_k\) in the variables \(X_1, \ldots, X_n\), such that the subalgebra \(A \subseteq \mathcal{M}([0,1]^n)\) generated by the McNaughton functions \(\hat{\tau}_1, \ldots, \hat{\tau}_k\) is separating, can be effectively transformed into a weighted triangulation \((\Delta, \mathbf{a}) = (\Delta, (a_1, \ldots, a_n))\) of the \(n\)-cube \([0,1]^n\), with vertices \(v_1, \ldots, v_z\), in such a way that \(\mathcal{H}_{\Delta, \mathbf{a}}\) is a basis generating set of \(A\).

*Proof.* Let \(g = (\hat{\tau}_1, \ldots, \hat{\tau}_k) : [0,1]^n \to [0,1]^k\), and \(R = \text{range of } g\). Since \(A\) is separating, \(g\) is a piecewise linear homeomorphism onto \(R\). The transformation proceeds as follows:

(i) Arguing as in the proof of Theorem 3.1, we first compute from \(\tau_1, \ldots, \tau_k\) a regular triangulation \(\Sigma\) of \([0,1]^n\) such that \(g\) is linear over every simplex of \(\Sigma\) (also see \([20, 18.1]\)). \(\Sigma\) is written down as the list of the sets of rational vertices of its simplexes.

(ii) We write down the rational triangulation \(g(\Sigma) = \{g(T) \mid T \in \Sigma\}\) of \(R\) given by the \(g\)-images of the simplexes of \(\Sigma\). This is effective, because the linear pieces of each function \(\hat{\tau}_i\) are computable from the MV-term \(\tau_i\), and so is the linear piece \(T_f\) of \(g(T)\).

(iii) Using desingularization \([20, 2.9, 18.1]\), we subdivide \(g(\Sigma)\) into a regular triangulation \(\nabla\) of \(R\). Let \(\Delta\) be the rational subdivision of \(\Sigma\) defined by \(g(\Delta) = \nabla\). Since \(g\) determines a computable one-to-one correspondence between the \(n\)-simplexes of \(\nabla\) and those of \(\Delta\), then also \(\Delta\) is computable.

(iv) Let us write \(\mathcal{H}_{\nabla}\) for the set of Schauder hats of \(\nabla\), \([20, 5.7]\). For each vertex \(w\) of \(\nabla\), we can effectively write down an MV-term \(\gamma_w(X_1, \ldots, X_k)\) such that the restriction to \(R\) of the associated McNaughton function \(\hat{\gamma}_w\) is the hat of \(\mathcal{H}_{\nabla}\) with vertex \(w\). Thus \(\mathcal{H}_{\nabla}\) can be effectively computed. The routine verification can be made arguing as in the proof of \([8, 9.1.4]\).

(v) Observe that \(g^{-1}\) is linear on every simplex of \(\nabla\), and is an explicitly given piecewise linear function. As a matter of fact, for each \(n\)-simplex \(U\) of \(\nabla\), the map \(g^{-1}(U)\) is an \(n\)-tuple of linear polynomials with rational coefficients that can be effectively computed from the \(k\)-tuple of linear polynomials with integer coefficients \((\hat{\tau}_1, \ldots, \hat{\tau}_k) | g^{-1}(U)\). For each \(i = 1, \ldots, k\), arguing by induction on the number of connectives of all subterms of \(\tau_i\), one effectively computes the linear function \(\hat{\tau}_i | g^{-1}(U)\).

(vi) Let \(v_1, \ldots, v_z\) be the vertices of \(\Delta\). For each \(j = 1, \ldots, z\), we set

\[
a_j = \text{den}(v_j)/\text{den}(g(v_j)).
\]
Since \( g \) is piecewise linear with integer coefficients, each \( a_j \) is an integer. Recalling that \( \circ \) denotes composition, the weighted triangulation \((\Delta, \mathbf{a})\) determines the basic set \( \mathcal{H}_{\Delta, \mathbf{a}} = \{ h \circ g \mid h \in \mathcal{H}_\mathcal{R} \} \). By [20, 5.8], \( \mathcal{H}_\mathcal{R} \) generates \( \mathcal{M}(\mathcal{R}) \), whence \( \mathcal{H}_{\Delta, \mathbf{a}} \) generates \( \mathcal{A} \).

Since \( \Delta \) is explicitly given by listing the sets of the vertices of its simplexes, and the integers \( a_1, \ldots, a_s \) associated to the vertices \( v_1, \ldots, v_z \) of \( \Delta \) are computed from (3), the proof of [8, 9.1.4] yields an effective procedure to write down MV-terms \( \sigma_1, \ldots, \sigma_z \) such that \( \{ \hat{\sigma}_1, \ldots, \hat{\sigma}_z \} = \mathcal{H}_{\Delta, \mathbf{a}} \).

\[ \square \]

7. Recognizing subalgebras of \( \mathcal{M}([0, 1]^n) \)

A main application of Theorem 6.3 is given by the following decidability result:

**Theorem 7.1.** The following problem is decidable:

**INSTANCE:** MV-terms \( \tau_1, \ldots, \tau_k \) and \( \sigma_1, \ldots, \sigma_l \) in the variables \( X_1, \ldots, X_n \), such that both sets \( \{ \tau_1, \ldots, \tau_k \} \) and \( \{ \sigma_1, \ldots, \sigma_l \} \) separate points (the separation property being decidable by Theorem 3.1).

**QUESTION:** Does the subalgebra \( \mathcal{A} \) of \( \mathcal{M}([0, 1]^n) \) generated by \( \tau_1, \ldots, \tau_k \) coincide with the subalgebra \( \mathcal{A}' \) of \( \mathcal{M}([0, 1]^n) \) generated by \( \sigma_1, \ldots, \sigma_l \)?

**Proof.** It is enough to decide \( \mathcal{A} \supseteq \mathcal{A}' \). By Theorem 6.3 we can safely suppose that for some weighted triangulations \((\Delta, \mathbf{a})\) and \((\Delta', \mathbf{a}')\), the MV-algebras \( \mathcal{A} \) and \( \mathcal{A}' \) are respectively generated by the basic sets

\[ \mathcal{H}_{\Delta, \mathbf{a}} = \{ h_1, \ldots, h_r \} \quad \text{and} \quad \mathcal{H}_{\Delta', \mathbf{a}'} = \{ h'_1, \ldots, h'_s \}. \] (4)

Let \( \mathbf{a} = (a_1, \ldots, a_r) \), \( \mathbf{a}' = (a'_1, \ldots, a'_s) \). Let \( h = (h_1, \ldots, h_r) : [0, 1]^n \to [0, 1]^k \) and \( R = h([0, 1]^n) \) be the range of \( h \). The separation hypothesis is to the effect that \( h \) is a piecewise linear homeomorphism of \([0, 1]^n\) onto \( R \). For each \( i = 1, \ldots, r \), all linear pieces of \( h_i \) have integer coefficients. As in the proof of Theorem 3.1 (also see [20, 18.1]), we now compute a rational subdivision \( \Delta^* \) of \( \Delta' \) such that \( h \) is linear on each simplex of \( \Delta' \) (by definition of \( \mathcal{H}_{\Delta, \mathbf{a}}, \Delta^* \) is automatically a subdivision of \( \Delta \)). Then the set

\[ h(\Delta^*) = \{ h(T) \mid T \in \Delta^* \} \]

is a rational triangulation of \( R \). Let \( \nabla \) be the regular subdivision of \( h(\Delta^*) \) obtained by the desingularization process in [20, 2.9]. Let \( w_1, \ldots, w_u \), be the vertices of \( \nabla \) and \( p_1, \ldots, p_u \) their respective Schauder hats. Since all steps of the desingularization process in [20, 2.9] are effective, \( \nabla \) is effectively computable. Upon writing

\[ \Sigma = h^{-1}(\nabla) = \{ h^{-1}(U) \mid U \in \nabla \}, \]

we get a rational triangulation \( \Sigma \) of \([0, 1]^n\) that jointly subdivides \( \Delta \) and \( \Delta' \) (whence \( h \) is linear on each simplex of \( \Sigma \)). Evidently, \( \Sigma \) is computable as the list of the sets of vertices of its simplexes. Let \( v_1 = h^{-1}(w_1), \ldots, v_u = h^{-1}(w_u) \) be the vertices of \( \Sigma \). The piecewise linear functions

\[ q_i = p_i \circ h, \ldots, q_u = p_u \circ h \]

have integer coefficients. Since \( \text{den}(h(v_i)) \) is a divisor of \( \text{den}(v_i) \), the rational number \( q_i(v_i) = 1/\text{den}(h(v_i)) \) is an integer multiple of \( 1/\text{den}(v_i) \), say \( Z \ni c_i = \text{den}(v_i)/\text{den}(h(v_i)), \) (\( i = 1, \ldots, u \)).

Letting now \( \mathbf{c} = (c_1, \ldots, c_u) \), we have a weighted triangulation \((\Sigma, \mathbf{c})\) such that \( \mathcal{H}_{\Sigma, \mathbf{c}} \) is a basic generating set of \( \mathcal{A} \), just as the set \( \{ p_1, \ldots, p_u \} \) generates \( \mathcal{M}(\mathcal{R}) \), by [20, 5.8].

Recalling (4), we are now ready to decide whether \( \mathcal{A} \supseteq \mathcal{A}' \) as follows:

For every \( j = 1, \ldots, s \) and \( n \)-simplex \( T \in \Sigma \) let \( f_{T, j} = h'_j \mid T \) be the restriction to \( T \) of the \( j \)th hat of the basic set \( \mathcal{H}_{\Delta', \mathbf{a}'} \). Observe that \( f_{T, j} \) is linear on \( T \) and has linear coefficients. We now check whether \( f_{T, j} \) is obtainable as a sum of positive (\( > 0 \)) integer multiples of some of the hats \( q_i \mid T \). Let \( V_T \) be the set of vertices of \( \Sigma \) lying in \( T \), with the corresponding set of hats

\[ H_T = \{ q_v \mid v \in V_T \} \subseteq \{ q_1, \ldots, q_u \} \]

the hats of the basic set \( \mathcal{H}_{\Sigma, \mathbf{c}} \) of \( \mathcal{A} \).

**Case 1:** For each \( j = 1, \ldots, s \), \( T \in \Sigma \) and \( v \in V_T \), \( q_v(v) \) is a divisor of \( f_{T, j}(v) \).
Then the linear function $f_{T,j}$ coincides (over $V_T$, and hence) over $T$ with a suitable sum of positive integer multiples of the hats of $H_T$. Direct inspection shows that $h'_j$ is a sum of integer multiples of some hats in $\mathcal{H}_{\Sigma,\Sigma}$. It follows that $\mathcal{H}_{\Sigma,\Sigma}$ generates $A'$, and we conclude $A \supseteq A'$.

Case 2: For some $j = 1, \ldots, s$, $T \in \Sigma$ and $v \in V_T$, $q_v(v)$ is not a divisor of $f_{T,j}(v)$.

The possible values of functions in $A$ at $v$ are integer multiples of $q_v(v)$, because all other hats of $\mathcal{H}_{\Sigma,\Sigma}$ vanish at $v$. Therefore, $h_j$ does not belong to $A$, whence the inclusion $A \supseteq A'$ fails.

This completes the decision procedure for $A = A'$.

$\square$

Problem 7.2. Prove or disprove the decidability of the following problem:

**INSTANCE**: MV-terms $\tau_1, \ldots, \tau_k$ and $\sigma_1, \ldots, \sigma_l$ in the variables $X_1, \ldots, X_n$.

**QUESTION**: Is the subalgebra of $\mathcal{M}([0,1]^n)$ generated by $\hat{\tau}_1, \ldots, \hat{\tau}_k$ equal to the subalgebra of $\mathcal{M}([0,1]^n)$ generated by $\hat{\sigma}_1, \ldots, \hat{\sigma}_l$?

8. Conclusions: Two types of presentations

Throughout this paper, MV-algebras $A \subseteq \mathcal{M}([0,1]^n)$ (resp., unital $\ell$-groups $(G, u) \subseteq \mathcal{M}_{\text{group}}([0,1]^n)$) have been “effectively presented” by a finite string of symbols $\tau_1, \ldots, \tau_u$, where each $\tau_i = \tau_i(X_1, \ldots, X_n)$ is an MV-term (resp., a unital $\ell$-group term) in the variables $X_1, \ldots, X_n$.

Traditional finite presentations are instead defined as we did in Section 2, by a single $k$-variable term $\sigma$, letting $Z_\sigma \subseteq [0,1]^k$ be the zeroset $\hat{\sigma}^{-1}(0)$ of the McNaughton function $\hat{\sigma} : [0,1]^k \to [0,1]$ associated to $\sigma$, and setting

$$A_\sigma = \mathcal{M}([0,1]^k)/j_\sigma \cong \mathcal{M}(Z_\sigma),$$

where $j_\sigma$ is the principal ideal of $\mathcal{M}([0,1]^k)$ generated by $\hat{\sigma}$. One similarly defines finite presentations of unital $\ell$-groups, [4, 6, 18].

By [20, 6.6], every finitely generated subalgebra of $\mathcal{M}([0,1]^n)$ is finitely presented, but not every finitely presented MV-algebra is isomorphic to a subalgebra of a free MV-algebra. For instance, $\{0,1\} \times \{0,1\}$ (and more generally, every non-simple finite MV-algebra) is finitely presented but is not isomorphic to a subfree MV-algebra, because its maximal spectral space is disconnected. Thus one may reasonably expect that decision problems that are unsolvable for finitely presented $\ell$-groups and open for finitely presented unital $\ell$-groups (equivalently, for finitely presented MV-algebras), turn out to be decidable for separating subalgebras of $\mathcal{M}([0,1]^n)$ presented via their generators $\hat{\tau}_1, \ldots, \hat{\tau}_u$. Here is an example of this state of affairs:

- As shown in [12, Theorem D], for each fixed $k \geq 6$ the property of being a free $k$-generator $\ell$-group is undecidable. This follows from Markov’s celebrated unrecognizability theorems (see [23] for a detailed account.)
- The same problem for unital $\ell$-groups and MV-algebras is open, except for $k = 1$, where the problem is decidable, (see [20, 18.3]).
- Theorem 4.8 shows the decidability of the problem whether a finitely generated separating subalgebra of $\mathcal{M}([0,1]^n)$ is isomorphic to $\mathcal{M}([0,1]^n)$.

The separation hypothesis plays a crucial role in most decidability results of the earlier sections. As a matter of fact, the final two results of this paper will show that (without the separation hypothesis), for all decision problems concerning finitely generated subalgebras $A$ of free algebras, it is immaterial whether $A$ is presented by a list of generators $\hat{\tau}_1, \ldots, \hat{\tau}_u$ or by a principal ideal $j_\sigma$ of some free algebra.

**Theorem 8.1.** There is a computable transformation of every presentation of an MV-algebra $A \subseteq \mathcal{M}([0,1]^n)$ by a list of generators $\hat{\tau}_1, \ldots, \hat{\tau}_u$, into a presentation of an isomorphic copy $A_\sigma$ of $A$ as a principal quotient of some finitely generated free MV-algebra as in (5).

**Proof.** Following [20, 18.1], from the input MV-terms $\tau_1, \ldots, \tau_u$ we first compute the rational polyhedron $R \subseteq [0,1]^u$ given by the range of the function $q = \langle \hat{\tau}_1, \ldots, \hat{\tau}_u \rangle$. By [20, 3.6], $A \cong \mathcal{M}(R)$. Next, in the light of [20, 2.9, 18.1], we list the sets of vertices of the simplexes of a regular triangulation $\Delta$ of $[0,1]^k$ such that the set $\Delta_R = \{ T \in \Delta \mid T \subseteq R \}$ is a triangulation of $R$. 
Without loss of generality, $\Delta_R$ is full: any simplex of $\Delta$ all of whose vertices lie in $\Delta_R$ is a simplex of $\Delta_R$. Following the proof of [8, 9.1.4(ii)], we compute $\text{MV}$-terms $\rho_1, \ldots, \rho_w$ in the variables $Y_1, \ldots, Y_u$, whose associated McNaughton functions $\hat{\rho}_1, \ldots, \hat{\rho}_w$ constitute the set $\mathcal{H}_\Delta$ of Schauder hats of $\Delta$, as defined in [8, 9.1.3]. The $\oplus$-sum of all hats with vertices not belonging to $R$ (coincides with their pointwise sum taken in the unital $\ell$-group $\mathcal{M}_{\text{group}}(R)$) and provides an $\text{MV}$-term $\sigma(Y_1, \ldots, Y_u)$, together with its associated McNaughton function $\hat{\sigma} \in \mathcal{M}([0, 1]^n)$. Since $\Delta_R$ is full, the zero set $Z_{\sigma}$ of $\hat{\sigma}$ coincides with $R$. The isomorphisms

$$A \cong \mathcal{M}(g([0, 1]^n)) = \mathcal{M}(R) = \mathcal{M}(Z_{\sigma}) \cong \mathcal{M}([0, 1]^n)/j_{\sigma} = A_{\sigma},$$

yield a finite presentation of $A$ as a principal quotient of $\mathcal{M}([0, 1]^n)$. \hfill \Box

Conversely, we can prove:

**Theorem 8.2.** For any arbitrary input $\text{MV}$-term $\sigma = \sigma(Y_1, \ldots, Y_k)$, we have:

(i) It is decidable whether the $\text{MV}$-algebra $A_{\sigma} = \mathcal{M}([0, 1]^k)/j_{\sigma} \cong \mathcal{M}(Z_{\sigma})$ is isomorphic to a subalgebra of a free $\text{MV}$-algebra.

(ii) In case $A_{\sigma}$ is isomorphic to a subalgebra of a free $\text{MV}$-algebra, $\sigma$ can be effectively transformed into a finite list of $\text{MV}$-terms $\tau_i$ in $n$ variables, in such a way that $A_{\sigma}$ is isomorphic to the subalgebra $A$ of the free $\text{MV}$-algebra $\mathcal{M}([0, 1]^n)$ generated by the set of $\tau_i$.

**Proof.** Using a variant of the algorithm $\text{Mod}$ of [20, 18.1], we first compute a regular triangulation $\Delta$ whose support is the zero set $Z_{\sigma} \subseteq [0, 1]^k$ of $\hat{\sigma}$. The proof of (i) and (ii) then proceeds as follows:

(i) In [4, 4.10] it is proved that $A_{\sigma}$ is isomorphic to a (necessarily finitely generated) subalgebra of a free $\text{MV}$-algebra iff the following three conditions hold:

(a) $Z_{\sigma}$ intersects the set of vertices of $[0, 1]^k$;

(b) $Z_{\sigma}$ is connected;

(c) $Z_{\sigma}$ is strongly regular.

From $\Delta$, explicitly given by the sets of vertices of its simplexes, conditions (a)-(b) can be immediately checked. To check condition (c), for each maximal simplex $M$ of $\Delta$, one checks whether the greatest common divisor of the denominators of the vertices of $M$ equals $1$. This completes the proof of (i).

(ii) Assuming all three checks (a)-(c) are successful, following the proof of the characterization theorem [4, 4.10] we output the desired $\text{MV}$-terms $\tau_i$ through the following steps:

(d) From $\Delta$ we compute a collapsible triangulation $\Delta'$ whose support lies in the cube $[0, 1]^n$, for some suitably large integer $n$.

(e) Next we compute a simplicial map from $\Delta'$ to $\Delta$, providing a $Z$-map $\eta$ from the support of $\Delta'$ onto the support $Z_{\sigma}$ of $\Delta$.

(f) Following now the proof of [6, 5.1], we compute a $Z$-map $\gamma$ which is a retraction of $[0, 1]^n$ onto the support of $\Delta'$.

(g) Letting $\pi_i : [0, 1]^k \to [0, 1]$ denote the $i$th coordinate function, we observe that for each $i \in \{1, \ldots, k\}$ the $Z$-map $\pi_i \circ \eta \circ \gamma$ belongs to $\mathcal{M}([0, 1]^n)$. Since both $Z$-maps $\gamma$ and $\eta$ are explicitly given, an application of [8, 9.1.5] yields $\text{MV}$-terms $\tau_1, \ldots, \tau_k$ in the variables $X_1, \ldots, X_n$ such that $\hat{\tau}_i = \pi_i \circ \gamma \circ \eta$.

Let $A$ be the subalgebra of $\mathcal{M}([0, 1]^n)$ generated by $\hat{\tau}_1, \ldots, \hat{\tau}_k$. The final part of the proof of [4, 4.10] yields

$$A_{\sigma} = \mathcal{M}([0, 1]^n)/j_{\sigma} \cong \mathcal{M}(Z_{\sigma}) = \mathcal{M}((\gamma \circ \eta)([0, 1]^n)) \cong A. \hfill \Box$$

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