Nonzero solutions for a system of variational inequalities in reflexive Banach spaces

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\begin{abstract}
In this paper, some existence theorems of nonzero solutions for a system of bilinear variational inequalities are proved by using the coincidence degree theory in reflexive Banach spaces. The results presented in this paper improve and extend some known results in the literature.
\end{abstract}

1. Introduction

Variational inequality theory with applications is an important part of nonlinear analysis. In recent years, variational inequalities have been generalized and extended in many different directions, such as in the field of mechanics, differential equations, control theory, game theory, optimization methods, etc. (see, for example, [1,2] and the references therein).

In 1987, Noor [3] studied the famous Signorini problem in mechanics in the framework of the following variational inequality:

\begin{equation}
a(u, v - u) + j(u, v) - j(u, u) \geq \langle g(u), v - u \rangle, \quad \forall v \in K, \quad (I)\ast
\end{equation}

and proved the existence of solutions of the Signorini problem in Hilbert spaces.

In 1991, Zhang and Xiang [4] studied the existence of solutions of bilinear variational inequality (1) in reflexive Banach space. As an application, they discussed the existence of solutions for the Signorini problem. Recently, Huang [5] studied some existence theorems of nonzero solutions and multiple solutions of the following bilinear variational inequality:

\begin{equation}
a(u, v - u) + j(u, v) - j(u, u) \geq \langle g(u), v - u \rangle + \langle f, v - u \rangle, \quad \forall v \in K. \quad (II)\ast
\end{equation}

On the other hand, the existence of nonzero solutions for variational inequalities is an important topic of variational inequality theory. Assume that $X$ is a reflexive Banach space and $X^*$ is the dual space of $X$. Denote by $\langle \cdot, \cdot \rangle$ the duality pairing. Suppose that $K$ is a nonempty closed convex subset of $X$.

In 1998, Zhu [6] introduced and studied the following system of variational inequalities involving the linear operators:

\begin{equation}
\text{Find } (u, w) \in K \times K : \begin{cases}
(Au, v - u) \geq \langle g(w), v - u \rangle, & \forall v \in X, \quad (I)\ast

(Bu, v - u) \geq \langle h(w), v - u \rangle, & \forall v \in K. \quad (II)\ast
\end{cases}
\end{equation}

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By using the coincidence degree theory, Zhu [6] proved the following result.

**Theorem 1.1** ([6]). Suppose that linear mappings $A, B : X \rightarrow X^*$ are continuous and coercive, $g : K \rightarrow X^*$ is one-to-one and linear continuous, $h : K \rightarrow X^*$ is completely continuous, $\text{Im} A$ (the image of $A$) $\subset \text{Im} g$. Assume that $(u_n, w_n)$ satisfies the inequality (1) with $u_n \neq 0$ and $w_n \neq 0$ such that the following conditions hold:

(a) $$\lim_{\|w_n\| \rightarrow +\infty} \frac{\langle h(w_n), u_n \rangle}{\|u_n\|^2} = 0;$$
(b) there exists $u_0 \in K$, $u_0 \neq 0$, such that
$$\lim_{\|w_n\| \rightarrow 0} \frac{\langle h(w_n), u_0 \rangle}{\|u_n\|} \geq M\|u_0\|.$$ 

Then the system of variational inequalities (3) has at least one positive solution.

Moreover, Lai, Zhu and Deng [7], and Wu and Huang [8] obtained some existence theorems of nonzero solutions for variational inequalities by the fixed point index approach under some different conditions.

Motivated and inspired by the above papers, in this paper, we consider the following problem for a system of variational inequalities:

Find $(u, w) \in K \times K$ such that

$$\begin{align*}
\begin{cases}
\quad a(u, v - u) + j_1(u, v) - j_1(u, u) \geq \langle g(w), v - u \rangle + \langle f_1, v - u \rangle, \quad \forall v \in X, \quad (I) \\
\quad b(u, v - u) + j_2(u, v) - j_2(u, u) \geq \langle h(w), v - u \rangle + \langle f_2, v - u \rangle, \quad \forall v \in K, \quad (II)
\end{cases}
\end{align*}$$

where $a, b : X \times X \rightarrow R$ are bilinear mappings and $g, h : X \rightarrow X^*$ are nonlinear mappings.

It is easy to see that the system of variational inequalities (4) includes variational inequalities (1) and (2) as special cases.

By using the coincidence degree theory, we prove some existence theorems of nonzero solutions for the system of variational inequalities (4) in reflexive Banach spaces under some suitable conditions. The results presented in this paper improve and extend some known results in [1,3–5].

2. Preliminaries

**Assumptions:**

(A) Bilinear mappings $a, b : X \times X \rightarrow R = (-\infty, +\infty)$ are continuous and coercive, that is, there exist constants $L, M, \alpha, \beta > 0$ such that
$$|a(u, v)| \leq L\|u\|\|v\|, \quad |b(u, v)| \leq M\|u\|\|v\|$$
and
$$a(u, u) \geq \alpha\|u\|^2, \quad b(u, u) \geq \beta\|u\|^2, \quad \forall u \in X.$$

(B) $j_i : X \times X \rightarrow R \cup \{+\infty\}$ $(i = 1, 2)$ satisfies the following conditions:

(i) $j_i$ is linear with respect to the first argument;
(ii) $j_i$ is convex lower semi-continuous with respect to the second argument;
(iii) there exist $\gamma_1 \in (0, \alpha)$ and $\gamma_2 \in (0, \beta)$ such that $j_i(u, v) \leq \gamma_1\|u\|\|v\|$ and $j_2(u, v) \leq \gamma_2\|u\|\|v\|$ for all $u, v \in X$;
(iv) for all $u, v, w \in K, j_i(u, v) - j_i(u, w) \leq j_i(u, v - w)$.

Obviously, conditions (iii) and (iv) of Assumption (B) imply that $j_i(u, 0) = 0$ for $i = 1, 2$.

(C) The mapping $g : K \rightarrow X^*$ is bounded continuous, i.e., there exists a constant $\xi > 0$ such that $\|g(w)\| \leq \xi\|w\|$ for all $w \in K$.

The mapping $h : K \rightarrow X^*$ is completely continuous.

**Lemma 2.1** ([4]). Let $a : X \times X \rightarrow R$ be a bilinear continuous and coercive mapping satisfying Assumption (A) and $j : X \times X \rightarrow R^* = [0, +\infty)$ satisfy Assumption (B). If $g : K \rightarrow X^*$ is a semi-continuous mapping and antimonotone (i.e., $\langle g(u) - g(v), u - v \rangle \leq 0$ for all $u, v \in K$), then there exists a unique solution of the variational inequality (1) in $K$.

For any $q \in X^*$, by Lemma 2.1, the following variational inequality:
$$a(u, v - u) + j(u, v) - j(u, u) \geq \langle q, v - u \rangle + \langle f, v - u \rangle, \quad \forall v \in K$$
has a unique solution $u \in K$.

Now, we define mappings $K_0 : X^* \rightarrow K$ and $K_0g : K \rightarrow K$, respectively, as follows:
$$K_0(q) = u, \quad (K_0g)(u) = K_0(g(u)).$$

Given $w \in K$, the inequality (1) has a unique solution $u = K_0g(w) \in K$ and the inequality (II) has a unique solution $\hat{u} = K_0h(w) \in K$. Hence the system (4) has a solution if and only if the mappings $K_0g$ and $K_0h$ have a common point in $K$, that is, there exist $(u, w) \in K \times K$ such that $u = K_0g(w)$ and $K_0h(w)$.
Lemma 2.2 ([5]). The mappings \( K_a, K_b : X^* \to K \) have the following properties:

\[
\|K_a(p) - K_a(q)\| \leq \frac{1}{\alpha - \gamma_1} \|p - q\|, \quad \|K_b(p) - K_b(q)\| \leq \frac{1}{\beta - \gamma_2} \|p - q\|, \quad \forall p, q \in X^*.
\]

Lemma 2.2 implies that \( K_a \) and \( K_b \) are continuous and bounded.

Let \( U \) be a relatively open subset of \( K \). A pair of mappings \((g, h)\) is said to be \((a, b)\)-admissible on \( U \) if \( K_a g \) and \( K_b h \) have no common point on \( \partial U \) (the boundary of \( U \) relative to \( K \)), i.e., the system of variational inequalities (4) has no solution in \( K \times \partial U \). Given a homotopy \( H : \overline{U} \times (0, 1) \to X^* \), \((g, H)\) is \((a, b)\)-admissible on \( U \) if for every \( t \in [0, 1] \), \( K_a g \) and \( K_b H(\cdot, t) \) have no common point on \( \partial U \), i.e., the following system of variational inequalities: \((u, w, t) \in K \times K \times [0, 1] \),

\[
a(u, v - u) + j_1(u, v) - j_1(u, u) \geq (g(w), v - u) + f_1(v, v - u), \quad \forall v \in X, \\
b(u, v - u) + j_2(u, v) - j_2(u, u) \geq (H(w, t), v - u), \quad \forall v \in K,
\]

has no solution in \( K \times \partial U \times [0, 1] \).

Suppose that \( U \) is bounded and \( \overline{U} \) denotes the closure of \( U \). Since \( K_b h \) is completely continuous, the coincidence degree \( D([K_a g, K_b h], U) \) of \( K_a g \) and \( K_b h \) in \( U \) is well defined [9].

Now, we define the solution index for the system (4) in \( U, I_{a,b}([g, h], U) \) by setting

\[ I_{a,b}([g, h], U) = D([K_a g, K_b h], U). \]

Lemma 2.3. If Assumptions (A)–(C) are satisfied, then the solution index \( I_{a,b} \) for the system (4) has the following properties:

(i) \( I_{a,b}([g, 0], U) = \{0\} \) if \( 0 \not\in U \), or \( \{1\} \) if \( 0 \in U \);
(ii) if \( I_{a,b}([g, h], U) \neq \{0\} \), then the system (4) has a solution \((u, w) \in K \times U \);
(iii) for every open subset \( V \subset U \) such that the system (4) has no solution on \( K \times (U \setminus V) \),

\[ I_{a,b}([g, h], U) = I_{a,b}([g, h], V) \];
(iv) if the homotopy \( H \) is completely continuous and \((g, h)\) is \((a, b)\)-admissible on \( U \), then \( I_{a,b}([g, H(\cdot, t)], U) \) is independent of \( t \in [0, 1] \).

Proof. It follows immediately from the above definition and the properties of the coincidence degree [9]. \( \square \)

3. Main results

For convex subset \( K \) of \( X \), the recession cone of \( K \) is defined by

\[ rc(K) = \{w \in X : w + u \in K, \forall u \in K\}. \]

Theorem 3.1. Suppose that Assumptions (A)–(C) are satisfied, \( f_1, f_2 \in X^* \), and \( g, h : K \to X^* \) are antimonotone. Assume that

(a) If \((u_n, w_n)\) satisfies the inequality (I) with \( u_n \neq 0 \) and \( w_n \neq 0 \), then

\[ \liminf_{\|w_n\|\to\pm\infty} \frac{\langle h(w_n) + f_2, u_n\rangle}{\|u_n\|^2} < \beta; \]

(b) If \((u, w)\) satisfies the inequality (I), then there exist \( u_0 \in rcK \setminus \{0\} \) and a neighborhood \( V(0) \) of zero point such that

\[ b(u, u_0) + j_2(u, u_0) < \langle h(w) + f_2, u_0\rangle, \quad \forall u, w \in K \cap V(0). \]

Then the system of variational inequalities (4) has a nonzero solution.

Proof. For \( r > 0 \), letting \( K' = \{x \in K : \|x\| < r\} \), we know that \( K' \) is a bounded open subset of \( K \) and \( \partial K' = \{x \in K : \|x\| = r\} \). Therefore, \( I_{a,b}([g, h], K') \) is well defined.

Next, we shall verify that \( I_{a,b}([g, h], K^R) = \{1\} \) for large enough \( R \) and \( I_{a,b}([g, h], K') = \{0\} \) for small enough \( r \).

First, we define a homotopy \( H_1 : K \times [0, 1] \to K \) by \( H_1(w, t) = K_b(t(h(w) + f_2)) \). By Lemma 2.2, We have

\[
\|H_1(w, t_1) - H_1(w, t_2)\| = \|K_b(t_1(h(w) + f_2)) - K_b(t_2(h(w) + f_2))\| \\
\leq \frac{1}{\beta - \gamma_2} |t_1(h(w) + f_2) - t_2(h(w) + f_2)| \\
= \frac{1}{\beta - \gamma_2} \|h(w) + f_2\| \cdot |t_1 - t_2|.
\]

It follows from (5) that \( H_1(w, t) \) is completely continuous at \( t \) for all \( w \in K \).
We claim that \((g, H_1)\) is \((a, b)\)-admissible on \(K^c\) for large enough \(R\). Arguing by contradiction, we find sequences \(\{t_n\}\), \(\{u_n\}\) and \(\{w_n\}\), such that \(\|w_n\| \to +\infty\) and

\[
\begin{align*}
\alpha(u_n, v - u_n) + j_1(u_n, v) - j_1(u_n, u_n) &\geq \langle g(w_n), v - u_n \rangle + \langle f_1, v - u_n \rangle, \quad \forall v \in X, \\
\beta(u_n, v - u_n) + j_2(u_n, v) - j_2(u_n, u_n) &\geq (h(w_n) + f_2), v - u_n, \quad \forall v \in K.
\end{align*}
\]

Letting \(v = 0\) in (7), we have

\[
b(u_n, u_n) + j_2(u_n, u_n) \leq t_n((h(w_n) + f_2), u_n).
\]

By Assumptions (A), (B) and (8),

\[
\beta\|u_n\|^2 \leq t_n(h(w_n), u_n) + t_n(f_2, u_n) - j_2(u_n, u_n) \leq t_n((h(w_n) + f_2), u_n).
\]

From (9), we know that \((h(w_n) + f_2, u_n) \geq 0\). Now condition (a) implies that

\[
\liminf_{\|w_n\| \to +\infty} \frac{\langle h(w_n) + f_2, u_n \rangle}{\|u_n\|^2} < \beta.
\]

Since \(t_n \in [0, 1]\), combining (9) with (10), we have

\[
\beta \leq \liminf_{\|w_n\| \to +\infty} \frac{\langle h(w_n) + f_2, u_n \rangle}{\|u_n\|^2} < \beta,
\]

which is a contradiction. Thus \((g, H_1)\) is \((a, b)\)-admissible on \(K^c\) for large enough \(R\) and so

\[
l_{a,b}[g, h, K^c] = l_{a,b}[g, h, K_1, K^c] = l_{a,b}[g, h, K_1, K^c] = l_{a,b}[g, 0, K^c] = 1. \tag{11}
\]

Let \(r > 0\) be small enough such that \(K^c \subset K \cap V(0)\). By condition (b), if \((u, w)\) satisfies the inequality (I), we know that there exists \(u_0 \in rK \setminus \{0\}\) such that, for all \(u, w \in K^c\),

\[
b(u, u_0) + j_2(u, u_0) < \langle (h(w) + f_2), u_0 \rangle. \tag{12}
\]

Next, we shall verify that \(l_{a,b}[g, h, K^c] = 0\) for small enough \(r\). In fact, otherwise, if \(l_{a,b}[g, h, K^c] \neq 0\), then, by (ii) of Lemma 2.3, we can find \((u, w) \in K \times K^c\) such that

\[
\begin{align*}
\alpha(u, v - u) + j_1(u, v) - j_1(u, u) &\geq \langle g(w), v - u \rangle + \langle f_1, v - u \rangle, \quad \forall v \in X, \\
b(u, v - u) + j_2(u, v) - j_2(u, u) &\geq \langle (h(w) + f_2), v - u \rangle, \quad \forall v \in K.
\end{align*}
\]

Since \(u_0 \in rK\) and \(u \in K\), it is easy to see that \(u_0 + u \in K\). Taking \(v = u_0 + u\) in (14), we have

\[
b(u, u_0) + j_2(u, u_0 + u) - j_2(u, u) \geq \langle (h(w) + f_2), u_0 \rangle. \tag{15}
\]

It follows from condition (iv) of Assumption (B) that

\[
j_2(u, u_0 + u) \leq j_2(u, u_0) + j_2(u, u)
\]

and so (15) implies that

\[
b(u, u_0) + j_2(u, u_0) \geq \langle (h(w) + f_2), u_0 \rangle,
\]

which contradicts (12). Therefore, \(l_{a,b}[g, h, K^c] = 0\).

Now, from (11), we know that \(l_{a,b}[g, h, K^c \setminus K^c] = 1\). Thus there exists a nonzero solution \((u, w)\) for the system of variational inequalities (4). This completes the proof. \(\square\)

**Theorem 3.2.** Suppose that Assumptions (A)–(C) are satisfied, \(f_1, f_2 \in X^\ast\), and \(g, h : K \to X^\ast\) are antimonotone. Assume that

(a) if \((u_n, w_n)\) satisfies inequality (I) with \(u_n \neq 0\) and \(w_n \neq 0\), then

\[
\liminf_{\|w_n\| \to 0} \frac{\langle h(w_n) + f_2, u_n \rangle}{\|u_n\|^2} < \beta;
\]

(b) if \((u, w)\) satisfies the inequality (I), then there exist \(u_0 \in rK \setminus \{0\}\) and a constant \(\rho > 0\) such that

\[
b(u, u_0) + j_2(u, u_0) < \langle (h(w) + f_2), u_0 \rangle, \quad \forall u, w \in K \text{ with } \|u\| > \rho.
\]

Then the system of variational inequalities (4) has a nonzero solution.
Proof. For \( r > 0 \), letting \( K' = \{ x \in K, \| x \| < r \} \), we know that \( K' \) is a bounded open subset of \( K \) and \( \partial K' = \{ x \in K, \| x \| = r \} \). Therefore, \( I_{a,b}(g, h), K' \) is well defined.

Next, we shall verify that \( I_{a,b}(g, h), K' \) = \( \{ 0 \} \) for large enough \( R \) and \( I_{a,b}(g, h), K' \) = \( \{ 1 \} \) for small enough \( r \).

First, we define a homotopy \( H_1 : K \times [0, 1] \to K \) by \( H_1(w, t) = K_0(t(h(w) + f_2)) \). It follows from (5) that \( H_1(w, t) \) is completely continuous at \( t \) for all \( w \in K \).

We claim that \( (g, H_1) \) is \((a, b)\)-admissible on \( K' \) for small enough \( r \). Arguing by contradiction, we find sequences \( \{t_n\}, \{u_n\} \) and \( \{w_n\} \), such that \( \|w_n\| \to 0 \) and

\[
a(u_n, v - w_n) + j_1(u_n, v) - j_1(u_n, u_n) \geq (g(w_n), v - u_n) - (f_1, v - u_n), \quad \forall v \in X, \quad (16)
\]

\[
b(u_n, v - w_n) + j_2(u_n, v) - j_2(u_n, u_n) \geq t_n((h(w_n) + f_2), v - u_n), \quad \forall v \in K. \quad (17)
\]

Letting \( v = 0 \) in (17), we have

\[
b(u_n, u_n) + j_2(u_n, u_n) \leq t_n((h(w_n) + f_2), u_n). \quad (18)
\]

By Assumptions (A), (B) and (18), we have

\[
\beta \|u_n\|^2 \leq t_n(h(w_n), u_n) + t_n(f_2, u_n) - j_2(u_n, u_n) \leq t_n((h(w_n) + f_2), u_n). \quad (19)
\]

From (19), we have \( (h(w_n) + f_2, u_n) \geq 0 \). By condition (a), we have

\[
\liminf_{\|w_n\| \to 0} \frac{(h(w_n) + f_2, u_n)}{\|u_n\|^2} < \beta. \quad (20)
\]

Since \( t_n \in [0, 1] \), combining (19) with (20), we have

\[
\beta \leq \liminf_{\|w_n\| \to 0} \frac{(h(w_n) + f_2, u_n)}{\|u_n\|^2} < \beta,
\]

which is a contradiction. This contradiction shows that \( (g, H_1) \) is \((a, b)\)-admissible on \( K' \) for small enough \( r \). Therefore, we have

\[
I_{a,b}(g, h), K' = I_{a,b}(g, H_1(\cdot, 1)), K' = I_{a,b}(g, H_1(\cdot, 0)), K' = I_{a,b}(g, 0), K' = \{ 1 \} \quad (21)
\]

Since \( b, h \) are bounded, then there exists two constants \( M > 0 \) and \( P > 0 \) such that

\[
\sup_{u \in K'} \|b(u, u)\| \leq M\|u\|^2, \quad \sup_{w \in K'} \|h(w)\| \leq P.
\]

So we have

\[
\sup_{u \in K'} \|b(u, u)\| \leq M\|u\|^2, \quad \sup_{w \in K'} \|h(w)\| \leq P\|u_0\|.
\]

Pick \( z \in X^* \) such that \( (z, u_0) > 0 \) and let \( N \) be large enough such that

\[
M\|u\|\|u_0\| + P\|u_0\| + j_2(u, u_0) < (f_2, u_0) + N\langle z, u_0 \rangle. \quad (22)
\]

We define a homotopy \( H_2 : K \times [0, 1] \to K \) as follows:

\[
H_2(w, t) = K_0(h(w) + f_2 + tNz), \quad \forall (w, t) \in K \times [0, 1]. \quad (23)
\]

It is easy to show that \( H_2(w, t) \) is completely continuous at \( t \) for all \( w \in K \).

We claim that \( (g, H_2) \) is \((a, b)\)-admissible on \( K' \) for large enough \( R \). Arguing by contradiction, we find sequences \( \{t_n\}, \{u_n\} \) and \( \{w_n\} \) such that \( \|w_n\| \to +\infty \) and

\[
a(u_n, v - w_n) + j_1(u_n, v) - j_1(u_n, u_n) \geq (g(w_n), v - u_n) + (f_1, v - u_n), \quad \forall v \in X, \quad (24)
\]

\[
b(u_n, v - w_n) + j_2(u_n, v) - j_2(u_n, u_n) \geq (h(w_n) + f_2 + t_nNz), v - u_n), \quad \forall v \in K. \quad (25)
\]

Since \( u_0 \in \text{rc}(K), u_n \in K \), we have \( u_0 + u_n \in K \). Taking \( v = u_0 + u_n \) in (24), it follows that

\[
b(u_n, u_0) + j_2(u_n, u_n) - j_2(u_n, u_n) \geq (h(w_n), u_0) + (f_2 + t_nNz), u_0). \quad (26)
\]

Since \( j_2(u_n, u_0 + u_n) \leq j_2(u_n, u_n) + j_2(u_n, u_n) \), by (25), we have

\[
b(u_n, u_0) + j_2(u_n, u_n) \geq (h(w_n), u_0) + (f_2 + t_nNz), u_0) \geq (h(w_n) + f_2, u_n),
\]

which contradicts condition (b). Thus, \( (g, H_2) \) is \((a, b)\)-admissible on \( K' \) for large enough \( R \).

We now claim that \( I_{a,b}(g, H_2(\cdot, 1)), K' \) = \( \{ 0 \} \). If \( I_{a,b}(g, H_2(\cdot, 1)), K' \) \( \neq \{ 0 \} \), then, by (ii) of Lemma 2.3, we can find \( (u, w) \in K \times K \) such that

\[
a(u, v - u) + j_1(u, v) - j_1(u, u) \geq (g(w), v - u) + (f_1, v - u), \quad \forall v \in X, \quad (27)
\]

\[
b(u, v - u) + j_2(u, v) - j_2(u, u) \geq (h(w) + f_2 + Nz), v - u), \quad \forall v \in K.
\]
Suppose that Assumptions (A)–(C) are satisfied, \( f_1, f_2 \in X^* \), and \( g, h : K \to X^* \) are antimonotone. Assume that \((u_n, w_n)\) satisfies the inequality (I) with \( u_n \neq 0 \) and \( w_n \neq 0 \) such that the following conditions hold:

\[
\text{(a)} \quad \liminf_{\|w_n\| \to +\infty} \frac{\langle h(w_n) + f_2, u_n \rangle}{\|u_n\|^2} < \beta; \\
\text{(b)} \quad \liminf_{\|w_n\| \to 0} \frac{\langle h(w_n) + f_2, u_0 \rangle}{\|u_n\|} > (M + \gamma_2)\|u_0\|.
\]

Then the system of variational inequalities (4) has a nonzero solution.

**Proof.** Similar to the first part of the proof of Theorem 3.1, we have \( I_{a,b}[g, h], K^c = \{1\} \) for large enough \( R \).

Pick \( z \in X^* \) such that \( (z, u_0) > 0 \) and let \( N \) be large enough. Define a homotopy \( H_3 : K^c \times [0, 1] \to K \) by \( H_3(w, t) = K_b((1-t)(h(w) + f_2) + tNz) \). It is easy to show that \( H_3 \) is completely continuous at \( t \) for all \( w \in K^c \).

Next, we shall verify that \((g, H_3)\) is \((a, b)\)-admissible on \( K^c \) for small enough \( r \). Suppose that there exist sequences \( \{u_n\}, \{w_n\} \) and \( \{t_n\} \) such that \( \|w_n\| \to 0 \) and

\[
\text{a}(u_n, v - u_n) + j_1(u_n, v) - j_1(u_n, u_0) \geq \langle g(w_n), v - u_n \rangle + \langle f_1, v - u_n \rangle, \quad \forall v \in X, \tag{29}
\]

\[
\text{b}(u_n, v - u_n) + j_2(u_n, v) - j_2(u_n, u_0) \geq (1 - t_n)\langle h(w_n) + f_2, u_0 \rangle + t_nNz - v - u_n, \quad \forall v \in K. \tag{30}
\]

Taking \( v = u_0 + u_n \) in (30), we obtain

\[
b(u_n, u_0) + j_2(u_n, u_0 + u_n) - j_2(u_n, u_0) \geq (1 - t_n)\langle h(w_n) + f_2, u_0 \rangle + t_nNz, \tag{31}
\]

Since \( j_2(u_n, u_0 + u_n) \leq j_2(u_n, u_0) + j_2(u_n, u_0) \) by Assumption (B), (31) implies that

\[
b(u_n, u_0) + j_2(u_n, u_0) \geq (1 - t_n)\langle h(w_n) + f_2, u_0 \rangle + t_nNz. \tag{32}
\]

By Assumptions (A), (B) and (32),

\[
(M + \gamma_2)\|u_0\| \geq (1 - t_n)\langle h(w_n) + f_2, u_0 \rangle + t_nNz. \tag{33}
\]

It follows from condition (b) that

\[
\liminf_{\|w_n\| \to 0} \frac{\langle h(w_n) + f_2, u_0 \rangle}{\|u_n\|} > (M + \gamma_2)\|u_0\|. \tag{34}
\]

Since \( t_n \in [0, 1] \), combining (33) with (34), we have

\[
(M + \gamma_2)\|u_0\| \geq \liminf_{\|w_n\| \to 0} \frac{\langle h(w_n) + f_2, u_0 \rangle}{\|u_n\|} > (M + \gamma_2)\|u_0\|,
\]

which is a contradiction. Thus \((g, H_3)\) is \((a, b)\)-admissible on \( K^c \) for small enough \( r \) and so

\( I_{a,b}[g, h], K^c = I_{a,b}[g, H_3(\cdot, 1)], K^c = I_{a,b}[g, H_3(\cdot, 0)], K^c \).

Now we prove that \( I_{a,b}[g, H_3(\cdot, 1)], K^c = \{0\} \). By (ii) of Lemma 2.3, it is sufficient to verify that the following system of variational inequalities:

\[
a(u, v - u) + j_1(u, v) - j_1(u, u) \geq \langle g(w), v - u \rangle + \langle f_1, v - u \rangle, \quad \forall v \in X, \tag{35}
\]

\[
b(u, v - u) + j_2(u, v) - j_2(u, u) \geq N(z, v - u), \quad \forall v \in K \tag{36}
\]
has no solution in $K \times K^r$. If $(u, w) \in K \times K^r$ is a solution of the above system, letting $v = u + u_0$ in (36), then

$$b(u_0, u_0) + j_2(u, u_0) \geq N(z, u_0).$$

On the other hand, $u = K_a g(w)$ from (35) and so

$$N(z, u_0) \leq (M + r_2) \|K_a g(w)\| \leq \frac{M + r_2}{\alpha - r_1} \|g(w)\| \leq \frac{(M + r_2)\xi}{\alpha - r_1} - r,$$

which is a contradiction for sufficiently small $r$. Thus $I_{b,1}(g, H^3(\cdot, 1)), K^r) = \{0\}$ and so we have $I_{a,1}(g, h), K^r \setminus K^r) = \{1\}$. Therefore, there exists a nonzero solution $(u, w)$ for the system of variational inequalities (4). This completes the proof. □

References


