Holey Segre Varieties

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Holey Segre varieties are introduced, which generalize classical Segre varieties and whose existence is suggested by the fact that any non-prime finite field contains proper subfields. More precisely, a holey Segre variety is the tensor product $\mathbb{P}G(n, q) \otimes \mathbb{P}G(m, q^k)$. An algebraic description of such an object is given and its structure is investigated.

Furthermore, special choices of the parameters are considered, in which cases a holey Segre variety admits a partition into classical Segre varieties. Also, some holey Segre varieties are isomorphic to non-canonical subgeometries embedded in suitable Segre varieties.

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1. INTRODUCTION

A Segre variety is defined as the tensor product of two projective spaces over the same field. Thus, a natural question is to ask whether it makes sense to take projective spaces over different fields. In such a general setting this question seems quite meaningless. However, in the case of finite fields and by considering the field $GF(q)$ and its extension field $GF(q^k)$, it is possible to generalize the idea of a Segre variety by taking the tensor product (over $GF(q)$) of two projective spaces, over $GF(q)$ and $GF(q^k)$, respectively. The so-defined object is what we call a holey Segre variety (briefly, an H-Segre variety). The name is motivated by the fact that an H-Segre variety can be viewed as a ‘classical’ Segre variety with something missing.

After defining a H-Segre variety and providing an algebraic description of it, we investigate its structure and show that such a variety can be described by (classical) Segre varieties. Also, for some choices of the parameters, it admits a partition into Segre varieties. Furthermore, for suitable indices, a holey Segre variety turns out to be a non-canonical subgeometry embedded into an appropriate Segre variety.

Recent results on fibrations of projective spaces into Segre varieties inspired us to a generalization of the hyperbolic fibration in [2] and we are able to define what we call a Segre fibration, i.e., a partition of an H-Segre variety (of certain indices) into a number of suitable Segre varieties and projective spaces.

Finally, both plane sections and behaviours w.r.t. subgeometries are examined for the smallest non-trivial holey Segre variety.

We would like to mention that a completely different generalization of a Segre variety is that provided by Herzer in [6], who introduced generalized Segre varieties and investigated their structure.

2. DEFINITIONS AND PRELIMINARY RESULTS

Let $PG(n, q)$ and $PG(m, q)$ be projective spaces over $GF(q)$, the field with $q$ elements, $q$ a prime power, with $n, m \geq 0$. Set $N := (n + 1)(m + 1) − 1$.

For all points $x = (x_0, \ldots, x_n) \in PG(n, q)$ and $y = (y_0, \ldots, y_m) \in PG(m, q)$, define

$$x \otimes y = (x_0y_0, \ldots, x_0y_m, x_1y_0, \ldots, x_1y_m, \ldots, x_ny_0, \ldots, x_ny_m).$$

The Segre variety, product of $PG(n, q)$ and $PG(m, q)$, is the variety $S_{n,m}(q)$, of $PG(N, q)$, briefly $S_{n,m}$ whenever it is clear from the context the field we are working in, consisting of

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all points represented by the vectors $x \otimes y$, as $x$ and $y$ range over all points of $PG(n, q)$ and $PG(m, q)$, respectively.

A more classical, and equivalent, definition of a Segre variety is via its parametric equations; namely,

$$\xi_{ij} = x_i y_j, \quad i = 0, 1, \ldots, n, \quad j = 0, 1, \ldots, m,$$

where the $\xi_{ij}$ are homogeneous projective coordinates in $PG(N, q)$.

For any non-negative integer $n$, put $\theta_n = q^n + q^{n-1} + \cdots + q + 1$. The variety $S_{n,m}$ contains $\theta_n \theta_m = (q^n + q^{n-1} + \cdots + q + 1)(q^m + q^{m-1} + \cdots + q + 1)$ points.

$S_{n,m}$ contains two systems (reguli) of spaces, say $M_1$ and $M_2$, such that $M_1$ (resp. $M_2$) consists of $\theta_n$ $m$-dimensional (resp. $\theta_m$ $n$-dimensional) spaces, each of which is an $S_{0,m}$ (resp. $S_{n,0}$). Spaces of the same system are pairwise skew and any two spaces of different systems meet in exactly one point. The spaces of each system partition $S_{n,m}$ and they form an $m$-spread ($n$-spread), respectively.

Furthermore, the full automorphism group of a Segre variety $S_{n,m}$ is isomorphic to $PG L(n+1, q) \times PG L(n+1, q) \times C_2$, and the full automorphism group of a Segre variety $S_{n,m}$, $n \neq m$, is isomorphic to $PG L(n+1, q) \times PG L(m+1, q)$.

For background and more details, see [5, 8, 10, 11].

**Definition 2.1.** Let $q$ be any prime power, and let $m, n, k$ be non-negative integers. A *holey Segre variety* (briefly, an H-Segre) $S_{n,m}(q, k)$ in $PG(nm + n + m, q^k)$ is the Segre product $PG(n, q) \otimes PG(m, q^k)$.

Note that, by definition, for $k = 1$ a holey Segre variety $S_{n,m}(q, 1)$ is a (classical) Segre variety $S_{n,m}(q)$ over the field $GF(q)$. Furthermore, if $k \geq 2$, $S_{n,m}(q, k)$ is not a subvariety of $S_{n,m}(q^k)$, (see [8]). Also, an H-Segre is not a Segre variety defined over a subfield of $GF(q^k)$.

The points of $S_{n,m}(q, k)$ have homogeneous projective coordinates, over $GF(q^k)$, given by

$$\eta_{ij} = x_i y_j, \quad i = 0, 1, \ldots, n, \quad j = 0, 1, \ldots, m,$$

with $x_i$ (resp. $y_j$) homogeneous projective coordinates in $PG(n, q)$ (resp. $PG(m, q^k)$). Moreover, these points can be described via parametric equations over $GF(q)$; if $w$ is an element of $GF(q^k)$ of degree $k$ over $GF(q)$, for each of the $y_j$ one can write $y_j = a_{0,j} + a_{1,j} w + \cdots + a_{k-1,j} w^{k-1}$ and substitute into the above equations.

Recall the well known result.

**Theorem 2.2 ([17, Theorem 4.29]).** The following are equivalent.

1. The projective space $PG(m, q^k)$ can be partitioned into subgeometries isomorphic to $PG(m, q)$;
2. $gcd(m+1, k) = 1$.

As a consequence, we prove the next result.

**Theorem 2.3.** Assume $gcd(m+1, k) = 1$. Then the holey Segre variety $S_{n,m}(q, k)$ can be partitioned into Segre varieties $S_{n,m}(q)$.

**Proof.** Fix a partition $P$ of $PG(m, q^k)$ into subgeometries $\Sigma \cong PG(m, q)$. Thus, $S_{n,m}(q, k)$ is the Segre product $PG(n, q) \otimes PG(m, q^k) = PG(n, q) \otimes (\bigcup_{\Sigma \in P} \Sigma) = \bigcup_{\Sigma \in P} (PG(n, q) \otimes \Sigma)$.  

**Theorem 2.4.** The automorphism group $Aut(S_{n,m}(q, k))$ of a holey Segre variety $S_{n,m}(q, k)$, $(n, k) \neq (m, 1)$, is isomorphic to $PG L(n+1, q) \times PG L(m+1, q^k)$. 

PROOF. By looking at the structure of the H-Segre variety (viewed as a disjoint union of \(\theta_n\ n\)-dimensional subspaces over \(GF(q)\) or \(\theta_n\ m\)-dimensional subspaces over \(GF(q^k)\)) it is straightforward that \(\text{Aut}(S_m,q,k) \times PGL(m+1,q) \times PGL(m+1,q^k)\), where the first factor fixes each member of the collection of \(n\)-subspaces, whereas the second factor acts transitively on the members of the \(m\)-spread. \qed

3. A DESCRIPTION OF HOLEY SEGRE VARIETIES VIA SEGRE VARIETIES

Recall that the points of \(S_{n,1}(q,k)\) have homogeneous projective coordinates, over \(GF(q^k)\), given by

\[
\eta_{ij} = x_i y_j, \quad i = 0, 1, \ldots, n, \quad j = 0, 1,
\]

with \(x_i\) (resp. \(y_j\)) homogeneous projective coordinates in \(P_G(n,q)\) (resp. \(P_G(1,q^k)\)).

For all \(x_i, i = 0, 1, \ldots, n\), and for either \(\eta_0 = 1\) or \(\eta_0^{-1} \in GF(q)\), the points of \(S_{n,1}(q,k)\) with coordinates \(\eta_{ij} = x_i y_j\) form a Segre variety \(S_{n,q}(q,k)\) over the field \(GF(q)\), and this will be considered as the 'part at infinity' of the H-Segre variety. The remaining points form the 'finite part' of \(S_{n,1}(q,k)\), which has parametric equations

\[
\eta_0 = x_0, \quad \eta_1 = x_1, \quad \ldots, \quad \eta_n = x_n, \quad \text{with } z \in GF(q^k) \setminus GF(q).
\]

Let \(w\) be a generator of \(GF(q^k)\) over \(GF(q)\) (not necessarily a primitive element). Put \(z = a_0 + a_1 w + \cdots + a_{k-1} w^{k-1}\) and note that not all of \(a_1, \ldots, a_{k-1}\) are zero. Hence, the points of the 'finite part' are the points \((x_0, x_1, \ldots, x_n, x_0 z, x_1 z, \ldots, x_n z)\) whose coordinates, homogeneous over \(GF(q)\) as the \(x_i\) are, can be rewritten as \(\eta_0 = x_0, \eta_1 = x_1, \ldots, \eta_n = x_n\), or \(x_0 a_0, x_1 a_1, \ldots, x_n a_n\), where \(a_0 = 1\) and not all of \(a_1, \ldots, a_{k-1}\) zero, i.e., such points are exactly the following: fix the Segre variety \(S_{n,k}(q)\); on each space of the system of \(k\)-dimensional subspaces, delete the points which belong either to the hyperplane with equation \(a_k = 0\) or to the line with equations \(a_1 = \cdots = a_{k-1} = 0\) (view the \(a_j\) as homogeneous projective coordinates in this \(PG(k,q)\)).

Actually, any point of \(PG(1,q^k) \setminus PG(1,q)\), on the finite part of the H-Segre, blows up to the above-mentioned part of a \(k\)-dimensional space over \(GF(q)\).

We give the general idea of the recursive construction of \(S_{n,m}(q,k)\) by showing in detail the first steps. Consider \(S_{n,2}(q,k) = PG(n,q) \setminus PG(1,q^k)\). As \(PG(1,q^k)\) can be viewed as \(AG(2,q^k)\) \(\times PGL(1,q^k)\), hence \(S_{n,2}(q,k)\) is the union of an \(S_{n,1}(q,k)\), which is called the 'part at infinity' of \(S_{n,2}(q,k)\), and can be described (as above) as the union of an \(S_{n,1}(q)\) and a suitable part of \(S_{n,k}(q)\), and a 'finite part', which is \(PG(n,q) \setminus AG(2,q^k)\). As \(AG(2,q^k) \cong AG(2,k)\), using a generator of \(GF(q^k)\) over \(GF(q)\), it is easy to show that \(PG(n,q) \setminus AG(2,k)\) is the following subset of the Segre variety \(S_{n,2k}(q)\): on each \(2k\)-dimensional space of one of the systems of subspaces, delete the points which belong to one of (1) the hyperplane with equation \(a_k = 0\), (2) the hyperplane with equation \(a_1 = 0\) or (3) the plane with equations \(a_1 = \cdots = a_{k-1} = 0\) (view the \(a_j\) as homogeneous projective coordinates in this \(PG(2k,q)\)).

Here there are two types of blowing up: the points on the part at infinity which are not on the \(S_{n,1}(q)\) blow-up, as in the previous case, to a 'piece' of a \(k\)-dimensional space over \(GF(q)\), whereas each point on the finite part of the H-Segre blows up to a suitable part of a \(2k\)-dimensional space over \(GF(q)\).

Next, \(S_{n,3}(q,k) = PG(n,q) \setminus PG(3,q^k)\) can be viewed as the union of an \(S_{n,2}(q,k) = PG(n,q) \setminus PG(2,q^k)\), which is the 'part at infinity' and can be described as before, and the 'finite part', which is \(PG(n,q) \setminus AG(3,q^k)\), which is described as a part of \(S_{n,3k}(q)\).
using $AG(3, q^k) \cong AG(3k, q)$. The H-Segre variety $S_{n,m}(q, k) = PG(n, q) \otimes PG(m, q^k)$ is obtained recursively with $m$ steps following the previous idea.

4. HOLEY SEGRE VARIETIES AND PROJECTIVE SUBGEOMETRIES

Consider the smallest non-trivial H-Segre $S_{1,1}(q, 2)$. This is a set of $(q^2 + 1)(q + 1)$ points in $PG(3, q^2)$ contained in a hyperbolic quadric $Q^+(3, q^2)$.

Let $S$ be a Singer cycle of $GL(2, q^2)$. In $GL(2, q^2)$, $S$ is similar to the diagonal matrix $D = \text{diag}(\omega, \omega^3)$, where $\omega$ is a primitive element of $GF(q^2)$ over $GF(q^2)$.

Since $\omega^{q^2 + 1}$ is a primitive element of $GF(q^2)$ over $GF(q)$, it follows that $\text{diag}(\omega^{q^2 + 1}, \omega^3)$ is the canonical form of a Singer cycle $T$ of $GL(2, q)$.

Consider the Kronecker product $R = T \otimes S$. Working over the field $GF(q^4)$, we have $\text{diag}(\omega^{q^2 + 1}, \omega^q) \otimes \text{diag}(\omega, \omega^3)$, which is the diagonal matrix $\text{diag}(\omega^{q^2 + 2}, \omega^{2q^2 + 1}, \omega^{q^2 + q}, \omega^{q^2 + q + 3})$ in $GL(4, q^4)$.

Note that the eigenvalues of $R$ in $GF(q^4)$ are not conjugate over $GF(q)$, but they are pairwise conjugate over $GF(q^2)$.

It follows that $(R)$ as a collineation group of $PG(3, q^2)$ fixes two lines, and as a collineation group of $PG(3, q)$ acts irreducibly.

**Lemma 4.1.** The subgroup $(R)$ of $PGL(4, q)$ has order $(q + 1)(q^2 + 1)$.

**Proof.** Let $\alpha = q^3 + q^2 + q + 1$. Then $\omega^{(q^2 + 2)\alpha} = \omega^{2(q^2 + 1)\alpha} = \omega^{q^3 + q\alpha + \alpha} = \omega^{(q^3 + q^2 + q)\alpha} = \beta^3$, where $\beta \in GF(q)^*$. and $\alpha$ is the smallest positive integer with this property.

Observe that the order of $R$ as a collineation of $PG(3, q^2)$ is $q^2 + 1$.

Over $GF(q)$, since $(R)$ is irreducible and the order of $R$ is $q^2 + q + 1$, $S_{1,1}(q, 2)$ is a subgeometry $U \cong PG(3, q)$, embedded in $Q^+(3, q^2)$. (Note that $U$ is not a subgeometry over the subfield $GF(q)$.

Indeed, $U$ is the subgeometry of $PG(3, q^2)$ given by $[\omega^{q^2 + 2i}, \omega^{q^2 + 1 + 2i}, \omega^{q^2 + q + 2i}, \omega^{q^2 + q^2 + 2i}, i = 0, 1, \ldots, q^3 + q^2 + q]$, which is the image of the point $P = (1, 1, 1, 1)$ under the group $R$. The same observation applies in the next sections when the argument involves Singer cycles.)

Also, by definition of a Segre product, $U$ is a union of $q^2 + 1$ disjoint lines defined over $GF(q)$ and so is a line spread of $PG(3, q)$.

The above construction can be generalized with only formal changes to the case of $S_{n,1}(q, n + 1)$, yielding an $n$-spread of $PG(2n + 1, q)$.

**Theorem 4.2.** An H-Segre variety $S_{n,1}(q, n + 1)$ is a subgeometry $U \cong PG(2n + 1, q)$ embedded in the Segre variety $S_{1,n}(q^{n+1})$. The subgeometry $U$ is naturally partitioned into $q^{n+1} + 1$ subspaces of dimension $n$ (an $n$-spread).

**Proof.** Let $S$ be a Singer cycle in $GL(2, q^{n+1})$. Hence, $S$ is similar in $GL(2, q^{2(n+1)})$ to the diagonal matrix $D = \text{diag}(\omega, \omega^{q^2})$, where $\omega$ is a primitive element of $GF(q^{2(n+1)})$ over $GF(q^{n+1})$. Since $\omega^{q^{n+1} + 1}$ is a primitive element of $GF(q^{n+1})$ over $GF(q)$, it follows that $\text{diag}(\omega^{n+1}, \omega^{q^2 + 1}, \omega^{n+1} + q + 1, \ldots, \omega^{q^2 + 2q^2 + 1})$ is the canonical form of a Singer cycle $T$ of $GL(n + 1, q)$. The Kronecker product $T \otimes S$ induces a collineation of $PG(2n + 1, q^{n+1})$ whose orbits are subgeometries $U \cong PG(2n + 1, q)$ embedded in the Segre variety $S_{1,n}(q^{n+1})$.

We explicitly point out that the above spread arises because of the particular choice of the indices.
Next, look at the H-Segre variety $S_{1,1}(q, 3)$, which is contained in the hyperbolic quadric $Q^+(3, q^3)$ clearly, $S_{1,1}(q, 3) \subset S_{2,1}(q, 3)$ in a canonical way.

As shown above, $S_{2,1}(q, 3)$ is a subgeometry $U \equiv PG(5, q)$ embedded in the Segre variety $S_{1,2}(q^3)$. Moreover, a spread of $U$ consisting of $q^3 + 1$ planes, each isomorphic to $PG(2, q)$, arises as in Theorem 4.2. By definition, $S_{1,1}(q, 3)$ is the union of a set $S$ of $q^3 + 1$ pairwise skew lines defined over $GF(q)$. Since by [8, Theorem 25.5.12 (iv)] $Q^+(3, q^3)$ can be viewed as the intersection of $S_{1,2}(q^3)$ with a suitable subspace $\Sigma \equiv PG(3, q^3)$ of $PG(5, q^3)$, it is clear that $S$ is the complete intersection $Q^+(3, q^3) \cap U$.

Let $S$ be a Singer cycle of $GL(2, q^3)$. Thus, in $GL(2, q^6)$, $S$ is similar to the diagonal matrix $D = \text{diag}(\omega, \omega^q, \omega^{q^2}, \omega^{q^3})$, where $\omega$ is a primitive element of $GF(q^6)$ over $GF(q^3)$. It follows that $\omega^{q^4+q^3+1}$ is a primitive element of $GF(q^7)$ over $GF(q)$. Hence $D' = \text{diag}(\omega^{q^4+q^3+1}, \omega^{q^2+q^3+1})$ is the canonical form of a Singer cycle $T$ of $GL(2, q)$. Consider the Kronecker product $S \otimes T$. When working in $GF(q^6)$, this gives the diagonal matrix $\text{diag}(\omega^{q^4+q^3+1}, \omega^{q^2+q^3+1}, \omega^{q^2+q^3+1})$.

The eigenvalues of this matrix are distinct elements in $GF(q^6)$, pairwise conjugate over $GF(q^3)$ (they are not conjugate over $GF(q^4)$). This means that $(S \otimes T)$ fixes two skew lines in $PG(3, q^3)$. By regarding $\omega$ as a primitive element of $GF(q^6)$ over $GF(q)$, the eigenvalues of $S \otimes T$ represent four fixed points in a subgeometry $U \equiv PG(5, q)$. This shows that $S_{1,1}(q, 3)$ is the complete intersection of $Q^+(3, q^3)$ with $U$.

Generalizing the argument above (the eigenvalues of the matrix $S \otimes T$ are, in general, distinct elements of $GF(q^{2n}))$ yields the following theorem.

**Theorem 4.3.** An H-Segre variety $S_{1,1}(q, n)$, $n \geq 3$, is the complete intersection of $Q^+(3, q^n)$ with the $(2n - 1)$-dimensional subgeometry $PG(2n - 1, q)$ representative of a H-Segre variety $S_{n-1,1}(q, n)$.

## 5. Segre Fibrations

Fibrations of projective spaces into Segre varieties appear, e.g., in [1–3].

Note that the H-Segre variety $S_{1,1}(q, 2)$ is the disjoint union of $q - 1$ hyperbolic quadrics and two skew lines; therefore, since $S_{1,2}(q, 2)$ is isomorphic to $PG(3, q)$, this partition yields a hyperbolic fibration of $PG(3, q)$ as defined in [2]. Indeed, take in $PG(3, q^3)$ two skew lines, say $l$ and $r$, with $l$ defined over $GF(q)$ and $s$ defined over $GF(q^3)$. With the exception of two points $P$ and $Q$, the point-set of $r$ can be partitioned into $q - 1$ Baer sublines, say $l_1, \ldots, l_{q-1}$. By taking all lines joining points of $l$ and points of $l_i$, we obtain $q - 1$ hyperbolic quadrics $Q^+(3, q^3)$. Finally, $PG(1, q) \otimes P$ and $PG(1, q) \otimes Q$ are the two skew lines of the hyperbolic fibration.

This can be generalized in two different ways.

The first one is mainly a remark. Observe that $S_{n,1}(q, n+1) = PG(n, q) \otimes PG(1, q^n+1) \equiv PG(2n + 1, q)$ can be partitioned into: (i) $q^n+1$ copies of $S_{1,n}(q)$ if $n$ is even, or (ii) $2^n+1$ copies of $S_{1,n+1}(q)$ and two copies of $PG(n, q)$ if $n$ is odd, by partitioning $PG(1, q^n+1)$ into copies of $PG(1, q)$, plus two points if $n$ is odd, and joining the relevant points as in the example above.

The second generalization is less straightforward. We give the details for the first step, then we sketch the general construction. First observe that $S_{2,3}(q, 3) = PG(2, q) \otimes PG(2, q^3) \equiv PG(8, q)$. Indeed, let $\omega$ be a primitive element of $GF(q^9)$ over $GF(q)$. Hence, $\omega^{q^4+q^3+1}$ is a primitive element of $GF(q^3)$ over $GF(q)$. A Singer cycle of $GL(3, q)$ is conjugate in $GL(3, q^3)$ to the diagonal matrix $\text{diag}(\omega^{q^4+q^3+1}, \omega^{q^2+q^3+1}, \omega^{q^2+q^3+1})$, and a Singer cycle of $GL(3, q^3)$ is conjugate in $GL(3, q^9)$ to the diagonal matrix $\text{diag}(\omega, \omega^9, \omega^{q^4})$. Also,


\[
\text{diag}(\omega^1, \omega^2, \omega^3) \otimes \text{diag}(\omega^4 + q^1 + q^2, \omega^5 + q^3 + q^4) = \text{diag}(\omega^6 + q^3 + q^4, \omega^6 + q^4 + q^1, \\
\omega^6 + q^4 + q^2, \omega^7 + q^3 + q^4 + q^1, \omega^8 + q^5 + q^6 + q^7)
\]

and the nine entries of the last vector are in threees conjugate over \(GF(q^3)\) but they are not conjugate over \(GF(q)\), hence they span a \(PG(8, q)\), which is a non-canonical subgeometry of \(PG(8, q^3)\) isomorphic to \(S_{2, 2}(q, 3)\). (Recall that \(PG(V, GF(q))\), \(V\) a vector space over \(GF(q)\), is a canonical subgeometry of \(PG(V^*, GF(q'))\) whenever \(V^* = GF(q') \otimes V\). For more details, see, e.g., [9].) On the other hand, by [4] there is a partition of \(PG(2, q^3)\) into a simplex (three points) and \((q^3 + 2)(q - 1)\) copies of \(PG(2, q)\). By joining the points of the given \(PG(2, q)\) with the points on each element of the above-mentioned partition of \(PG(2, q^3)\), we obtain a partition of the H-Segre variety \(S_{2, 2}(q, 3)\) into \((q^3 + 2)(q - 1)\) Segre varieties \(S_{2, 2}(q)\) and three pairwise skew planes over \(GF(q)\).

This construction can be generalized to any \(n\) and provides a sort of generalized hyperbolic fibration, that we call a Segre fibration, which partitions \(S_{n, n}(q, n + 1) \cong PG(n^2 + 2n, q)\) into a number of Segre varieties \(S_{n, n}(q)\) and \(n + 1\) pairwise skew \(n\)-dimensional spaces. Indeed, let \(\omega\) be a primitive element of \(GF(q(n^2 + 2n + 1)) \otimes GF(q)\). Hence, \(\omega^{q^2} + q^2 - 1 + q^{n^2 - n - 2} + \cdots + q^{n + 1} + 1\) is a primitive element of \(GF(q(n^2 + 2n + 1))\) over \(GF(q)\). As before, the tensor product of the diagonal form of (a conjugate of) a Singer cycle of \(GL(n^2 + 2n + 1, q)\) over \(GF(q)\) and one of \(GL(n + 1, q)\) over \(GF(q)\) gives a diagonal matrix whose \(q^2 + 2n + 1\) non-zero entries are not conjugate over \(GF(q)\), thus they span a \(PG(n^2 + 2n, q)\), which is a non-canonical subgeometry of \(PG(n^2 + 2n + 1, q)\) isomorphic to \(S_{n, n}(q, n + 1)\). On the other hand, by [4] there is a partition of \(PG(n, q(n^2 + 1))\) into a simplex \((n + 1)\) points and the right number of copies of \(PG(n + 1, q)\). By joining the relevant points and constructing the H-Segre variety \(S_{n, n}(q, n + 1) = PG(n, q(n^2 + 1)) \otimes PG(n, q(n^2 + 1))\), we obtain a partition of \(S_{n, n}(q, n + 1)\) into \((q^2 + q^2 - 1 + q^{n^2 - n - 2} + \cdots + q^{n + 1} + 1)/(q^n - 1)\) Segre varieties \(S_{n, n}(q)\) and \(n + 1\) pairwise skew \(n\)-dimensional spaces over \(GF(q)\).

6. SOME REMARKS ON THE SMALLEST HOLEY SEGRE VARIETY \(S_{1, 1}(q, 2)\)

First, we want to study an arbitrary plane section of the ‘smallest’ non-trivial H-Segre variety \(S := S_{1, 1}(q, 2)\).

In \(PG(3, q^2)\), with homogeneous projective coordinates \(x_1, x_2, x_3, x_4\), consider the lines \(l : x_3 = x_4 = 0\) and \(m : x_1 = x_2 = 0\). Choose on \(l\) a Baer subline, say \(\tilde{l} = [(a, b, 0, 0) : a, b \in GF(q)\), \(a, b\) not both zero], and let \(m\) be described by the point \((0, 0, c, d)\), with \(c, d \in GF(q^2)\) and \(c, d\) not both zero. The lines joining a point of \(\tilde{l}\) and a point of \(m\) under the Plücker map form the chosen \(S\), embedded in \(Q^+(3, q^2)\), the latter viewed as a three-dimensional section of the Klein quadric \(Q^+(5, q^2)\) with equation \(y_1y_6 - y_2y_5 + y_3y_4 = 0\), where the \(y_i\) are homogeneous projective coordinates in \(PG(5, q^2)\). Next, consider all lines in \(PG(3, q^2)\) meeting \(l\) and \(m\) and the line \(t\) with equations \(x_1 = x_3\) and \(x_2 = x_4\). On the Klein quadric, we obtain a conic section, say \(C\), of \(Q^+(3, q^2)\). By considering the lines meeting \(l\) and \(m\), we obtain \(a = c\) and \(b = d\), and so \(c, d\) must lie in \(GF(q)\). It follows that the conic \(C\) and \(S\) meet in a Baer conic, the same holding for any secant plane. Obviously, the section of \(S\) with a tangent plane (at a point of \(S\)) is the union of a line over \(GF(q)\) and a line over \(GF(q^2)\), both incident with the tangency point, and the section of \(S\) with a plane tangent to \(Q^+(3, q^2)\) at a point not on \(S\) is a line over \(GF(q)\).

In what follows, we describe some intersections of \(S\) with (possibly non-canonical) subgeometries \(PG(3, q)\).

Clearly, the canonical subgeometry \(PG(3, q)\) of \(PG(3, q^2)\) intersects \(S\) in an \(S_{1, 1}(q)\).
On the other hand, as extreme cases, a non-canonical subgeometry $PG(3, q)$ may be either completely contained (cf. Theorem 4.2) or completely disjoint from $S$. The latter can be obtained as follows. Take any hyperbolic fibration of $PG(3, q^2)$ and fix two of its hyperbolic quadrics, say $\mathcal{H}_1$ and $\mathcal{H}_2$. Without loss of generality, we may assume that $S \subset \mathcal{H}_1$, which is by construction disjoint from $\mathcal{H}_2$, in which an $S_{1,1}(q, 2) \cong PG(3, q)$ (and this 3-space is non-canonical) is canonically embedded.

However, other cases do occur and some of the possible intersections contain an elliptic quadric $Q^-(3, q)$ (cf. [1, Theorem 4.1]), or a rational curve of any degree. A construction for the latter is the following.

Let $S$ be a Singer cycle of $GL(2, q^2)$. Then $S$ is conjugate in $GL(2, q^4)$ to the diagonal matrix $D = \text{diag}(\omega, \omega^{q^2})$, where $\omega$ is a primitive element of $GF(q^4)$ over $GF(q^2)$. Then, $\omega^{q^2+1}$ is a primitive element of $GF(q^2)$ over $GF(q)$ and $\text{diag}(\omega^{q^2+1}, \omega^{q^2+2})$ is the canonical form of a Singer cycle $T$ of $GL(2, q)$. Consider the Kronecker product $D^{q^2+1} \otimes D^{n(q^2+1)}$, $n \geq 1$. Then, $\text{diag}(\omega^{q^2+1}, \omega^{q^2+2}) \otimes \text{diag}(\omega^{q^2+n}, \omega^{q^2+n+1}, \omega^{q^2+n+q}, \omega^{n+1})$ is a Singer cycle $T \otimes T^n$, and so are the second and the third one. From a geometric point of view, the collineation group $\langle T \otimes T^n \rangle$ fixes two skew lines over $GF(q)$, hence there is a fixed non-canonical subgeometry $PG(3, q)$. All the other non-linear orbits of $T \otimes T^n$ are rational curves of degree $n + 1$. Observe that the trivial case in which the rational curve is a line is not included in the above construction. However, lines can be obtained by taking a hyperbolic fibration $F$ of $PG(3, q^2)$, two hyperbolic quadrics $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{F}$ such that $S \subset \mathcal{H}_1$, two lines over $GF(q)$, say $l_1$ and $l_2$ with $l_i \subset \mathcal{H}_i$, $i = 1, 2$, and by intersecting with $S$ the 3-space over $GF(q)$ spanned by $l_1$ and $l_2$.

Finally, note that we restricted our investigation to the smallest case, as the study of sections of higher-dimensional H-Segre varieties by subspaces of the ambient space relies on the knowledge of analogous sections of classical Segre varieties and the latter is still extremely poor.

7. CONCLUDING REMARKS

An alternative approach to a generalization of Segre varieties consists in starting with the following definition.

**Definition 7.1.** Let $q$ be any prime power, and let $m, n, h, k$ be non-negative integers. A **double holey Segre variety** (briefly, a DH-Segre) $S_{n,m}(q, h, k)$ is the Segre product $PG(n, q^h) \otimes PG(m, q^k)$.

Note that if $h|k$, then $S_{n,m}(q, h, k) = S_{n,m}(q^h, k)$, $S_{n,m}(q, h, k)$ is naturally embedded in the Segre variety $S_{t,m}(q^t)$ in $PG(nm + n + m, q^t)$, for $t = [h, k]$ the least common multiple of $h$ and $k$.

Many results on DH-Segre varieties can be given using techniques similar to those in the previous sections. To clarify this idea, we discuss the following example.

Put $H := S_{1,1}(q, 2, 3) = PG(1, q^2) \otimes PG(1, q^3)$. Let $S_1$ ($S_2$) be a Singer cycle of $PG(1, q^2)$ ($PG(1, q^3)$) and denote by $D_1$ ($D_2$) its canonical form in $PGL(2, q^2)$ ($PGL(2, q^3)$). Note that the smallest field extension containing both $GF(q^2)$ and $GF(q^3)$ is $GF(q^{12})$. Let $\omega$ be a primitive element of $GF(q^{12})$ over $GF(q)$. Then $\omega^{q^8} \otimes \omega^{q^2}$ is a primitive element of $GF(q^4)$ ($GF(q^3)$) over $GF(q^2)$ ($GF(q^3)$). Hence,
diag(\(\omega^{q^{8}+q^{4}+1}, \omega^{q^{11}+q^{6}+q^{2}}\)) \otimes \text{diag}(\omega^{q^{8}+1}, \omega^{q^{8}+q^{4}})\) is a diagonal matrix whose four non-zero entries are not conjugate over \(GF(q)\). The corresponding four independent points can be completed to a simplex of \(\Sigma \equiv PG(11, q)\). As in Section 4, we conclude that \(H\) is the complete intersection of \(Q^{+}(3, q^{6})\) and \(\Sigma\).

Further, observe that, following the ideas in [1], nested holey Segre varieties and nested double holey Segre varieties can be defined and their properties turn out to be quite similar to those of nested Segre varieties, when the structure of the new objects is taken into account.

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