Simultaneous Partial And Backward Decoding Approach for Two-Level Relay Networks

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Abstract—In this paper, we propose a new achievable rate for two-level relay networks by introducing a new strategy named simultaneous partial and backward decoding. In the proposed method, benefitting from regular encoding/backward decoding strategy, different message parts transmitted in the network are simultaneously decoded at the relays and the receiver. The proposed strategy is shown to achieve better rates than the previously proposed methods based on decode-and-forward and partial decode-and-forward.

I. INTRODUCTION

Decode-and-forward protocol, first introduced by Cover and El Gamal in 1979 for relay channel [1], is a relaying protocol in which the relay can decode the transmitted message by the sender. Partial decode-and-forward was defined in [2] as a special case of the proposed coding scheme by Cover and El Gamal [1, Theorem 7]. In this scheme, the relay can partially decode the transmitted message by the sender. In [3], generalized backward decoding strategies were proposed for the relay channel based on sequential backward decoding and simultaneous backward decoding. The achievable rate of the second strategy was shown to include that of [1, Theorem 7].

There have been a lot of works which apply the proposed decode-and-forward schemes in [1] and [2], to multiple relay networks [4]-[17]. In [5], Gupta and Kumar applied irregular encoding/sequential decoding to multirelay networks in a manner similar to [4]. In [6] and [7], Xie and Kumar developed regular encoding/sliding-window decoding for multiple relays, and showed that their scheme achieves better rates than those of [4] and [5]. Regular encoding/backward decoding first proposed in [8], was generalized to the relay networks in [9]. The achievable rates of the two regular encoding strategies turn out to be the same, however, the delay of sliding-window decoding is much less than that of backward decoding. In [10] and [11], parity-forwarding protocol is introduced and a structured generalization of decode-and-forward strategy for multiple relay networks with feed-forward structure based on such protocol is proposed. In their method, each relay chooses a selective set of previous nodes in the network and decodes all messages of those nodes. Parity forwarding was shown to improve previous decode-and-forward strategies, and it achieves the capacity of new forms of degraded multiple relay networks. In [12], mixed strategy consisting of partial decoding scheme and compress-and-forward is developed for multiple relay networks. In [13] and [14], partial decode-and-forward scheme was extended to multiple relay network and the capacity of multilevel semi-deterministic and orthogonal relay network were established. In [15], a comprehensive partial decoding scheme based on regular encoding/sliding window decoding analysis is proposed, despite [12]-[14], all possible partial decoding states are considered between the different parts of the messages of the source and the relays in a two-level relay network. In [16] and [17], an achievable rate was proposed for two-relay networks based on symmetric relaying, in which the relays are arranged in the same position and no priorities are considered for any of them over the others in receiving the message of the source or the message of the other relays.

In this work, we propose a new achievable rate for two-level relay networks based on using simultaneous partial and backward decoding strategy at the relays and the receiver. In the proposed method, each relay in the network and the receiver can partially decode the message parts transmitted by the previous relay and the transmitter and also these different message parts are simultaneously decoded at the relays and the receiver. Similar to [15] and in contrast with [12], we consider a comprehensive scenario where all possible partial decoding states are considered between different parts of the messages of the source and the relays in a two-level relay network. Simultaneous partial and backward decoding strategy is shown to yield better rates than the previously proposed methods based on decode-and-forward [7], partial decode-and-forward based on regular encoding/sliding window decoding [15] and also the rate proposed in [12] with the omission of compress-and-forward part from it.

II. THE PROPOSED ACHIEVABLE RATE FOR TWO-LEVEL RELAY NETWORKS

In this section, we propose a partial decode-and-forward scheme based on simultaneous regular encoding/backward decoding. In the proposed method, each relay in the network can partially decode the message transmitted by the sender and the previous relay and also different message parts are simultaneously decoded at the relays and the receiver. Individual
parts of the messages transmitted by the sender and the sender are shown in Fig. 1. In this figure, $U_0^1$ denotes common part of the source message that is decoded by both relays. $U_0^2$ denotes private part of the source message that is decoded by the first relay. $U_2$ denotes part of the transmitted message by the first relay that is decoded by the second relay. The role of $U_1^2$ is to transmit $U_0^1$ to the second relay. The role of $U_2$ is to transmit $U_0^1$ to the receiver. The rates of $U_0^1$, $U_2$ are shown by $R_0^1$. The rates of $U_0^1$ and $U_0^2$ are shown by $R_0^1$ and $R_0^2$ respectively. The rate of the part of the message source that is directly decoded by the receiver is shown by $R_0^0$. As it is shown next we don’t need to define any auxiliary random variable for this part of the source message. In the proposed method, $U_0^1$ and $U_0^2$ are decoded simultaneously at the first relay. $U_0^1$ and $U_0^2$ are decoded simultaneously at the second relay. $U_0^1$, $U_0^2$ and $U_0^2$ are decoded simultaneously at the receiver. The proposed rate is shown by the next theorem.

**Theorem 1:** For any relay networks $(X_0 \times X_1 \times X_2, p(y_0, y_1, y_2|x_0, x_1, x_2), Y_0 \times Y_1 \times Y_2)$, the capacity $C$ is lower bounded by

$$C \geq \sup_{p(u_{01}^2, u_{02}^2, u_{12}, u_{20}, x_{01}, x_{12}, x_{20})} \min \{I(X_0 X_1 X_2; Y_0), I(U_0^2 U_1^1; Y_1|X_1 U_1^2 U_2 U_0^2) + I(U_0^2 Y_0^2 U_0^1 U_1^2 X_1 U_2 U_2^2) + I(U_0^2 U_1^1 X_1 U_1^2 U_2 U_0^2) + I(U_0^2 U_0^1 U_1^2 X_1 U_2 U_2^2) + I(U_0^2 U_0^1 U_1^2 Y_2 U_0^2 X_1 U_2 U_2^2) + I(U_0^2 U_0^1 U_1^2 X_1 U_1^2 U_2 U_0^2) + I(U_0^2 U_0^1 U_1^2 Y_2 U_0^2 U_0^1 U_2 U_2^2) + I(U_0^2 U_0^1 X_1 U_1^2 U_2 U_0^2) + I(U_0^2 X_1 X_2 U_0^2 U_0^1 U_2 U_2^2) + I(U_0^2 X_1 X_2 U_0^2 U_0^1 U_2 U_2^2) \} \tag{1}$$

where the supremum is over all joint probability mass functions

$$p(x_0, u_{01}^2, u_{02}^2, u_{12}, u_{20}, x_{01}, x_{12}, x_{20}) = p(x_0 u_{01}^2 u_{02}^2 u_{12} u_{20}) p(u_{01}^2 u_{12} u_{20}) p(x_1 u_{12} u_{20}) p(u_{01}^2 u_{12} u_{20}) p(x_1 u_{12} u_{20}) \tag{2}$$

on $U_0^1 \times U_0^2 \times U_0^1 \times U_1^2 \times U_2 \times X_0 \times X_1 \times X_2$ such that

$$(U_0^1, U_0^2, U_1^1, U_1^2, U_2^0) \rightarrow (X_0, X_1, X_2) \rightarrow (Y_0, Y_1, Y_2) \tag{3}$$

form a Markov chain.

**Proof:** In this encoding scheme, the message of the transmitter is divided into four parts. The first part is decoded only by two relays, the second part is decoded by the first relay, the third part is decoded by the second relay and the fourth part is directly decoded by the receiver. The message of first relay is also divided into two parts. The first part is decoded by second relay and the receiver can only make an estimate of it, while the second part is directly decoded by the receiver. The sender and the relays cooperate in next transmission blocks to remove the receiver’s uncertainty about the previous parts of the message.

We make use of regular block Markov superposition encoding and for decoding we make use of backward decoding scheme [7]. We consider $B$ blocks of transmission, each of $n$ symbols. A sequence of $B - 2$ messages, $w_{00,i} \times w_{01,i} \times w_{02,i} \times w_{03,i} \times w_{04,i}, i \in [1, 2^n R_0^2] \times [1, 2^n R_0^2] \times [1, 2^n R_0^2], i = 1, 2, \ldots, B - 2$, will be sent over the channel in $n B$ transmissions. $w_{01,i}$ is decoded by two relays, $w_{01,i}$ is decoded by the first relay. $w_{02,i}$ is decoded by the second relay. In each $n$-block $b = 1, 2, \ldots, B$, we shall use the same set of codewords. The random codewords to be used in each block are generated as follows:

**Random Coding:**

For any joint probability mass function satisfying (2) and (3), Choose $2^n R_0^2$ i.i.d. $w_{02}$ each with probability $p(w_{02}) = \prod_{i=1}^{n} p(w_{02,i})$. Label these as $w_{02}^{1_1}(m_{20}), m_{20} \in [1, 2^n R_0^2]$. For each $w_{02}(m_{20})$, generate $2^n R_0^2$ i.i.d. $x_{2}^n$ each with probability $p(x_{2}^n|w_{02}) = \prod_{i=1}^{n} p(x_{2,i}|w_{02,i})$. Label these as $x_{2}^n (m_{02}|m_{20}), m_{02} \in [1, 2^n R_0^2]$. For each $w_{02}(m_{20}), m_{20}$, generate $2^n R_0^2$ i.i.d. $x_{2}^n$ with probability $p(u_{12}^n|w_{02}) = \prod_{i=1}^{n} p(u_{12,i}|w_{02,i})$. Label these as $x_{2}^n (m_{12}|m_{20}), m_{12} \in [1, 2^n R_0^2]$. For every $w_{02}(m_{20})$ and $x_{2}^n (m_{12}|m_{20})$, generate $2^n R_0^2$ i.i.d. $x_{1}^n$ each with probability $p(x_{1}^n|w_{02}, x_{2}^n) = \prod_{i=1}^{n} p(x_{1,i}|w_{02,i}, x_{2,i})$. Label these as $x_{1}^n (m_{10}|m_{12}, m_{20}), m_{10} \in [1, 2^n R_0^2]$. For every $w_{02}(m_{20})$ and $x_{1}^n (m_{10}|m_{12}, m_{20})$, generate $2^n R_0^2$ i.i.d $w_{01}$ each with probability $p(w_{01}|w_{02}, x_{1}^n, x_{2}^n, u_{12}) = \prod_{i=1}^{n} p(w_{01,i}|w_{02,i}, x_{1,i}, x_{2,i}, u_{12,i})$. Label these as $w_{01}^{1_1}(m_{10}|m_{12}, m_{20}), m_{10} \in [1, 2^n R_0^2]$. For every $w_{01}(m_{10}, m_{12}, m_{20})$, $x_{1}^n (m_{10}|m_{12}, m_{20})$ and $w_{01}^{1}(w_{01}|m_{12}, m_{20})$, generate $2^n R_0^2$ i.i.d $u_{12}$ each with probability $p(u_{12}^n|w_{01}, x_{1}^n, x_{2}^n, u_{12}) = \prod_{i=1}^{n} p(u_{12,i}|w_{01,i}, x_{1,i}, x_{2,i}, u_{12,i})$.
In the defined codewords, the index \( m_{20} \) represents the index \( m_{12} \) of the previous block. The index \( m_{12} \) represents the index \( w_{01} \) of the previous block. The index \( m_{10} \) represents the index \( w_{00} \) of the previous block. The index \( m_{02} \) represents the index \( w_{02} \) of the previous block.

Based on this encoding scheme, the total rate transmitted by the sender is expressed as,

\[
R = R_{00} + R_{01} + R_{02} + R_{01}^2
\]  
(4)

The transmitter and the relay encoders send the following codewords,

\[
x_0^n(w_{00,i}|1,1, w_{01,i}, 1, 1, 1, 1), x_1^n(1|1,1,1), x_2^n(1|1)
\]

in block \( i = 1 \), the following codewords

\[
x_0^n(w_{00,i}|w_0^{01,i}, w_{02,i}, w_{01,i}, 1, w_{01,i-1}, 1, 1),
\]

\[
x_1^n(1|w_{01,i-1},1), x_2^n(1)
\]

in each block \( i = 2 \), the following codewords

\[
x_0^n(w_{00,i}|w_0^{01,i}, w_{02,i}, w_{01,i}, w_{01,i-1}, w_{01,i-1}, 1, w_{01,i-2}, w_{02,i-1}),
\]

\[
x_1^n(w_{01,i-1}, w_{01,i-2}, w_{02,i-1}), x_2^n(w_{02,i-1}|w_{01,i-2})
\]

in each block \( i = 3, ..., B-2 \), the following codewords

\[
x_0^n(1|w_{01,B-1}, w_{02,B-1}, w_{01,B-2}, w_{01,B-3}),
\]

\[
x_1^n(w_{01,B-2}, w_{01,B-3}), x_2^n(w_{01,B-3})
\]

in block \( i = B-1 \), and the following codewords

\[
x_0^n(1,1,1, w_{01,B-1}, w_{02,B-1}, 1),
\]

\[
x_1^n(w_{02,B-1}), x_2^n(w_{02,B-1}|w_{01,B-2})
\]

at the end of block \( i = B \).

The message parts \( w_{00} \) and \( w_{02} \) are only decoded by one relay but the message part \( w_{01} \) is decoded by both relays. Hence, to be able to simultaneously decode these message parts, we need to send codewords in an asynchronous manner as shown above. This is the keynote of the proposed strategy.

1) Encoding at the first relay: Assume that at the end of block \( (i-1) \), the first relay knows \( \{w_{01,i-1}, \hat{w}_{01,i-1}, \ldots, \hat{w}_{01,i-1}, \hat{w}_{01,i-1}, \ldots, \hat{w}_{01,i-1}\} \), \( \{\tilde{m}_{12,i}, \tilde{m}_{12,i}, \ldots, \tilde{m}_{12,i}\} \), \( \{\tilde{m}_{20,i}, \tilde{m}_{20,i}, \ldots, \tilde{m}_{20,i}\} \) and \( \{\hat{m}_{10,i}, \hat{m}_{10,i}, \ldots, \hat{m}_{10,i}\} \).

At the end of block \( i \), by knowing \( \hat{w}_{01,i}, \tilde{m}_{12,i}, \tilde{m}_{20,i} \) from the previous block, the first relay determines \( \hat{w}_{01,i}, \tilde{w}_{01,i} \) such that

\[
\begin{align*}
\hat{w}_{01,i} = \hat{w}_{01,i} \\
\tilde{w}_{01,i} = \tilde{w}_{01,i}
\end{align*}
\]  
(5)

\( \hat{w}_{01,i} = \hat{w}_{01,i} \) and \( \tilde{w}_{01,i} = \tilde{w}_{01,i} \) with high probability if

\[
R_{01}^2 + R_{01}^1 < I(U_0^2 U_0^1 Y_1 | X_1 U_1 U_2 U_2) \quad \text{and} \quad n \text{ is sufficiently large.}
\]  
(6)

\[
R_{01}^1 < I(U_0^1 Y_1 | U_0^2 U_1 U_2 U_2) \quad \text{and} \quad n \text{ is sufficiently large.}
\]  
(7)

Proof: See Appendix A.

2) Simultaneous Backward decoding at the second relay: Decoding of \( w_{02,i} \) and \( w_{01,i} \) is performed simultaneously in backward manner at the second relay by starting from the block \( i = B \) to \( i = 1 \). Assume that \( \hat{w}_{01,i}, \tilde{w}_{01,i-1} \) and \( \hat{w}_{02,i} \) have been decoded accurately. The second relay determines \( \hat{w}_{01,i-2} \) and \( \tilde{w}_{01,i-1} \) as follows such that

\[
\begin{align*}
\hat{w}_{02,i} & : \hat{w}_{01,i} \rightarrow \hat{w}_{01,i-1} \rightarrow \hat{w}_{01,i-2} \\
\tilde{w}_{02,i} & : \tilde{w}_{01,i} \rightarrow \tilde{w}_{01,i-1} \rightarrow \tilde{w}_{01,i-2}
\end{align*}
\]  
(8)

This can be done with arbitrary small probability of error if \( n \) is sufficiently large and

\[
R_{02} < I(U_0^2 Y_2 | U_0^1 U_1 U_2 U_2) \quad \text{and} \quad n \text{ is sufficiently large.}
\]  
(9)

Proof: See Appendix B.

3) Simultaneous Backward decoding at the receiver: Decoding of \( \hat{w}_{01,i} \), \( \tilde{w}_{01,i} \) and \( \hat{w}_{02,i} \) at the receiver is performed simultaneously in a backward manner by starting from block \( i = B \) down to block \( i = 1 \).

Assume that \( \hat{w}_{01,i}, \tilde{w}_{01,i-1}, \hat{w}_{02,i} \) and \( \tilde{w}_{02,i} \) have been decoded accurately. The sink determines \( \tilde{w}_{01,i-2}, \hat{w}_{01,i-1} \), and \( \tilde{w}_{02,i-1} \) were sent such that

\[
\begin{align*}
\hat{w}_{02,i} & : \hat{w}_{01,i} \rightarrow \hat{w}_{01,i-1} \rightarrow \hat{w}_{01,i-2} \\
\tilde{w}_{02,i} & : \tilde{w}_{01,i} \rightarrow \tilde{w}_{01,i-1} \rightarrow \tilde{w}_{01,i-2}
\end{align*}
\]  
(10)

This can be done with arbitrary small probability of error if \( n \) is sufficiently large and

\[
R_{01}^2 + R_{02}^1 < I(U_0^1 X_1 Y_1 | U_0^1 U_1 X_1 U_2 U_2) \quad \text{and} \quad n \text{ is sufficiently large.}
\]  
(11)

Proof: See Appendix C.

4) By knowing \( m_{12,i}, w_{01,i}, w_{02,i} \) and \( w_{01,i} \), decoding of \( w_{00,i} \) is performed in forward manner at the receiver by starting from the block \( i = 1 \) to \( i = B \). The receiver declares that \( \hat{w}_{00,i} = \hat{w}_{00,i} \) if \( w_{00} \) is the unique value in \( \{1, \ldots, 2^n R_{00}\} \) such that in the block \( i \),

\[
\begin{align*}
\hat{w}_{00,i} & : \hat{w}_{01,i} \rightarrow \hat{w}_{02,i} \rightarrow \hat{w}_{01,i} \rightarrow \hat{m}_{10,i} \rightarrow \hat{m}_{12,i} \rightarrow \hat{m}_{20,i} \\
\tilde{w}_{00,i} & : \tilde{w}_{01,i} \rightarrow \tilde{w}_{02,i} \rightarrow \tilde{w}_{01,i} \rightarrow \tilde{m}_{10,i} \rightarrow \tilde{m}_{12,i} \rightarrow \tilde{m}_{20,i}
\end{align*}
\]  
(12)

\[
R_{00} < I(X_0^1 Y_0 | U_0^2 U_0^1 U_1 X_1 U_1 U_2 U_2) \quad \text{and} \quad n \text{ is sufficiently large.}
\]  
(13)

Now the relations of Theorem 1 can be obtained by the following argument:

(3), (4), (14) and (15) the first term of (1), (4), (6) and (15) the second term of (1), (4), (7), (10) and (15) the third
term of (1), (3), (4), (10), (12) and (15) the fourth term of (1), (3), (4), (6), (13) and (15) the fifth term of (1).

Remarks:
1) By replacing $U_{01}^1 = U$ and $U_{01}^2 = U_{12} = U_{20} = U_{02} = X_2 = Y_2 = 0$ in (1), the achievable rate for the relay channel by partial decode-forward [4] is obtained as follows,\\
$\mathcal{R} \leq \sup \min \{I(X_0 X_1; Y_0), I(U; Y_1 | X_1) + I(X_0; Y_0 | X_1 U)\}$

2) If we choose $U_{01}^1 = U_{02} = X_0, U_{12} = X_1, U_{20} = X_2$ and $U_{01}^2 = 0$, in (1), the achievable rate for relay network by decode-and-forward [7, Theorem 3.2] is obtained as follows,
$\mathcal{R} \leq \sup \min \{I(X_0 X_1; X_2 Y_0), I(X_0 Y_1 | X_1 X_2), I(X_0 X_1 Y_2 | X_2)\}$

3) If we choose $X_0 = U_3 U_2 U_1, U_{01}^3 = 0, U_{01}^1 = U_1, U_{02} = U_2, X_1 = U_2 V_1, U_{20} = V_2$ and $U_3 = V_2^2$ in (1), the achievable rate for the relay network with a structure like that considered in [12] is obtained as follows,
$\mathcal{R} = \sup \min \{I(U_3 U_2 U_1 V_1^1 V_2^1 V_2^2; Y_0), I(U_1; Y_1 V_1^1 V_2^2) + I(U_2; U_2 V_1^1 V_2^2), I(U_3; U_3 U_2 U_1 V_1^1 V_2^2), I(U_3 U_2 U_1 V_1^1 V_2^2; Y_0), I(U_3; U_3 U_2 U_1 V_1^1 V_2^2), I(U_1; Y_1 V_1^1 V_2^2) + I(U_3 U_2 U_1 V_1^1 V_2^2; Y_0)\}$

which is greater than the proposed rate in [12] by neglecting the compress-and-forward variables, as follows,
$\mathcal{R} \leq \sup \min \{I(U_3 U_2 U_1 V_1^1 V_2^1 V_2^2; Y_0), I(U_1; Y_1 V_1^1 V_2^2) + I(U_2; U_2 V_1^1 V_2^2), I(U_3; Y_0 U_2 U_1 V_1^1 V_2^2), I(U_3 U_2 U_1 V_1^1 V_2^2; Y_0)\}$

4) Similarly it can easily be shown that the rate of Theorem 1 is higher than the rate proposed in [15] and it has less terms.

III. CONCLUSION

This paper presents a new achievable rate based on a comprehensive partial decoding scheme for two-level relay network. An application of regular encoding/simultaneous backward decoding is presented to implement the proposed rate. The proposed partial decoding strategy is called simultaneous partial and backward decoding scheme due to the fact that in this strategy, we decode different parts of the messages simultaneously at the relays and the receiver. It is shown that the proposed rate includes previously presented rates based on sequential decode-and-forward and partial decode-and-forward strategies.

APPENDIX A: PROOF OF (6), (7)

Proof can be done by the analysis of error probability. By the symmetry of the random code construction, the conditional probability of error does not depend on which pair of indices is sent. so without loss of generality, we can assume that $(w_{01,i}^1, w_{01,i}^2) = (1, 1)$ was sent. we have error if either the correct codewords are not typical with the received sequence or there is a pair of incorrect codewords that are typical with the received sequence. Define the event $E(w_{01,i}^1, w_{01,i}^2)$ such that (5) is satisfied. Then by the union of event bound, we have,
$P_e^{(n)} = P(E^c(1, 1) \cup \cup_{(w_{01,i}^1, w_{01,i}^2)} \neq (1, 1) E(w_{01,i}^1, w_{01,i}^2))$

\[ \leq P(E^c(1, 1)) + \sum_{(w_{01,i}^1, w_{01,i}^2) \neq (1, 1)} E(w_{01,i}^1, w_{01,i}^2) \]

where $P$ is the conditional probability given that (1, 1) was sent. From [18, Thm 15.2.1], $P(E^c(1, 1)) \to 0$. And due to the fact that $w_{01,i}^1$ is conditionally defined given $w_{01,i}^2$, it is impossible to correctly decode $w_{01,i}^1$ when $w_{01,i}^2$ is decoded incorrectly, thus 
$\sum_{(w_{01,i}^1, w_{01,i}^2) \neq (1, 1)} E(w_{01,i}^1, w_{01,i}^2) = 0$.

From [18, Thm 15.2.2], for $w_{01,i}^1 \neq 1$, we have
$E(w_{01,i}^1) \leq 2^{-n(I(U_3; Y_1 U_1 U_2 U_3; X_1 U_2))}$

Similarly, for $(w_{01,i}^1, w_{01,i}^2) \neq (1, 1)$
$E(w_{01,i}^1, w_{01,i}^2) \leq 2^{-n(I(U_3; U_1 U_2 U_3; X_1 U_2 U_3; X_1 U_2))}$

Since $\epsilon > 0$ is arbitrary, the conditions of (6) and (7) imply that each term of (16) tends to 0 as $n \to \infty$.

APPENDIX B: PROOF OF (9), (10)

Proof can be done by the analysis of error probability. By the symmetry of the random code construction, the conditional probability of error does not depend on which pair of indices is sent. So without loss of generality, we can assume that $(w_{01,i-1}^1, w_{01,i-1}^2) = (1, 1)$ was sent.

We have error if either the correct codewords are not typical with the received sequence or there is a pair of incorrect codewords that is typical with the received sequence. Define the event $E(w_{01,i-1}^1, w_{01,i-1}^2)$ such that (8) is satisfied.

Then by the union of event bound, we have,
$P_e^{(n)} = P(E^c(1, 1) \cup \cup_{(w_{01,i-1}^1, w_{01,i-1}^2) \neq (1, 1) E(w_{01,i-1}^1, w_{01,i-1}^2))}$

\[ \leq P(E^c(1, 1)) + \sum_{(w_{01,i-1}^1, w_{01,i-1}^2) \neq (1, 1)} E(w_{01,i-1}^1, w_{01,i-1}^2) \]

where $P$ is the conditional probability given that (1, 1) was sent. From [18, Thm 15.2.1], $P(E^c(1, 1)) \to 0$. Since according to the random code construction, $w_{02,i}^2$ is conditionally defined given $w_{01,i}^1$, it is impossible to correctly decode $w_{02,i-1}$ when $w_{01,i-1}^2$ isn’t decoded correctly. Hence $E(1, w_{01,i-1}^2) \to 0$.

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From [18, Thm 15.2.2], for \( w_{02,i-1} \neq 1 \), we have
\[
E(w_{02,i-1}, 1, 1) \leq 2^{-n(I(U_{01}Y_2|U_{01}U_{12}X_2X_{20})-4\epsilon)}
\]
Similarly, for \( (w_{02,i-1}, w_{01,i-2}) \neq (1, 1) \), we have
\[
E(w_{02,i-1}, w_{01,i-2}) = 2^{-n(I(U_{02}U_{12}Y|U_{01}X_2X_{20})-4\epsilon)}.
\]
Since \( \epsilon > 0 \) is arbitrary, the conditions of (9) and (10) imply that each term of (17) tends to 0 as \( n \to \infty \).

**APPENDIX C: PROOF OF (12)-(14)**

Proof can be done by the analysis of error probability. By the symmetry of the random code construction, the conditional probability of error does not depend on which pair of indices is sent. So without loss of generality, we can assume that \( (w_{01,i-1}, w_{02,i-1}, w_{01,i-2}) = (1, 1, 1) \) was sent.

We have error if either the correct codewords are not typical with the received sequence or there is a pair of incorrect codewords that is typical with the received sequence. Define the event \( E(w_{01,i-1}, w_{02,i-1}, w_{01,i-2}) \) such that (11) is satisfied.

Then by the union of event bound, we have,
\[
P_e(n) = P \left( E^c(1, 1, 1) \bigcup E(w_{01,i-1}, w_{02,i-1}, w_{01,i-2}) \right) \leq P(E(1, 1, 1)) + \sum_{(w_{01,i-1}, w_{02,i-1}) \neq (1, 1)} E(w_{01,i-1}, w_{02,i-1}, w_{01,i-2}) \leq 2^{-n(I(U_{01}Y_2|U_{01}U_{12}X_2X_{20})-4\epsilon)}.
\] (18)

where \( P \) is the conditional probability given that \( (1, 1, 1) \) was sent. From [18, Thm 15.2.1], \( P(E(1, 1, 1)) \to 0 \). Since according to the random code construction, \( w_{01,i-1} \) and \( w_{02,i-1} \) are conditionally defined given \( w_{01,i-2} \) it is impossible to have \( w_{01,i-1} = 1 \) or \( w_{02,i-1} = 1 \) when \( w_{01,i-2} \neq 1 \). Hence \( E(1, 1, 1, w_{01,i-2}) \to 0 \) and \( E(w_{01,i-1}, 1, w_{01,i-2}) \to 0 \) and \( E(1, w_{02,i-1}, 1, 1) \to 0 \) for \( (w_{01,i-1}, 1, w_{01,i-2}) \neq 1 \).

From [18, Thm 15.5.2.2, for \( w_{02,i-1} \neq 1 \), we have
\[
E(w_{01,i-1}, 1, 1, 1) \leq \sum_{(w_{02,i-1}, w_{01,i-2}) \neq (1, 1)} E(w_{01,i-1}, w_{02,i-1}, w_{01,i-2}) \leq \|A^*_n\|^{2-n(H(Y_0U_{02}|U_{01}U_{12}X_2X_{20})-\epsilon)}.
\]

**REFERENCES**


