From Competition to Complementarity: Comparative Influence Diffusion and Maximization

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ABSTRACT
Influence maximization is a well-studied problem that asks for a small set of influential users from a social network, such that by targeting them as early adopters, the expected total adoption through influence cascades over the network is maximized. However, almost all prior work focuses on cascades of a single propagating entity or purely-competitive entities. In this work, we propose the Comparative Independent Cascade (Com-IC) model that covers the full spectrum of entity interactions from competition to complementarity. In Com-IC, users’ adoption decisions depend not only on edge-level information propagation, but also on a node-level automaton whose behavior is governed by a set of model parameters, enabling our model to capture not only competition, but also complementarity, to any possible degree. We study two natural optimization problems, Self Influence Maximization and Complementary Influence Maximization, in a novel setting with complementary entities. Both problems are NP-hard, and we devise efficient and effective approximation algorithms via non-trivial techniques based on reverse-reachable sets and a novel “sandwich approximation” strategy. The applicability of both techniques extends beyond our model and problems. Our experiments show that the proposed algorithms consistently outperform intuitive baselines in four real-world social networks, often by a significant margin. In addition, we learn model parameters from real user action logs.

1. INTRODUCTION
Online social networks are ubiquitous and play an essential role in our daily lives. Fueled by popular applications such as viral marketing, there has been extensive research in influence and information propagation in social networks, from both theoretical and practical points of view. A key computational problem in this field is influence maximization, which asks to identify a small set of $k$ influential users (also known as seeds) from a given social network, such that by targeting them as early adopters of a new technology, product, or opinion, the expected number of total adoptions triggered by social influence cascade (or, propagation) is maximized [9, 16].

The dynamics of an influence cascade are typically governed by a stochastic diffusion model, which specifies how adoptions propagate from one user to another in the network.

Most existing work focuses on two types of diffusion models — single-entity models and pure-competition models. A single-entity model has only one propagating entity for social network users to adopt: the classic Independent Cascade (IC) and Linear Thresholds (LT) models [16] belong to this category. These models, however, ignore complex social interactions involving multiple propagating entities. Considerable work has been done to extend IC and LT models to study competitive influence maximization, but almost all models assume that the propagating entities are in pure competition and users adopt at most one of them [1, 3, 4, 6–8, 14, 17, 22].

In reality, the relationship between different propagating entities is certainly more general than pure competition. In fact, consumer theories in economics have two well-known notions: substitute goods and complementary goods [23]. Substitute goods are similar ones that compete, and can be purchased, one in place of the other, e.g., smartphones of various brands. Complementary goods are those that tend to be purchased together, e.g., iPhone and its accessories, computer hardware and software, etc. There are also varying degrees of substitutability and complementarity: buying a product could lessen the probability of buying the other without necessarily eliminating it; similarly, buying a product could boost the probability of buying another to any degree. Pure competition only corresponds to the special case of perfect substitute goods.

The limitation of pure-competition models can be exposed by the following example. Consider a viral marketing campaign featuring iPhone 6 and Apple Watch. It is vital to recognize the fact that Apple Watch generally needs an iPhone to be usable, and iPhone’s user experience can be greatly enhanced by a pairing Apple Watch (see, e.g., http://bit.ly/1GOqesc). Clearly none of the pure-competition models is suitable for this campaign because they do not even allow users to adopt both the phone and the watch! This motivates us to design a more powerful, expressive, yet reasonably tractable model that captures not only competition, but also complementarity, and to any possible degrees associated with these notions.

To this end, we propose the Comparative Independent Cascade model, or Com-IC for short, which, unlike most existing diffusion models, consists of two critical components that work jointly to govern the dynamics of diffusions: edge-level information propagation and a Node-Level Automaton (NLA) that ultimately makes adoption decisions based on a set of model parameters, known as the Global Adoption Probabilities (GAPs). Of these, edge-level propagation is similar to the propagation captured by the classical IC and LT models, but only controls information awareness. The NLA is a novel feature and is unique to our proposal. Indeed, the term “comparative” comes from the fact that a user makes a comparison between them by “running” her NLA. Notice that “comparative” subsumes “competitive” and “complementary” as special cases. In theory, the Com-IC model is able to accommodate any number of propagating entities (items) and cover the entire spectrum from competition to complementarity between pairs of items, reflected by the values of GAPs. In this work, as the first step toward comparative influence diffusion and viral marketing, we focus on the case of two items. At any time, w.r.t. any item $A$, a user in the network is in one of the following four states: $A$-idle, $A$-suspended, $A$-rejected, or $A$-adopted. The NLA sets out probabilistic transition rules between states, and different GAPs are
applied based on a given user’s state w.r.t. the other item B and the relationship between A and B. Intuitively, competition (complementarity) is modeled as reduced probability (resp., increased probability) of adopting the second item after the first item is already adopted. After a user adopts an item, she propagates this information to her neighbors in the network, making them aware of the item. The neighbor may adopt the item with a certain probability, as governed by her NLA.

We then define two novel optimization problems for two complementary items A and B. Our first problem, Self Influence Maximization (SELFINFMAX), asks for k seeds for A such that given a fixed set of B-seeds, the expected number of A-adopted nodes is maximized. The second one, Complementary Influence Max‐imization (COMPINFMAX), considers the flip side of SELFINFMAX: given a fixed set of A-seeds, find a set of k seeds for B such that the expected increase in A-adopted nodes thanks to B is maximized. To the best of our knowledge, we are the first to systematically study influence maximization for complementary items.

We show that both problems are NP-hard under the Com-IC model. Moreover, two important properties, submodularity and monotonicity (see §2), which allow greedy a approximation algorithm frequently used for influence maximization, do not hold in unrestricted Com-IC model. Even when restricting Com-IC to mutual complementarity, submodularity does not hold in general.

To circumvent the aforementioned difficulties, we first show that submodularity holds for a subset of the complementary parameter space. We then make a non-trivial extension to the Reverse-Reachable Set (RR-set) techniques [2, 24, 25], originally proposed for influence maximization with single-entity models, to obtain effective and efficient approximation solutions to both SELFINFMAX and COMPINFMAX. Next, we propose a novel Sandwich Approximation (SA) strategy which, for a given non-submodular set function, provides an upper bound function and/or a lower bound function, and uses them to obtain data-dependent approximation solutions w.r.t. the original function. We further note that both techniques are applicable to a larger context beyond the model and problems studied in this paper: for RR-sets, we provide a new definition and general sufficient conditions not covered by [2, 24, 25] that apply to a large family of influence diffusion models, while SA applies to the maximization of any non-submodular functions that are upper- and/or lower-bounded by submodular functions.

In experiments, we first learn GAPs from user action logs from two social networking sites – Flixster.com and Douban.com. We demonstrate that our approximation algorithms based on RR-sets and SA techniques consistently outperform some intuitive baselines, typically by a significant margin on real-world networks.

To summarize, we make the following contributions:

- We propose the Com-IC model to characterize influence diffusion dynamics of products with arbitrary degree of competition or complementarity, and identify a subset of the parameter space under which submodularity and monotonicity of influence spread hold, paving the way for designing approximation algorithms (§3 and §5).

- We propose two novel problems – Self and Complementary Influence Maximization – for complementary products under the Com-IC model (§4).

- We show that both problems are NP-hard, and devise efficient and effective approximation solutions by non-trivial extensions to RR-set techniques and by proposing Sandwich Approximation, both having applicability beyond this work (§6).

- We conduct empirical evaluations on four real-world social networks and demonstrate the superiority of our algorithms over intuitive baselines, and also propose a methodology for learning global adoption probabilities for the Com-IC model from user action logs of social networking sites (§7).

To maintain compactness of the main text, we move some technical proofs and additional results to the appendix.

2. BACKGROUND & RELATED WORK

Given a graph \( G = (V, E, p) \) where \( p : E \rightarrow [0, 1] \) specifies pairwise influence probabilities (or weights) between nodes, and \( k \in \mathbb{Z}_+ \), the influence maximization problem asks to find a set \( S \subseteq V \) of k seeds, activating which leads to the maximum expected number of active nodes (denoted \( \sigma(S) \)) [16]. Under both IC and LT models, this problem is NP-hard: Chen et al. [10, 11] showed computing \( \sigma(S) \) exactly for any \( S \subseteq V \) is \#P-hard. Fortunately, \( \sigma(\cdot) \) is a submodular and monotone function of \( S \) for both IC and LT, which allows a simple greedy algorithm with an approximation factor of \( 1 - 1/e - \epsilon \), for any \( \epsilon > 0 \) [16, 21]. A set function \( f : 2^U \rightarrow \mathbb{R}_{\geq 0} \) is submodular if for any \( S \subseteq T \subseteq U \) and any \( x \in U \setminus T \), \( f(S \cup \{x\}) - f(S) \geq f(T \cup \{x\}) - f(T) \), and monotone if \( f(S) \leq f(T) \) whenever \( S \subseteq T \subseteq U \). Tang et al. [24, 25] proposed new randomized approximation algorithms which are orders of magnitude faster than the original greedy algorithms in [16].

In competitive influence maximization [1, 3, 4, 6–8, 14, 17, 22] (also surveyed in [9]), a common thread is the focus on pure competition, which only allows users to adopt at most one product or opinion. Most works are from the follower’s perspective [1, 6–8, 14], i.e., given competitor’s seeds, how to maximize one’s own spread, or minimize the competitor’s spread. [3, 17] aim to maximize the collective influence spread of all competitors.

For viral marketing with non-competing items, Datta et al. [12] studied influence maximization with items whose propagations are independent. Narayamath et al. [20] studied a setting with two sets of products, where a product can be adopted by a node only when it has already adopted a corresponding product in the other set. Their model extends LT. We depart by defining a significantly more powerful and expressive model in Com-IC, compared to theirs which only covers the special case of perfect complementarity. Our technical contributions for addressing the unique challenges posed by Com-IC are substantially different from [20], which follows the typical route as in [16].

[19] analyzed Twitter data to study the effect of different cascades on users and predicted the likelihood of a user adopting a piece of information (e.g., URLs in tweets) given cascades that the user was previously exposed to. More recently, McAuley et al. [18] used logistic regression to learn substitute/complementary relationships between products from user reviews. Both studies focus on data analysis and behavior prediction and do not provide diffusion modeling for competing and complementary items, nor do they study the influence maximization problem in this context.

3. COMPARATIVE IC MODEL

Review of Classical IC Model. In the IC model [16], there is just one entity (e.g., idea or product) being propagated through the network. An instance of the model has a directed graph \( G = (V, E, p) \) where \( p : E \rightarrow [0, 1] \), and a seed set \( S \subseteq V \). For convenience, we use \( p_{u,v} \) for \( p(u, v) \). At time step 0, the seeds are active and all other nodes are inactive. Propagation proceeds in discrete time steps. At time \( t \), every node \( u \) that became active at \( t - 1 \) makes one attempt to activate each of its inactive out-neighbors \( v \). This can be seen as node \( u \) “testing” if the edge \( (u, v) \) is “live” or “blocked”. The out-neighbor \( v \) becomes active at \( t \) iff the edge is live. The propagation ends when no new nodes become active.

Key differences from IC model. In the Comparative IC model
A we focus on just two products (Com-IC), there are at least two products. For ease of exposition, a reconsideration probability $\rho_v$ node level automaton (NLA) to decide which state to transit to.

1. Edge transition. For an untested edge $(u, v)$, flip a biased coin independently: $(u, v)$ is live w.p. $p_{uv}$, and blocked w.p. $1 - p_{uv}$. Each edge is tested at most once in the entire diffusion process.

2. Node tie-breaking. Consider a node $v$ to be tested at time $t$. Generate a random permutation $\pi$ of $v$'s in-neighbors (with live edges) that adopted at least one product at $t - 1$. Then, test $v$ with each such in-neighbor $u$ and $u$’s adopted item ($A$ and/or $B$) following $\pi$. If there is a $w \in N^+(v)$ adopting both $A$ and $B$, then test both products, following their order of adoption by $w$.

3. Node adoption. Consider the case of testing an $A$-idle node $v$ for adopting $A$ (Figure 1). If $v$ is not $B$-adopted, then w.p. $q_{A|B}$, it becomes $A$-adopted and w.p. $1 - q_{A|B}$ it becomes $A$-suspended. If $v$ is $B$-adopted, then w.p. $q_{A|B}$, it becomes $A$-adopted and w.p. $1 - q_{A|B}$ it becomes $A$-rejected. The case of adopting $B$ is symmetric.

4. Node reconsideration. Consider an $A$-suspended node $v$ that just adopts $B$ at time $t$. Define $\rho_A = \max\{q_{A|B} - q_{A|B|0}, 0\}/(1 - q_{A|B}).$ Then, $v$ reconsiders to become $A$-adopted w.p. $\rho_A$, or $A$-rejected w.p. $1 - \rho_A$. The case of reconsidering $B$ is symmetric.

Figure 2: Com-IC model: diffusion dynamics

(Com-IC), there are at least two products. For ease of exposition, we focus on just two products $A$ and $B$ below. Each node can be in any of the states {idle, suspended, adopted, rejected} w.r.t. each of the products. All nodes are initially in the joint state of $(A$-idle, $B$-idle). One of the biggest differences between Com-IC and IC is the separation of information diffusion (edge-level) and the actual adoption decisions (node-level). Edges only control the information that flows to a node: e.g., when $u$ adopts a product, its out-neighbor $v$ may be informed of this fact. Once that happens, $v$ uses its own node level automaton (NLA) to decide which state to transit to. This depends on $v$’s current state w.r.t. the two products as well as parameters corresponding to the state transition probabilities of the NLA, namely the Global Adoption Probabilities, defined below.

A concise representation of the NLA is shown in Figure 1. Each state is indicated by the label. The state diagram is self-explanatory. E.g., with probability $q_{A|B}$, a node transits from a state where it’s $A$-idle to $A$-adopted, regardless of whether it was $B$-idle or $B$-suspended. From the $A$-suspended state, it transits to $A$-adopted w.p. $\rho_A$ and to $A$-rejected w.p. $1 - \rho_A$. The probability $\rho_A$, called reconsideration probability, as well as the reconsideration process will be explained below. Note that in a Com-IC diffusion process defined below, not all joint state is reachable from the initial $(A$-idle, $B$-idle) state, e.g., $(A$-idle, $B$-rejected). Since all unreachable states are irrelevant to adoptions, they are negligible (details in appendix).

Global Adoption Probability (GAP). GAPs, consisting of four parameters $Q = \{q_{A|B}, q_{A|B|0}, q_{B|A}, q_{B|A|0}\} \in [0, 1]^4$ are important parameters of the NLA which decide the likelihood of adoptions after a user is informed of an item. $q_{A|B}$ is the probability that a user adopts $A$ given that she is informed of $A$ but not $B$-adopted, and $q_{A|B}$ is the probability that a user adopts $A$ given that she is already $B$-adopted. A similar interpretation applies to $q_{B|A}$ and $q_{B|A}$. Intuitively, GAPs reflect the overall popularity of products and how they are perceived by the entire market. They are considered aggregate estimates and hence are not user specific in our model. We provide further justifications at the end of this section and describe a way to learn GAPs from user action log data in §7.

GAPs enable Com-IC to model competition and complementarity, to arbitrary degrees. We say that $A$ competes with $B$ iff $q_{B|A} \leq q_{B|B}$. Similarly, $A$ complements $B$ iff $q_{B|A} \geq q_{B|B}$. We include the special case of $q_{B|A} = q_{B|B}$ in both cases above for convenience of stating our technical results, and it actually means that the propagation of $B$ is completely independent of $A$ (cf. Lemma 3). Competition and complementarity in the other direction are similar. The degree of competition and complementarity is determined by the difference between the two relevant GAPs, i.e., $|q_{B|A} - q_{B|B}|$ and $|q_{B|A} - q_{B|B}|$. For convenience, we use $Q^+$ to refer to any set of GAPs representing mutual complementarity: $\{q_{B|A} \leq q_{B|B} \& q_{B|A} \leq q_{B|B}\}$, and similarly, $Q^-$ for GAPs representing mutual competition: $\{q_{B|A} \leq q_{B|B} \& q_{B|A} \geq q_{B|B}\}$. Diffusion dynamics. Let $G = (V, E, p)$ be a directed social graph with pairwise influence probabilities. Let $S_A, S_B \subset V$ be the seed sets for $A$ and $B$. Influence diffusion under Com-IC proceeds in discrete time steps. Initially, every node is $A$-idle and $B$-idle. At time step 0, every $u \in S_A$ becomes $A$-adopted and every $u \in S_B$ becomes $B$-adopted. If $u \in S_A \cap S_B$, we randomly decide the order of $u$ adopting $A$ and $B$ with a fair coin. For ease of understanding, we describe the rest of the diffusion process in a modular way in Figure 2. We use $N^+(v)$ and $N^-(v)$ to denote the set of out-neighbors and in-neighbors of $v$, respectively.

We draw special attention to tie-breaking and reconsideration. Tie-breaking is used when a node’s in-neighbors adopt different products and try to inform the node at the same step. Node reconsideration concerns the situation that a node $v$ did not adopt $A$ initially but later after adopting $B$ it may reconsider adopting $A$: when $B$ competes with $A$ ($q_{A|B} \geq q_{A|B}$), $v$ will not reconsider adopting $A$, but when $B$ complements $A$ (specifically, $q_{A|B} < q_{A|B}$), $v$ will reconsider adopting $A$. In the latter case, the probability of adopting $A$, $\rho_A$, is defined in such a way that the overall probability of adopting $A$ is equal to $q_{A|B}$ (since $q_{A|B} = q_{A|B} + (1 - q_{A|B})\rho_A$).

Design Considerations. The design of Com-IC not only draws on the essential elements from a classical diffusion model (IC) stemming from mathematical sociology, but also closes a gap between theory and practice, in which diffusions typically do not occur just for one product or with just one mode of pure competition. With GAPs in the NLA, the model can characterize any possible relationship between two propagating entities: competition, complementarity, and any degree associated with them. GAPs are fully capable of handling asymmetric relationship between products. E.g., an Apple Watch ($A$) is complemented more by an iPhone ($B$) than the other way round: many functionalities of the watch are not usable without a pairing iPhone, but an iPhone is totally functional without a watch. This asymmetric complementarity can be expressed by any GAPs (satisfying $(q_{A|B} - q_{A|B}) > (q_{B|A} - q_{B|A}) \geq 0$). Furthermore, introducing NLA with GAPs and separating the propagation of product information from actual adoptions reflects Kalish’s famous characterization of new product adoption [15]: customers go through two stages — awareness followed by actual adoption. In Kalish’s theory, product awareness is propagated through word-of-

1 No generality is lost in assuming seeds adopt an item without watching the NLA: for every $v \in V$, we can create two dummy nodes $v_A, v_B$ and edges $(v_A, v)$ and $(v_B, v)$ with $p_{v_A, v} = p_{v_B, v} = 1$. Requiring seeds to go through NLA is equivalent to constraining that $A$-seeds ($B$-seeds) be selected from all $v_A$’s (resp. $v_B$’s).
Self Influence Maximization

In what follows, we propose two problems. The first one, since competitive viral marketing has been studied extensively (see the expressive Com-IC model. In this work, we focus on in-

4. FORMAL PROBLEM STATEMENTS

Many interesting optimization problems can be formulated under the expressive Com-IC model. In this work, we focus on influence maximization with complementary propagating entities, since competitive viral marketing has been studied extensively (see §2). In what follows, we propose two problems. The first one, Self Influence Maximization (SELFINFMAX), is a natural extension to the classical influence maximization problem [16]. The second one is the novel Complementary Influence Maximization (COMPINFMAX), where the objective is to maximize complementary effects (or “boost” the expected number of adoptions) by selecting the best seeds of a complementary good.

Given the seed sets $S_A, S_B$, we first define $\sigma_A(S_A, S_B)$ and $\sigma_B(S_A, S_B)$ to be the expected number of $A$-adopted and $B$-adopted nodes, respectively under the Com-IC model. Clearly, both $\sigma_A$ and $\sigma_B$ are real-valued bi-set functions mapping $2^V \times 2^V$ to $[0, |V|]$, for any fixed $Q$. Unless otherwise specified, GAPs are not considered as arguments to $\sigma_A$ and $\sigma_B$ as $Q$ is constant in a given instance of Com-IC. Also, for simplicity, we may refer to $\sigma_A(\cdot, \cdot)$ as $A$-spread and $\sigma_B(\cdot, \cdot)$ as $B$-spread. The following two problems are defined in terms of $A$-spread, without loss of generality.

**Problem 1 (SELFINFMAX).** Given a directed graph $G = (V, E, p)$ with pairwise influence probabilities, $B$-seed set $S_B \subset V$, a cardinality constraint $k$, and a set of GAPs $Q^*$, find an $A$-seed set $S_A^* \subset V$ of size $k$, such that the expected number of $A$-adopted nodes is maximized under Com-IC: $S_A^* \in \arg \max_{T \subseteq V, |T| = k} \sigma_A(T, S_B).

**Problem 2 (COMPINFMAX).** Given a directed graph $G = (V, E, p)$ with pairwise influence probabilities, $A$-seed set $S_A \subset V$, a cardinality constraint $k$, and a set of GAPs $Q^*$, find a $B$-seed set $S_B^* \subset V$ of size $k$ such that the expected increase (boost) in $A$-adopted nodes maximized under Com-IC: $S_B^* \in \arg \max_{T \subseteq V, |T| = k} \sigma_B(\sigma_A(T) - \sigma_A(S_A))/|T|).

**Theorem 1.** SELFINFMAX & COMPINFMAX are NP-hard.

**Proof.** SELFINFMAX subsumes influence maximization under classic IC when $S_B = \emptyset$ and $q_A(0) = q_A(1) = 1$. COMPINFMAX also subsumes influence maximization under classic IC: we set $q_{A(0)} = 0$ (thus without $B$-seeds, any $A$-seed set would have an $A$-spread of 0) and set $q_{A(1)} = q_{B(1)} = q_{B(0)} = 1$. We then create a special node $a$, which has a directed edge to all $v \in V$ with influence probability 1. When $S_A = \{a\}$, finding a $B$-seed set to maximize the increase in $A$-spread is equivalent to maximizing $B$-spread. This completes the proof.

In §7, we show that COMPINFMAX leads to a significant boost to the spread achieved by an $A$-seed set alone. We also note that computing the exact value of $\sigma_A(S_A, S_B)$ and $\sigma_B(S_A, S_B)$, for any given $S_A$ and $S_B$, is #P-hard (by similar arguments in the proof of Theorem 1).

5. PROPERTIES OF Com-IC

Since neither of SELFINFMAX and COMPINFMAX can be solved in PTIME unless $P = NP$, we explore approximation algorithms by studying submodularity and monotonicity for Com-IC, which may pave the way for designing approximation algorithms. Note that $\sigma_A$ is a bi-set function taking arguments $S_A$ and $S_B$, so we analyze the properties w.r.t. each of the two arguments. As appropriate, we refer to the properties of $\sigma_A$ w.r.t. $S_A$ ($S_B$) as self-monotonicity (resp., cross-monotonicity) and self-

5.1 An Equivalent Possible World Model

To facilitate a better understanding of Com-IC and our analysis on submodularity, we define a Possible World (PW) model that provides an equivalent view of the Com-IC model. Given a graph $G = (V, E, p)$ and a diffusion model, a possible world consists of a deterministic graph sampled from a probability distribution over all subgraphs of $G$. For Com-IC, we also need some variables for each node to fix the outcomes of random events in relation to the NLA (adoption, tie-breaking, and reconsideration), so that influence cascade is fully deterministic in a single possible world.

**Generative Rules.** Retain each edge $(u,v) \in E$ w.p. $p_{u,v}$ (live edge) and drop it w.p. $1 - p_{u,v}$ (blocked edge). This generates a possible world $W = (E_W, E_W)$, $E_W$ being the set of live edges. Next, $\forall v \in V: (i)$ choose “thresholds” $\alpha_u^{A,W}$ and $\alpha_u^{B,W}$ independently and uniformly at random from $[0, 1]$, for comparison with GAPs in adoption decisions (when $W$ is clear from context, we write $\alpha_u^A$ and $\alpha_u^B$); $(ii)$ generate a random permutation $\pi_v$ of all in-neighbors $u \in N^+(v)$ (tie-breaking); $(iii)$ sample a discrete value $\tau_v \in \{A, B\}$, where each value has a probability of $0.5$ (used for tie-breaking in case $v$ is a seed of both $A$ and $B$).

**Deterministic cascade in a PW.** At time step 0, nodes in $S_A$ and $S_B$ first become $A$-adopted and $B$-adopted, respectively (ties, if any, are broken based on $\tau_v$). Then, iteratively for each step $t \geq 1$, a node $v$ becomes “reachable” by $A$ at time step $t$ if $t$ is the length of a shortest path from any seed $u \in S_A$ to $v$ consisting entirely of live edges and $A$-adopted nodes. Node $v$ then becomes $A$-adopted at step $t$ if $\alpha_u^A \leq x$, where $x = q_{A\mid B}$ if $v$ is not $B$-adopted, otherwise $x = q_{A\mid B}$. For re-consideration, suppose $v$ just becomes $B$-adopted at step $t$, while being $A$-suspected (i.e., $v$ was reachable by $A$ before step $t$ but $\alpha_u^A > q_{A\mid B}$). Then, $v$ adopts $A$ if $\alpha_u^A \leq q_{A\mid B}$. The reachability and reconsideration tests of $B$ are symmetric. For tie-breaking, if $v$ is reached by both $A$ and $B$ at $t$, the permutation $\tau_v$ is used to determine the order in which $A$ and $B$ are considered. In addition, if $v$ is reached by $A$ and $B$ from the same in-neighbor, e.g., $u$, then the informing order follows the order in which $u$ adopts $A$ and $B$.

The following lemma establishes the equivalence between this possible world model and Com-IC. This allows us to analyze monotonicity and submodularity using the PW model only.

**Lemma 1.** For any fixed $A$-seed set $S_A$ and $B$-seed set $S_B$, the joint distributions of the sets of $A$-adopted nodes and $B$-adopted nodes obtained (i) by running a Com-IC diffusion from $S_A$ and $S_B$ and (ii) by randomly sampling a possible world $W$ and running a deterministic cascade from $S_A$ and $S_B$ in $W$, are the same.

**Remarks on Monotonicity.** It turns out that when $A$ competes with $B$ while $B$ complements $A$, monotonicity does not hold in general (see appendix for counter-examples). However, these cases are rather unnatural, and thus we next focus on mutual competition ($Q^+$) and mutual complementary cases ($Q^-$), for which we can show self- and cross-monotonicity do hold.

mouth effects; after an individual becomes aware, she would decide whether to adopt the item based on other considerations. Edges in the network can be seen as information channels from one user to another. Once the channel is open (live), it remains so. This modeling choice is reasonable as competitive goods are typically of the same kind and complementary goods tend to be adopted together.

We remark that Com-IC encompasses previously-studied single-entity and pure-competition models as special cases. When $q_{A(0)} = q_{B(1)} = 1$ and $q_{A(1)} = q_{B(0)} = 0$, Com-IC reduces to the (purely) competitive Influence Cascade model [9]. If, in addition, $q_{B(0)} = 0$, the model further reduces to the classic IC model.
Theorem 2. For any fixed B-seed set $S_B$, $\sigma_A(S_A, S_B)$ is monotonic in $S_A$ for any set of GAPs in $Q^+$ and $Q^-$. Also, $\sigma_A(S_A, S_B)$ is monotonic in $S_B$ for any GAPs in $Q^+$ and monotonically decreasing in $S_B$ for any GAPs in $Q^-$. 

5.2 Submodularity in Complementary Setting

Next, we analyze self-submodularity and cross-submodularity for mutual complementary cases ($Q^+$) that has direct impact on SelfInfMax and CompInfMax. The analysis for $Q^-$ is in the appendix.

For self-submodularity, we show that it is satisfied in the case of “one-way complementarity”, i.e., $B$ complements $A$ ($q_{A|B} \leq q_{B|A}$), but $A$ does not affect $B$ ($q_{B|A} = q_{B|A}$), or vice versa (Theorem 3). We will also show the $\sigma_A$ is cross-submodular in $S_B$ when $q_{B|A} = 1$ (Theorem 4). However, both properties are not satisfied in general (see appendix for counter-examples). We give two useful lemmas first, and thanks to Lemma 2 below, we may assume w.l.o.g. that tie-breaking always favors $A$ in complementary cases.

Lemma 2. Consider any Com-IC instance with $Q^+$. Given fixed $A$- and $B$-seed sets, for all nodes $v \in V$, all permutations of $v$’s in-neighbors are equivalent in determining if $v$ becomes $A$-adopted and $B$-adopted, and thus the tie-breaking rule is not needed for mutual complementary case.

Lemma 3. In the Com-IC model, if $B$ is indifferent to $A$ (i.e., $q_{B|A} = q_{B|A}$), then for any fixed $B$ seed set $S_B$, the probability distribution over sets of $B$-adopted nodes is independent of $A$-seed set. Symmetrically, the probability distribution over sets of $A$-adopted nodes is also independent of $B$-seed set if $A$ is indifferent to $B$.

Theorem 3. For any instance of Com-IC model with $q_{A|B} \leq q_{B|A}$ and $q_{B|A} = q_{B|A}$, (i) $\sigma_A$ is self-submodular w.r.t. A seed set $S_A$ for any fixed $B$-seed set $S_B$. (ii) $\sigma_B$ is self-submodular w.r.t. $B$ seed set $S_B$ and is independent of $A$-seed set.

Proof. Showing (ii) holds trivially. By Lemma 3, A does not affect B’s diffusion in any sense. Thus, $\sigma_B(S_A, S_B) = \sigma_B(S_A)$. It can be shown that the function $\sigma_B(S_A)$ is both monotone and submodular w.r.t. $S_A$ for any $q_{B|A}$, through a straightforward extension to the proof of Theorem 2.2 in Kempe et al. [16].

For (i), we first fix a possible world $W$ and a $B$-seed set $S_B$. Let $\Phi_B^W(S_B)$ be the set of $A$-adopted nodes in possible world with $A$-seed set $S_A$ ($S_B$ omitted when it is clear from the context). Consider two sets $S, S' \subseteq T \subseteq V$, some node $u \in V \setminus T$, and finally a node $v \in \Phi_B^W(T \cup \{u\}) \setminus \Phi_B^W(T)$. There must exist a live-edge path $P_A$ from $T \cup \{u\}$ consisting entirely of $A$-adopted nodes. We denote by $w_{0} \in T \cup \{u\}$ the origin of $P_A$.

We first prove a key claim: $P_A$ remain $A$-adopted when $S_A = \{w_{0}\}$. Consider any node $w_i \in P_A$. In this possible world, if $\alpha_{w_i}^A \leq q_{A|B}$, then regardless of the diffusion of $B$, $w_i$ will adopt $A$ as long as its predecessor $w_i-1$ adopts $A$. If $q_{A|B} < \alpha_{w_i}^A \leq q_{A|B}$, then there must also be a live-edge path $P_B$ from $S_B$ to $w_i$ that consists entirely of $B$-adopted nodes, and it boosts $w_i$ to adopt $A$. Since $q_{B|A} = q_{B|A}$, $A$ has no effect on $B$-propagation (Lemma 3), and $P_A$ always exists and all nodes on $P_B$ would still be $B$-adopted through $S_B$ (fixed) irrespective of $A$-seeds. Thus, $P_B$ always boosts $w_i$ to adopt $A$ as long as $w_i-1$ is $A$-adopted. Hence, the claim is held by a simple induction on $P_A$ starting from $w_0$.

Then, it is easy to see $w_0 = u$. Suppose otherwise, then $w_0 \in T$ must be true. By the claim above and self-monotonicity of $\sigma_A$ (Theorem 2), $v \in \Phi_B^W(\{w_0\})$ implies $w \in \Phi_B^W(T)$, a contradiction. Therefore, we have $v \notin \Phi_B^W(S)$ and $v \in \Phi_B^W(S \cup \{u\})$. This by definition implies $\Phi_B^W(\cdot)$ is submodular for any $W$ and $S_B$, which is sufficient to show that $\sigma_A(S_A, S_B)$ is submodular in $S_A$. □

Theorem 4. In any instance of Com-IC with mutual complementarity $Q^+$, $\sigma_A$ is cross-submodular w.r.t. $B$-seed set $S_B$ for any fixed $A$-seed set, as long as $q_{B|A} = 1$.

Proof. We first fix an $A$-seed set $S_A$. Consider any possible world $W$. Let $\Phi_B^W(S_B)$ be the set of $A$-adopted nodes in $W$ with $B$ seed-set $S_B$ and $A$ seed-set $S_A$. Consider $B$-seed sets $S \subseteq T \subseteq V$ and another $B$-seed $u \in V \setminus T$. It suffices to show that for any $v \in \Phi_B^W(T \cup \{u\}) \setminus \Phi_B^W(T)$, we have $v \in \Phi_B^W(S \cup \{u\}) \setminus \Phi_B^W(S)$.

Let an $A$-path be a live-edge path from some $A$-seed such that all nodes on the path adopt $A$ and $B$-path is defined symmetrically. If a node $w$ has $\alpha_{w,A}^A \leq q_{A|B}$, we say that $w$ is $A$ ready, meaning that $w$ is ready for $A$ and will adopt $A$ if it is informed of $A$, regardless of its status on $B$. We say a path from $S_A$ is an $A$-ready path if all nodes on the path (except the starting $A$-seed) are $A$-ready. It is clear that all nodes on an $A$-ready path would always adopt $A$ regardless of $B$-seeds. We define $B$-ready nodes and paths symmetrically. We can show the following claim (proofs in appendix).

Claim 1. On any $A$-path $P_A$, if some node $w$ adopts $B$ and all nodes before $w$ on $P_A$ are $A$-ready, then every node following $w$ on $P_A$ adopts both $A$ and $B$, regardless of the actual $B$-seed set.

Now consider the case of $S_B = T \cup \{u\}$ first. Since $v \in \Phi_B^W(T \cup \{u\})$, there must be an $A$-path $P_A$ from some node $w_0 \in S_A$ to $v$. If path $P_A$ is $A$-ready, then regardless of $B$ seeds, all nodes on $P_A$ would always be $A$ adopted, but this contradicts the assumption that $v \notin \Phi_B^W(T)$. Therefore, there exists some node $w$ that is not $A$ ready, i.e., $\alpha_{w,A}^A < q_{A|B}$. Let $w$ be the first non-$A$-ready node on path $P_A$. Then $w$ must have adopted $B$ to help it adopt $A$, and $\alpha_{w,B}^B \leq q_{B|A}$. We can show the following key claim.

Claim 2. There is a $B$-path $P_B$ from some $B$-seed $x_0$ to $T \cup \{u\}$, such that even if $x_0$ is the only $B$-seed, $w$ still adopts $B$.

With the key Claim 2, the rest of the proof follows the standard argument as in the other proofs. In particular, since even when $x_0$ is the only $B$-seed, $w$ can still be $B$-adopted, then by Claim 1, $v$ would be $A$-adopted in this case. Thus we know that $x_0$ must be $u$, because otherwise it contradicts our assumption that $v \notin \Phi_B^W(T)$ (also relying on the cross-monotonicity proof made for Theorem 2). Then again by the cross-monotonicity, we know that $v \notin \Phi_B^W(S \cup \{u\})$, but $v \notin \Phi_B^W(S)$. This completes our proof. □

6. APPROXIMATION ALGORITHMS

We first review the state-of-the-art in influence maximization and then derive a general framework (§6.1) to obtain approximation algorithms for SelfInfMax (§6.2) and CompInfMax (§6.3).

TIM algorithm. For influence maximization, Tang et al. [25] proposed a randomized algorithm, Two-phase Influence Maximization (TIM), which produces a $(1 - 1/e - \epsilon)$-approximation with at least $1 - 1/k$ probability in $O(k + \log |V|)$ runs. Because of the concept of Reverse-Reachable sets (RR-sets) [2], and applies to the Triggering model [16] that generalizes both IC and LT. TIM is orders of magnitude faster than greedy algorithm with Monte Carlo simulations [16], while still giving approximation solutions with high probability. Very recently they also propose a new improvement [24], which significantly reduces the number of RR-sets generated using martingale analysis. To tackle SelfInfMax and CompInfMax, we primarily focus on the challenging task of correctly generating RR-sets in Com-IC and other more general models, which is orthogonal to the contributions of [24]. Hereafter we focus on the framework of [25].
to the much more complex dynamics involved in Com-IC, adapting TIM to solve \textsc{SelfInfMax} and \textsc{CompInfMax} is far from trivial, as we shall show.

**Reverse-Reachable Sets.** In a deterministic (directed) graph \(G' = (V', E')\), for a fixed \(v \in V'\), all nodes that can reach \(v\) form an RR-set rooted at \(v\) \cite{2}, denoted \(R(v)\). A random RR set encapsulates two levels of randomness: (i) a “root” node \(v\) is randomly chosen from the graph, and (ii) a deterministic graph is sampled according to a certain probabilistic rule that retains a subset of edges from the graph. E.g., for the IC model, each edge \(\{u, v\} \in E\) is removed w.p. \((1 - p_{u,v})\), independently. TIM first computes a lower bound on the optimal solution value and uses this bound to derive the number of random RR-sets to be sampled, denoted \(\theta\). To guarantee approximation solutions, \(\theta\) must satisfy:

\[
\theta \geq \epsilon^{-2}(8 + 2\epsilon)|V| \cdot \frac{\ell \log |V| + \log (\frac{|V|}{k}) + \log 2}{\text{OPT}_k} ,
\]

where \(\text{OPT}_k\) is the optimal influence spread achievable amongst all size-\(k\) sets, and \(\epsilon\) represents the trade-off between efficiency and quality: a smaller \(\epsilon\) implies more RR-sets (longer running time), but gives a better approximation factor. The approximation guarantee of TIM relies on a key result from \cite{2}, re-stated here:

**Proposition 1.** (Lemma 9 in \cite{25}). Fix a set \(S \subseteq V\) and a node \(v \in V\). Under the Triggering model, let \(\rho_1\) be the probability that \(S\) activates \(v\) in a cascade, and \(\rho_2\) be the probability that \(S\) overlaps with a random RR-set \(R(v)\) rooted at \(v\). Then, \(\rho_1 \leq \rho_2\).

### 6.1 A General Solution Framework

We use Possible World (PW) models to generalize the theory in \cite{2, 25}. For a generic stochastic diffusion model \(M\), an equivalent PW model \(M'\) is a model that specifies a distribution over \(W\), the set of all possible worlds, where influence diffusion in each possible world in \(W\) is deterministic. Further, given a seed set (or two seed sets \(S_A\) and \(S_B\) as in Com-IC), the distribution of the sets of active nodes (or \(A\)- and \(B\)-adopted nodes in Com-IC) in \(M\) is the same as the corresponding distribution in \(M'\). Then, we define a generalized concept of RR-set through the PW model:

**Definition 1 (General RR-set).** For each possible world \(W \in W\) and a given node \(v\) (a.k.a. root), the reverse reachable set (RR-set) of \(v\) in \(W\), denoted by \(R_W(v)\), consists of all nodes \(u\) such that the singleton set \(\{u\}\) would activate \(v\) in \(W\). A random RR-set of \(v\) is a set \(R_W(v)\) where \(W\) is randomly sampled from \(W\) using the probability distribution given in \(M'\).

It is easy to see that Definition 1 encompasses the RR-set definition in \cite{2, 25} for IC, LT, and Triggering models as special cases. For the entire solution framework to work, the key property that RR-sets need to satisfy is the following:

**Definition 2. (Activation Equivalence Property).** Let \(M\) be a stochastic diffusion model and \(M'\) be its equivalent possible world model. Let \(G = (V, E, p)\) be a graph. Then, \(RR\)-sets have the Activation Equivalence Property if for any fixed \(S \subseteq V\) and any fixed \(v \in V\), the probability that \(S\) activates \(v\) according to \(M\) is the same as the probability that \(S\) overlaps with a random RR-set generated from \(v\) in a possible world in \(M'\).

As shown in \cite{25}, the entire correctness and complexity analysis is based on the above property, and in fact in their latest improvement \cite{24}, they directly use this property as the definition of general RR-sets. Proposition 1 shows that the activation equivalence property holds for the triggering model. We now provide a more general sufficient condition for the activation equivalence property to hold (Lemma 5), which gives concrete conditions on when the RR-set based framework would work. More specifically, we show that for any diffusion model \(M\), if there is an equivalent PW model \(M'\) of which all possible worlds satisfy the following two properties, then RR-sets have the activation equivalence property.

**Possible World Properties.** (P1): Given two seed sets \(S \subseteq T\), if a node \(v\) can be activated by \(S\) in a possible world \(W\), then \(v\) shall also be activated by \(T\) in \(W\). (P2): If a node \(v\) can be activated by \(S\) in a possible world \(W\), then there exists \(u \in S\) such that the singleton seed set \(\{u\}\) can also activate \(v\) in \(W\). In fact, (P1) and (P2) are equivalent to monotonicity and submodularity, as we formally state below.

**Lemma 4.** Let \(W\) be a fixed possible world. Let \(f_{v,W}(\cdot)\) be an indicator function that takes on 1 if \(S\) can activate \(v\) in \(W\), and 0 otherwise. Then, \(f_{v,W}(\cdot)\) is monotone and submodular for all \(v \in V\) if and only if both (P1) and (P2) are satisfied in \(W\).

**Lemma 5.** Let \(M\) be a stochastic diffusion model and \(M'\) be its equivalent possible world model. If \(M'\) satisfies Properties (P1) and (P2), then the RR-sets as defined in Definition 1 have the activation equivalence property as in Definition 2.

Comparing with directly using the activation equivalence property as the RR-set definition in \cite{24}, our RR-set definition provides a more concrete way of constructing RR-sets, and our Lemmas 4 and 5 provide general conditions under which such constructions can ensure algorithm correctness. Algorithm 1, GeneralTIM, outlines a general solution framework based on \(RR\)-sets and TIM. It provides a probabilistic approximation guarantee for any diffusion models that satisfy (P1) and (P2). Note that the estimation of a lower bound \(LB\) of \(\text{OPT}_k\) (line 1) is orthogonal to our contributions and we refer the reader to \cite{25} for details. Finally, we have:

**Theorem 5.** Suppose for a stochastic diffusion model \(M\) with an equivalent PW model \(M'\), that for every possible world \(W\) and every \(v \in V\), the indicator function \(f_{v,W}(\cdot)\) is monotone and submodular. Then for the influence maximization problem under \(M\) with graph \(G = (V, E, p)\) and seed size \(k\), GeneralTIM (Algorithm 1) applied on the general RR-sets (Definition 1) returns a \((1 - 1/(e - \epsilon))\)-approximate solution with at least \(1 - |V|^{-\epsilon}\) probability.

Theorem 5 follows from Lemmas 4 and 5, and the fact that all theoretical analysis of TIM relies only on the Chernoff bound and the activation equivalence property, “without relying on any other results specific to the IC model” \cite{25}. Next, we describe how to generate RR-sets correctly for \textsc{SelfInfMax} and \textsc{CompInfMax} under Com-IC (line 3 of Algorithm 1), which is much more complicated than IC/LT models \cite{25}. We will first focus on submodular settings for \textsc{SelfInfMax} (Theorem 3) and \textsc{CompInfMax} (Theorem 4). In \S 6.4, we propose Sandwich Approximation to handle general \(Q\) where submodularity does not hold.

### 6.2 Generating RR-sets for \textsc{SelfInfMax}

We present two algorithms, \textsc{RR-SIM} and \textsc{RR-SIM+}, for generating random RR-sets per Definition 1. The overall algorithm for \textsc{SelfInfMax} can be obtained by plugging \textsc{RR-SIM} or \textsc{RR-SIM+} into GeneralTIM (Algorithm 1).

According to Definition 1, for \textsc{SelfInfMax}, the RR-set of a root \(v\) in a possible world \(W\), \(R_W(v)\), is the set of nodes \(u\) such that if \(u\) is the only \(A\)-seed, \(v\) would be \(A\)-adopted in \(W\), given any fixed \(B\)-seed set \(S_B\). By Theorems 2 and 3 (whose proofs indeed show that the indicator function \(f_{v,W}(\cdot)\) is monoton and submodular), along with Lemmas 4 and 5, we know that RR-sets following Definition 1 have the activation equivalence property. We now focus on how to construct RR-sets following Definition 1. Recall that
### Algorithm 1: GeneralTIM (G = (V, E, p), k, ε, ℓ)
1. \( LB \leftarrow \) lower bound of OPT₁, estimated by method in [25];
2. Compute \( \theta \) using Eq. (1) with \( LB \) replacing \( OPT₁ \);
3. \( \mathcal{R} \leftarrow \) generate \( \theta \)-random RR-sets according to Definition 1; // for SELFINFMAX use RR-SIM or RR-SIM+: for COMINFMAX use RR-CIM
4. for \( i = 1 \) to \( k \) do
5. \( v_i \leftarrow \) the node appearing in the most RR-sets in \( \mathcal{R} \);
6. \( S \leftarrow S \cup \{v_i\} \); // \( S \) was initialized as \( \emptyset \)
7. Remove all RR-sets in which \( v_i \) appears;
8. return \( S \) as the seed set;

### Algorithm 2: RR-SIM (G = (V, E), v, S₈)
1. create an empty FIFO queue \( Q \) and empty set \( R \);
2. enqueue all nodes in \( S₈ \) into \( Q \); // start forward labeling
3. while \( Q \) is not empty do
4. \( u \leftarrow Q \). dequeue(); \( \) mark \( u \) as \( B \)-adopted;
5. foreach \( v \in N^+(u) \) such that (\( (u, v) \) is live do
6. if \( q_{B,W}^{u,v} \leq q_{A,B}(v) \) \&\& \( v \) is visited then
7. \( Q \). enqueue(v); // mark \( v \) as visited
8. clear \( Q \), and then enqueue \( v \); // start backward BFS
9. while \( Q \) is not empty do
10. \( u \leftarrow Q . \) dequeue();
11. \( R \leftarrow R \cup \{u\} \);
12. if \( u \) is \( B \)-adopted \&\& \( \alpha_A^{u,W} \leq q_{A,B}(v) \) \&\& \( v \) is not \( B \)-adopted \&\& \( \alpha_A^{u,W} \leq q_{A,B}(v) \) then
13. foreach \( w \in N^-(u) \) such that (\( (w, u) \) is live do
14. if \( w \) is not visited then
15. \( Q \). enqueue(w); // if mark \( w \) visited
16. return \( R \) as the seed set;

in Com-IC, adoption decisions for \( A \) are based on a number of factors such as whether \( v \) is reachable via a live-edge path from \( S_A \) and its state w.r.t. \( B \) when reached by \( A \). Note that \( q_{B,W}^{u,v} = q_{B,A}(u) \) implies that \( B \)-diffusion is independent of \( A \) (Lemma 3). Our algorithms take advantage of this fact, by first revealing node states w.r.t. \( B \), which gives a sound basis for generating RR-sets for \( A \).

#### 6.2.1 The RR-SIM Algorithm

Conceptually, RR-SIM (Algorithm 2) proceeds in three phases. Phase I samples a possible world according to §5.1 (omitted from the pseudo-code). Phase II is a forward labeling process from the input \( B \)-seed set \( S_B \) (lines 2 to 7): a node \( v \) becomes \( B \)-adopted if \( q_{B,W}^{u,v} \leq q_{B,B}(v) \) and \( v \) is reachable from \( S_B \) via a path consisting entirely of live edges and \( B \)-adopted nodes. In Phase III (lines 8 to 15), we randomly select a node \( v \) and generate RR-set \( R_W(v) \) by running a Breath-First Search (BFS) backwards (following incoming edges). Note that the RR-set generation for IC and LT models [25] is essentially a simpler version of Phase III.

**Backward BFS.** Given \( W \), an RR-set \( R_W(v) \) includes all nodes explored in the following backward BFS procedure. Initially, we enqueue \( v \) into a FIFO queue \( Q \). We repeatedly dequeue a node \( u \) from \( Q \) for processing until the queue is empty.

**Case 1:** \( u \) is \( B \)-adopted. There are two sub-cases: (i) If \( \alpha_A^{u,W} \leq q_{A,B}(u) \), then \( u \) is allowed to transit from \( A \)-informed to \( A \)-adopted. Thus, we continue to examine \( u \)’s in-neighbors. For all unexplored \( w \in N^-(u) \), if edge \((w, u)\) is live, then enqueue \( w \); (ii) If \( \alpha_A^{u,W} > q_{A,B}(u) \), then \( u \) cannot transit from \( A \)-informed to \( A \)-adopted, and thus \( u \) has to be an \( A \) seed to become \( A \)-adopted. In this case, \( u \)’s in-neighbors will not be examined.

**Case 2:** \( u \) is not \( B \)-adopted. Similarly, if \( \alpha_A^{u,W} \leq q_{A,B}(u) \), perform actions as in 1(i); otherwise perform actions as in 1(ii).

**Theorem 6.** Under one-way complementarity \( q_{A,B}(u) \leq q_{A,B}(v) \) and \( q_{B,W}^{u,v} = q_{B,W}^{v,u} \), the RR-sets generated by the RR-SIM algorithm satisfy Definition 1 for the SELFINFMAX problem. As a result, Theorem 5 applies to GeneralTIM with RR-SIM in this case.

**Lazy Sampling.** For RR-SIM to work, it is not necessary to sample all edge- and node-level variables (i.e., the entire possible world) upfront, as the forward labeling and backward BFS are unlikely to reach the whole graph. Hence, we can simply reveal edge and node states on demand (“lazy sampling”), based on the principle of deferred decisions. In light of this observation, the following improvements are made to RR-SIM. First, the first phase is simply skipped. Second, in Phase II, edge states and \( \alpha \)-values are sampled as the forward labeling from \( S_B \) goes on. We record the outcomes, as it is possible to encounter certain edges and nodes again in phase (III). Next, for Phase III, consider any node \( u \) dequeued from \( Q \). We need to perform an additional check on every incoming edge \((w, u)\). If \((w, u)\) has already been tested live in Phase II, then we just enqueue \( w \). Otherwise, we first sample its live/blockwise status, and enqueue \( w \) if it is live, Algorithm 2 provides the pseudo-code for RR-SIM, where sampling is assumed to be done whenever we need to check the status of an edge or the \( \alpha \)-values of a node.

**Expected time complexity.** For the entire seed selection (Algorithm 1 with RR-SIM) to guarantee approximate solutions, we must estimate a lower bound \( LB \) of \( OPT₁ \) and use it to derive the minimum number of RR-sets required, defined as \( \theta \) in Eq. (1). In expectation, the algorithm runs in \( O(\theta \cdot EPT) \) time, where \( EPT \) is the expected number of edges explored in generating one RR-set. Clearly, \( EPT = EPT_F + EPT_B \), where \( EPT_F \) (EPT_B) is the expected number of edges examined in forward labeling (resp., backward BFS). Thus, we have the following result.

**Lemma 6.** In expectation, GeneralTIM with RR-SIM runs in \( O((k + \ell)(|V| + |E|)\log |V| \cdot (1 + EPT_F/EPT_B)) \) time.

EPT_F increases when the input \( B \)-seed set grows. Intuitively, it is reasonable that a larger \( B \)-seed set may have more complementary effect and thus it may take longer time to find the best \( A \)-seed set. However, it is possible to reduce \( EPT_F \) as described below.

#### 6.2.2 The RR-SIM+ Algorithm

The RR-SIM algorithm may incur efficiency loss because some of the work done in forward labeling (Phase II) may not be used in backward BFS (Phase III). E.g., consider an extreme situation where all nodes explored in forward labeling are in a different connected component of the graph than the root \( v \) of the RR-set. In this case, forward labeling can be skipped safely and entirely! To leverage this, we propose RR-SIM+ (pseudo-code included in appendix), of which the key idea is to run two rounds of backward BFS from the random root \( v \). The first round determines the necessary scope of forward labeling, while the second one generates the RR-set.

**First backward BFS.** As usual, we create a FIFO queue \( Q \) and enqueue the random root \( v \). We also sample \( q_{B,W}^{u,v} \) uniformly at random from \([0, 1]\). Then we repeatedly dequeue a node \( u \) until \( Q \) is empty; for each incoming edge \((w, u)\), we test its live/blockwise status based on probability \( p_{w,u} \), independently. If \((w, u)\) is live and \( w \) has not been visited before, enqueue \( w \) and sample its \( q_{B,W}^{u,v} \).

Let \( T₁ \) be the set of all nodes explored. If \( T₁ \cap S_B = \emptyset \), then none of the \( B \)-seeds can reach the expanded nodes, so that forward labeling can be completely skipped. The above extreme example falls into this case. Otherwise, we run a residual forward labeling only from \( T₁ \cap S_B \) along the explored nodes in \( T₁ \): if a node \( u \in T₁ \cap S_B \) is reachable by some \( s \in T₁ \cap S_B \) via a live-edge path with all \( B \)-adopted nodes, and \( q_{B,W}^{u,v} \leq q_{B,B}(u) \), \( u \) becomes \( B \)-adopted.
Second backward BFS. This round is largely the same as Phase III in RR-SIM, but there is a subtle difference. Suppose we just dequeued a node $u$. It is possible that there exists an incoming edge $(w, u)$ whose status is not determined. This is because we do not enqueue previously visited nodes in BFS. Hence, if in the previous round, $w$ is already visited via an out-neighbor other than $u$, $(w, u)$ would not be tested. Thus, in the current round we shall test $(w, u)$, and decide if $w$ belongs to $R_{Bu}(v)$ accordingly. To see RR-SIM+ is equivalent to RR-SIM, it suffices to show that for each node explored in the second backward BFS, its adoption status w.r.t. $B$ is the same in both algorithms. We prove this fact in appendix.

The analysis on expected time complexity is similar: We can show that the expected running time of RR-SIM+ is $O((k + \epsilon)(|V| + |E|)\log |V| (1 + EPT_{B1}/EPT_{B2}))$, where $EPT_{B1}$ ($EPT_{B2}$) is the expected number of edges explored in the first (resp., second) backward BFS. Compared to RR-SIM, $EPT_{B2}$ is the same as $EPT_B$ in RR-SIM, so RR-SIM+ will be faster than RR-SIM if $EPT_{B1} < EPT_F$, i.e., if the first backward BFS plus the residual forward labeling explores fewer edges, compared to the full onward labeling in RR-SIM.

### 6.3 Generating RR-Sets for COMPINFMAX

In COMPINFMAX, by Definition 1, a node $u$ belongs to an RR-set $R_B(v)$ if $v$ is not $A$-adopted without any $B$-seed, but turns $A$-adopted when $u$ is the only $B$-seed. It turns out that constructing RR-sets for COMPINFMAX following the above definition is significantly more difficult than that for SELFINFMAX. This is because when $q_{A|B} \leq q_{A|\emptyset}$ and $q_{B|A} \leq q_{B|\emptyset}$, $A$ and $B$ complement each other, and thus a simple forward labeling from the fixed $A$-seed set, without knowing anything about $B$, will not be able to determine the $A$ adoption status of all nodes. This is in contrast to SELFINFMAX with one-way complementarity for which $B$-diffusion is fully independent of $A$. Thus, when generating RR-sets for COMPINFMAX, we have to determine more complicated status in a forward labeling process from $A$-seeds, as shown below.

**Phase I: forward labeling.** The nature of COMPINFMAX requires us to identify nodes with the potential to be $A$-adopted with the help of $B$. To this end, we first conduct a forward search from $S_A$ to label the nodes their status of $A$. As in RR-SIM, we also employ lazy sampling. The algorithm first enqueues all $A$-seeds (and labels them $A$-adopted) into a FIFO queue $Q$. Then we repeatedly dequeue a node $u$ for processing until $Q$ is empty. Let $v$ be an out-neighbor of $u$. Flip a coin with bias $p_{u,v}$ to determine if edge $(u,v)$ is live. If yes, we determine the label of $v$ to be one of the following:

- **$A$-adopted.** if $u$ is $A$-adopted and $\alpha_u^A \geq q_{A|\emptyset}$
- **$A$-rejected.** if $\alpha_u^A > q_{A|\emptyset}$ regardless of $u$'s status
- **$A$-suspended.** if $u$ is $A$-adopted and $\alpha_u^A \leq q_{A|\emptyset}$ or $q_{A|B}$ (2)
- **$A$-potential.** if $u$ is $A$-suspended/potential and $\alpha_u^A \leq q_{A|B}$

Here, $A$-potential is just a label used for bookkeeping and is not a state. Then node $v$ is added to $Q$ unless it is $A$-rejected. Note that both $A$-suspended and $A$-potential nodes can turn into $A$-adopted with the complementary effect of $B$. The main difference is an $A$-suspended node is informed of $A$, while an $A$-potential is not and the informing action must be triggered by $B$-propagation. Also, unlike a typical BFS, the forward labeling may need to revisit a node: if $u$ is $A$-adopted (just dequeued) and $v$ is previously labeled $A$-potential, $v$ should be “promoted” to $A$-suspended. This occurs when $v$ is first reached by a live-edge path through an $A$-suspended/potential in-neighbor, but later $v$ is reached by a longer path through an $A$-adopted in-neighbor.

To facilitate the second phase, we define additional node labels $AB$-diffusible and $B$-diffusible. Node $v$ is $AB$-diffusible if $v$ can adopt both $A$ and $B$ when $v$ is informed about both $A$ and $B$; while $v$ is $B$-diffusible if $v$ can adopt $B$ when it is informed about $B$. Accordingly, the technical conditions for them are given below:

- **$AB$-diffusible.** if $\alpha_v^A \leq q_{A|\emptyset} \lor ((q_{A|\emptyset} < q_{A|B} \land (\alpha_v^A \leq q_{A|B} \land \alpha_v^B \leq q_{B|\emptyset})))$
- **$B$-diffusible.** if $\alpha_v^B \leq q_{B|\emptyset} \lor v$ is $A$-adopted as labeled in Eq (2)

Note that these diffusible labels are only based on a node’s local state, and they are not limited to the nodes explored in the first phase — some nodes may only be explored in the second phase and they also need to be checked for these diffusible labels.

**Phase II: RR-set generation.** The second phase features a primary backward search from a random root $v$. Also, a number of secondary searches (from certain nodes explored in the primary search) may be necessary to find all nodes qualified for the RR-set. Intuitively, the primary backward search is to locate $A$-suspended nodes $u$ via $AB$-diffusible and $A$-potential nodes, since once such a node $u$ adopts $B$, it will adopt $A$ and then through those $AB$-diffusible and $A$-potential nodes, the root $v$ will adopt $A$ and $B$. Thus such a node $u$ can be put into the RR-set of $v$. In addition, if such node $u$ is also $AB$-diffusible, then any $B$-seed $w$ that can activate $u$ to adopt $B$ via $B$-diffusible nodes can also be put into the RR-set of $v$, and we find such nodes $w$ using a secondary backward search from $u$ via $B$-diffusible nodes. However, some additional complication may arise during the search process, and we cover all cases in Algorithm 3 and explain them below.

We first sample a root $v$ randomly from $V$. In case $v$ is labeled $A$-adopted or $A$-rejected, we simply return $R$ as $\emptyset$ because no $B$-seed set can change $v$’s adoption status of $A$ (lines 2 to 3). The primary search then starts. It first enqueues $v$ into a FIFO queue $Q$. Now consider a node $u$ dequeued from $Q$. Four cases arise.

**Case 1:** $u$ is $A$-suspended and $AB$-diffusible (lines 8 to 10). We
add \( u \) to \( R \). Moreover, any node \( w \) that can propagate \( B \) to \( u \) by itself should also be added to \( R \). To find all such \( w \)'s, we launch a secondary backward BFS from \( u \) via \( B \)-diffusible nodes. In particular, we conduct a reverse BFS from \( u \) to explore all nodes that could reach \( u \) via \( B \)-diffusible nodes, and put all of them into set \( R \).

If this secondary search touches a node \( w \) that is not \( B \)-diffusible, we put \( w \) in \( R \) but do not further explore the in-neighbors of \( w \).

Case 2: \( u \) is \( A \)-suspended but not \( AB \)-diffusible (line 11). Add \( u \) to \( R \), but do not initiate a secondary search, because \( u \) cannot adopt \( A \) or \( B \) even if it is informed of both \( A \) and \( B \), and thus the only way to make it adopt \( B \) is to make it a \( B \)-seed.

Case 3: \( u \) is \( A \)-potential and \( AB \)-diffusible (lines 13 to 15). We enqueue all \( R \) to \( u \) itself should also be added to \( R \). To find all such \( w \)'s, we launch a secondary backward BFS from \( u \) via \( B \)-diffusible nodes. In particular, we conduct a reverse BFS from \( u \) to explore all nodes that could reach \( u \) via \( B \)-diffusible nodes, and put all of them into set \( R \).

This is the most complicated case that needs a special treatment. In general, we should stop the primary backward search at \( u \) and try other branches, because \( u \) is not yet \( A \)-informed and \( u \) cannot help in diffusing \( A \) and \( B \) even when informed of \( A \) and \( B \). However, there is a special case in which we can still put \( u \) in \( R \) (making \( u \) a \( B \)-seed): \( u \) can reach an \( A \)-suspended and \( AB \)-diffusible node \( v_0 \) via a \( B \)-diffusible path such that \( u \) can activate \( v_0 \) in adopting \( B \) through this path, and then \( u \) can reach back \( u \) via an \( AB \)-diffusible path, such that \( u \) can activate \( u \) in adopting \( A \). For example, consider Figure 3 (all edges are live): \( \alpha \) is an \( A \)-seed, \( u \) is \( A \)-potential but not \( AB \)-diffusible, and \( u \) is \( AB \)-suspended.

To identify such \( u \), we start two secondary BFSs from \( u \), one traveling forwards, one backwards. The forward search explores all \( B \)-diffusible nodes reachable from \( u \) and puts them in a set \( S_f \), and stops at a node \( w \) when \( w \) is not \( B \)-diffusible, but also puts \( w \) in set \( S_f \). The backward search explores all \( AB \)-diffusible and \( A \)-potential/suspended/adopted nodes that can reach \( u \) and puts them in set \( S_b \). If there is a node \( u_0 \in S_f \cap S_b \) that is \( A \)-suspended, then we can put \( u \) into \( R \). After this special treatment, we stop exploring the in-neighbors of \( u \) in the primary search and continue the primary search elsewhere.

**Theorem 7.** Suppose that \( q_{A|B} \leq q_{A|AB} \) and \( q_{B|\emptyset} \leq q_{B|A} = 1 \). The RR-sets generated by the RR-CIM algorithm satisfies Definition 1 for the COMPINFMAX problem. As a result, Theorem 5 applies to GeneralT1IM with RR-CIM in this case.

**Expected Time Complexity.** Both phases of RR-CIM require more computations compared to RR-SIM. First, the number of edges explored in Phase I, namely \( EPT_E \), is larger in RR-CIM, as the forward labeling here needs to continue beyond just \( A \)-adopted nodes. For Phase II, let \( EPT_{BS} \) be the expected number of edges pointing to nodes in \( R \) and \( EPT_{BS} \) be the expected number of all other edges examined in this phase (including both primary and secondary searches). Thus, we have:

**Lemma 7.** In expectation, GeneralT1IM with RR-CIM runs in \( O\left(\left(k + \ell\right)|V| + |E|\right) \log |V| \left(1 + \frac{EPT_E + EPT_{BS}}{EPT_{BS}}\right)\) time.

## 6.4 The Sandwich Approximation Strategy

We present the Sandwich Approximation (SA) strategy that leads to algorithms with data-dependent approximation factors for SELFINF MAX and COMPINF MAX in the general mutual complement case of Com-IC \( (q_{AB} \leq q_{A|B} \) and \( q_{B|\emptyset} \leq q_{B|A}) \) when submodularity may not hold. In fact, SA can be seen as a general strategy, applicable to any non-submodular maximization problems for which we can find submodular upper or lower bound functions.

Let \( \sigma : 2^V \rightarrow \mathbb{R}_{\geq 0} \) be non-submodular. Let \( \mu \) and \( \nu \) be submodular and defined on the same ground set \( V \) such that \( \mu(S) \leq \sigma(S) \leq \nu(S) \) for all \( S \subseteq V \). That is, \( \mu \) is a lower (resp., upper) bound on \( \sigma \) everywhere. Consider the problem of maximizing \( \sigma \) subject to a cardinality constraint \( k \). Notice that if the objective function were \( \mu \) or \( \nu \), the problem would be approximable within \( 1 - 1/e \) (e.g., max-k-cover) or \( 1 - 1/e - \epsilon \) (e.g., influence maximization) by the greedy algorithm [16, 21]. A natural question is: Can we leverage the fact that \( \mu \) and \( \nu \) "sandwich" \( \sigma \) to derive an approximation algorithm for maximizing \( \sigma \)? The answer is "yes".

### Sandwich Approximation

First, run the greedy algorithm on all three functions. It produces an approximate solution for \( \mu \) and \( \nu \). Let \( S_\mu, S_\nu, S_\sigma \) be the solution obtained for \( \mu, \sigma, \nu \) respectively. Then, select the final solution to \( \sigma \) to be

\[
S_{sand} = \arg \max_{S \in \{S_\mu, S_\nu, S_{\sigma}\}} \sigma(S).
\]

**Theorem 8.** Sandwich Approximation solution gives:

\[
\sigma(S_{sand}) \geq \max \frac{\sigma(S_\mu) \mu(S_{\sigma} \cdot \nu(S_{\sigma})) \cdot (1 - 1/e) \cdot \sigma(S_{\nu})}{\nu(S_{\mu})}, \quad (4)
\]

where \( S_{\sigma} \) is the optimal solution maximizing \( \sigma \) (subject to cardinality constraint \( k \)).

**Remarks.** While the factor in Eq. (4) involves \( S_{\sigma} \), generally not computable in polynomial time, the first term inside max(...) involves \( S_\mu, S_\nu \) can be computed efficiently and can be of practical value (see Table 3 in §7). We emphasize that SA is much more general, not restricted to cardinality constraints. E.g., for a general matroid constraint, simply replace \( 1 - 1/e \) with \( 1/2 \) in (4), as the greedy algorithm is a 1/2-approximation in this case [21]. Furthermore, monotonicity is not important, as maximizing general submodular functions can be approximated within a factor of \( 1/2 \) [5], and thus SA applies regardless of monotonicity. On the other hand, the true effectiveness of SA depends on how close and \( \mu \) and \( \nu \) are to \( \sigma \), e.g., a constant function can be a trivial submodular upper bound function but would only yield trivial data-dependent approximation factors. Thus, an interesting question is how to derive \( \nu \) and \( \mu \) that are as close to \( \sigma \) as possible, while maintaining submodularity.

**SelfInfMax.** GeneralT1IM with RR-SIM or RR-SIM+ provides a \( (1 - 1/e) \)-approximate solution with high probability, when \( q_{A|B} \leq q_{A|AB} \) and \( q_{B|\emptyset} = q_{B|A} \). When \( q_{B|\emptyset} < q_{B|A} \), function \( \nu \) (upper bound) can be obtained by increasing \( q_{B|\emptyset} \) to \( q_{B|A} \), while \( \mu \) (lower bound) can be obtained by decreasing \( q_{B|A} \) to \( q_{B|\emptyset} \).

**COMPINFMax.** GeneralT1IM with RR-CIM provides a \( (1 - 1/e - \epsilon) \)-approximate solution with high probability, when \( q_{A|B} \leq q_{A|AB} \) and \( q_{B|\emptyset} \leq q_{B|A} = 1 \). When \( q_{B|A} \) is not necessarily 1, we obtain an upper bound function by increasing \( q_{B|A} \) to 1.

The correctness of the above approaches is ensured by the following theorem.

**Theorem 9.** Suppose \( q_{A|B} \leq q_{A|AB} \) and \( q_{B|\emptyset} \leq q_{B|A} \). Then, under the Com-IC model, for any fixed \( A \) and \( B \) seed sets \( S_A \) and \( S_B \), \( \sigma_{\mu}(S_A, S_B) \) is monotonically increasing w.r.t. any one of \( q_{A|B}, q_{A|AB}, q_{B|\emptyset}, q_{B|A} \) with other three GAPS fixed, as long as after the increase the parameters are still in \( Q^+ \).

Putting it all together, the final algorithm for SelfInfMax is GeneralT1IM with RR-SIM/RR-SIM+ and SA. Similarly, the final algorithm for CompInfMax is Algorithm 1 with RR-CIM and SA. It is important to see just how useful and effective SA is in practice. We address this question head on in §7, where we “stress test” the idea behind SA. Intuitively, if the GAPS are such that \( q_{B|\emptyset} \) and \( q_{B|A} \)
7. EXPERIMENTS

We perform extensive experiments on three real-world social networks to evaluate our algorithms: Flixster, Douban, and Last.fm. We also conduct scalability tests on larger synthetic graphs.

Flixster is collected from a social movie site\(^2\), and we extract a strongly connected component. Douban is collected from a Chinese social network\(^{[26]}\), where users rate books, movies, music, etc. We crawl all movie and book ratings of the users in the graph, and then derive two datasets from book and movie ratings respectively (details in §7.1). Last.fm is taken from the popular music website with social networking features. For all graphs, we learn the influence probability using the method proposed in\(^{[13]}\), widely adopted in prior work\(^{[9]}\). The basic statistics of all datasets are in Table 1.

### 7.1 Learning Global Adoption Probabilities

**Finding Signals from Data.** For Flixster and Douban, we learn GAPs from their timestamped rating data, which can be viewed as action logs. Each entry is a quadruple \((u, i, a, t_{u,i,a})\), indicating user \(u\) performed action \(a\) on item \(i\) at time \(t_{u,i,a}\). We count a rating quadruple as one adoption action and one informing action: if someone rated an item, she must have been informed of it first; we assume only adopters rate items. The key challenge is how to find actions that can be mapped to informing events that do not lead to adoptions. Fortunately, there are also special ratings providing such signals in both Flixster and Douban. Flixster.com allows users to indicate if they “want to see” a movie, or are “not interested” in one. We map both signals to the actions of a user being informed of a movie. Douban.com allows users to put items into a wish list. Thus, if a book/movie is in a user’s wish list, we treat it as an informing action. For Douban, we separate actions on books and movies to derive two datasets: Douban-Book and Douban-Movie.

**Learning Method.** Consider two items \(A\) and \(B\) in an action log. Let \(R_{A}\) and \(I_{A}\) be the set of users who rated \(A\) and who were informed of \(A\), respectively. Clearly, \(R_{A} \subseteq I_{A}\). Thus,

\[
q_{A|\emptyset} = |R_{A} \setminus R_{B <_{\text{rate}} A}| / |I_{A} \setminus R_{B <_{\text{inform}} A}|
\]

where \(R_{B <_{\text{rate}} A}\) is the set of users who rated both items \(B\) rated first, and \(R_{B <_{\text{inform}} A}\) is the set of users who rated \(B\) before being informed of \(A\). Next, \(q_{A|B}\) is computed as follows:

\[
q_{A|B} = |R_{B <_{\text{rate}} A}| / |R_{B <_{\text{inform}} A}|
\]

\(q_{B|\emptyset}\) and \(q_{B|A}\) can be computed in a symmetric way.

Table 2 presents the GAPs learned for a few pairs of popular movies in Flixster dataset. More examples, including those in Douban-Book and Douban-Movie and 95%-confidence intervals of those estimates, can be found in the appendix.

### 7.2 Experimental Settings

**Baseline Algorithms.** (i) HighDegree: selecting the \(k\) highest degree nodes as seeds; (ii) PageRank: selecting the \(k\) nodes with highest PageRank score; (iii) Random: selects \(k\) random nodes as seeds; (iv) Greedy\(^{[16]}\): the greedy algorithm with 10000 iterations of Monte Carlo (MC) simulations and lazy-forward optimization. Note that Greedy is known to be prohibitive: in one week, it only finishes on Flixster (both SelfInfMax and CompInfMax) and Douban-Book (SelfInfMax only).

**Parameters.** For Flixster we select Monster Inc and Shrek as \(A\) and \(B\). For Douban-Book we select The Unbearable Lightness of Being by M. Kundera as \(A\) and Norwegian Wood by H. Murakami as \(B\), with \(Q = \{.75, .85, .92, .97\}\). For Douban-Movie, we choose Fight Club as \(A\) and Seven as \(B\), with \(Q = \{.84, .89, .89, .95\}\). For Last.fm, since there is no signal in the data to indicate informing events (that do not lead to adoptions), we test with \(Q = \{.5, .75, .5, .75\}\). Note that in all datasets, the two items are mutually complementary, for which self-cross-submodularity does not hold (§5). Therefore, Sandwich Approximation (SA) are used by default for GeneralTIM and Greedy\(^{[16]}\).

In all cases, \(k = 50\). For GeneralTIM, we set \(\epsilon\) in such a way that a success probability of \(1 - 1/|V|\) is ensured\(^{[25]}\). For SelfInfMax, \(B\)-seeds are the top-10 highest degree nodes so that Greedy can at least finish on the smallest dataset (Flixster). For CompInfMax, \(A\)-seeds are top-50 highest degree nodes, due to that \(q_{B|\emptyset}\) and \(q_{A|B}\) are relatively close in our setting, thus more \(A\)-seeds would allow greater complementary effects (from \(B\)) to be mined. Our implementations are in C++ and we run experiments on a Linux server with 2.93GHz CPUs and 128GB RAM.

### 7.3 Results and Analysis

**Effects of \(\epsilon\).** We first evaluate the effects of \(\epsilon\) on GeneralTIM. As mentioned in §6, \(\epsilon\) controls the trade-off between approximation ratio and efficiency. Figure 4 plots influence spread and running time (log-scale) side-by-side, as a function of \(\epsilon\), on Flixster and Douban-Book for both problems. The results on other datasets are very similar and thus omitted. We can see that as \(\epsilon\) goes up from 0.1 to 0.5 and 1 (in fact, \(\epsilon = 1 > (1 - 1/e)\) means theoretical approximation guarantees are lost), the running time of all versions of GeneralTIM (RR-SIM, RR-SIM+, RR-CIM) decreases dramatically, by orders of magnitude: while, in practice, influence spread (SelfInfMax) and boost (CompInfMax) are almost completely unaffected (the largest difference among all test cases is only 0.45%). This allows us to set a reasonably large \(\epsilon\): unless otherwise stated, we use \(\epsilon = 0.5\) for RR-SIM and RR-SIM+, and \(\epsilon = 1\) for RR-CIM.

**Quality of Seeds.** The quality of seeds (of an algorithm) is measured by the influence spread or boost achieved. We evaluate the spread of seed sets computed by all algorithms by MC simulations with 10000 iterations to ensure a fair comparison.

As can be seen from Figures 5 and 6, our RR-set algorithms are consistently the best in all test cases, often leading by a significant margin. Note that Greedy results are omitted, and we note the spread it achieves is almost identical to GeneralTIM, matching the observations in prior work\(^{[25]}\). Similarly, RR-SIM results are identical to RR-SIM+, thus omitted.

For SelfInfMax, GeneralTIM with RR-SIM+ is 14%, 14%,

### Table 1: Statistics of graph data (all directed)

<table>
<thead>
<tr>
<th></th>
<th>Flixster</th>
<th>Douban-Book</th>
<th>Douban-Movie</th>
<th>Last.fm</th>
</tr>
</thead>
<tbody>
<tr>
<td># nodes</td>
<td>12.9K</td>
<td>23.3K</td>
<td>34.9K</td>
<td>61K</td>
</tr>
<tr>
<td># edges</td>
<td>192K</td>
<td>141K</td>
<td>274K</td>
<td>584K</td>
</tr>
<tr>
<td>avg. out-degree</td>
<td>14.8</td>
<td>6.5</td>
<td>7.9</td>
<td>9.6</td>
</tr>
<tr>
<td>max. out-degree</td>
<td>189</td>
<td>1690</td>
<td>545</td>
<td>1073</td>
</tr>
</tbody>
</table>

### Table 2: Selected GAPs learned for movies from Flixster

| Movie       | \(q_{A|\emptyset}\) | \(q_{A|B}\) | \(q_{B|\emptyset}\) | \(q_{B|A}\) |
|-------------|---------------------|--------------|---------------------|--------------|
| Monster Inc | .88                 | .89          | .84                 | .86          |
| Shrek       | .92                 | .92          | .92                 | .96          |
| Gone in 60 Seconds | .63     | .77          | .67                 | .82          |
| Prisoner of Azkaban | .85 | .84          | .66                 | .67          |
| What a Girl Wants | .92   | .94          | .80                 | .79          |
6%, and 12% better than the next best algorithm on Flixster, Douban-Book, Douban-Movie, and Last.fm respectively. For COMPINFMAX, GeneralTIM with RR-CIM is 180%, 22%, 40%, and 19% better (same order of datasets). The boost in $A$-spread provided by $B$-seeds (mined by GeneralTIM) is at least 10% to 49% (Last.fm) of the original $A$-spread by $S_A$ only. HighDegree typically has good performance, especially in graphs with many nodes having very large out-degrees (Douban-Movie, Last.fm), while PageRank has good quality seeds only on Last.fm. Random is consistently the worst. The performances of these baselines are generally consistent with observations in previous works [10, 11, 25], albeit for different diffusion models.

**Running Time and Scalability.** We compare the running time of GeneralTIM to Greedy, shown in Figure 7(a). Note that the top of $Y$-axis is set to be 604800 seconds (one week), and touching it means the algorithm runs beyond this. As can be seen, for SELF-INFMAX, GeneralTIM with RR-SIM, RR-SIM+ is about two to three orders of magnitude faster than Greedy; for COMPINFMAX, GeneralTIM with RR-CIM is also about two orders of magnitude faster than Greedy. In addition, we observe that RR-SIM+ is 2.6, 5.5, and 2.2 times as fast as RR-SIM on Flixster, Douban-Book, and Last.fm respectively, while on Douban-Movie, RR-SIM is 3.4 times faster. This is not surprising: As mentioned in §6, though the adjustment made in RR-SIM+ is intuitive and confirmed effective in three of the datasets, there is no theoretical guarantee that RR-SIM+ is always faster than RR-SIM.

We then use larger synthetic graphs to test the scalability of GeneralTIM with our RR-set generation algorithms. We generate power-law random graphs of 0.2, 0.4, ..., up to 1 million nodes with a power-law degree exponent of 2.16 [10]. These graphs have an average degree of about 5. Settings and parameters are set per §7.2, and we use $Q^a$ from Flixster. The difference between RR-SIM+ and RR-SIM is minimal so we only show RR-SIM+. We found that typically high-degree nodes in power-law graphs, chosen to be $B$-seeds per §7.2, have much larger degrees compared to real graphs, and hence $B$-seeds will reach a significant portion of the graph, making the adjustment of RR-SIM+ less effective than on real graphs. We can see that GeneralTIM with RR-SIM+ within 7.5 hours for the 1-million node graph, and its running time grows linearly in graph size, which indicates great scalability. RR-CIM is slower due to the inherent intricacy of COMPINFMAX, but it also scales linearly. To further put the numbers in perspective, Greedy—the only other known approximation algorithm for COMPINFMAX—takes about 48 hours on Flixster (12.9K nodes),
while GeneralTIM with RR-CIM is 4 hours faster on a 200K-node graph.

### 8. CONCLUSIONS & FUTURE WORK

In this work, we propose the Comparative Independent Cascade (Com-IC) model that allows any degree of competition or complementarity between two different propagating items, and study the novel SELFINFMAX and COMPINFMAX problems for complementary products. We develop non-trivial extensions to the RR-set techniques to achieve approximation algorithms. For non-submodular settings, we propose Sandwich Approximation to achieve data-dependent approximation factors. Our experiments demonstrate the effectiveness and efficiency of proposed solutions.

This work opens up a number of interesting avenues for future research. One direction is to design more efficient algorithms or heuristics for SELFINFMAX and (especially) COMPINFMAX: e.g., whether near-linear time algorithm is still available for these problems is still open. Another direction is to fully characterize the entire GAP space Q in terms of monotonicity and submodularity properties. Moreover, an important direction is to extend the model to multiple items. Given the current framework, Com-IC can be extended to accommodate k items, if we allow \( k \cdot 2^{k-1} \) GAP parameters — for each item, we specify the probability of adoption for every combination of other items that have been adopted. However, how to simplify the model and make it tractable, how to reason about the complicated two-way or multi-way competition and complementarity, how to analyze monotonicity and submodularity, and how to learn GAP parameters from real-world data, etc. remain interesting challenges.

### Acknowledgments

We would like to thank Lewis Tzeng for some early discussions on modeling influence propagations for partially competing and partially complementary items.

### 9. REFERENCES

APPENDIX

A. REMARKS ON COM-IC MODEL

A.1 Unreachable States

Recall that in the Com-IC model, before an influence diffusion starts, all nodes are in the initial joint state (A-idle, B-idle). According to the diffusion dynamics defined in Figure 2, there exist five unreachable joint states, which are not material to our analysis and problem-solving, since none of these is relevant to actual adoptions, the objectives studied in SELFINFMAX and COMPINFMAX. For completeness, we list these states here.

1. (A-idle, B-rejected)
2. (A-suspended, B-rejected)
3. (A-rejected, B-idle)
4. (A-rejected, B-suspended)
5. (A-rejected, B-rejected)

LEMMA 8. In any instance of the Com-IC model (no restriction on GAPs), no node can reach the state of (A-idle, B-rejected), from its initial state of (A-idle, B-idle).

PROOF. Let \( v \) be an arbitrary node from graph \( G = (V, E, p) \). Note that for \( v \) to reject \( B \), it must be first be informed of \( B \) (otherwise it remains B-idle, regardless of its state w.r.t. \( A \)), and then becomes B-suspended (otherwise it will be B-adopted, a contradiction). Now, note that \( v \) is never informed of \( A \), and hence it will not be triggered to reconsider \( B \), the only route to the state of B-rejected, according to the model definition. Thus, (A-idle, B-rejected) is unreachable.

The argument for (A-rejected, B-idle) being unreachable is symmetric, and hence omitted.

LEMMA 9. In any instance of the Com-IC model (no restriction on GAPs), no node can reach the state of (A-suspended, B-rejected), from its initial state of (A-idle, B-idle).

PROOF. Let \( v \) be an arbitrary node from graph \( G = (V, E, p) \). Note that for \( v \) to reject \( B \), it must be first be informed of \( B \) (otherwise it remains B-idle, regardless of its state w.r.t. \( A \)), and then becomes B-suspended (otherwise it will be B-adopted, a contradiction). Now, \( v \) transits from A-idle to A-suspended, meaning that \( v \) does not adopt \( A \). This will not further trigger reconsideration, and hence \( v \) stays at B-suspended. This completes the proof.

The argument for (A-rejected, A-suspended) being unreachable is symmetric, and hence omitted. Finally, it is evident from the proof of Lemma 9 that, the joint state of (A-suspended, A-suspended) is a sunken state, meaning the node will not get out it to adopt or reject any product. This implies that (A-rejected, B-rejected) is also unreachable.

A.2 Additional Counter-Examples

The first two counter-examples show that self-monotonicity and cross-monotonicity may not hold in general for the Com-IC model when there is no restriction on GAPs.

EXAMPLE 1 (NON-Self-MONOTONICITY). Consider Figure 8. All edges have probability 1. GAPs are \( q_{A|B} = q \in (0, 1) \), \( q_{B|A} = 0 \), which means that \( A \) competes with \( B \) but \( B \) complements \( A \). Let \( S_B = \{y\} \). If \( S_A = S = \{s_1\} \), the probability that \( v \) becomes \( A \)-adopted is \( 1 \), because \( v \) is informed of \( A \) from \( s_1 \), and even if it does not adopt \( A \) at the time, later it will surely adopt \( B \) propagated from \( y \), and then \( v \) will reconsider \( A \) and adopt \( A \). If it is \( T = \{s_1, s_2\} \), that probability is \( 1 - q + q^2 < 1 \): \( w \) gets \( A \)-adopted w.p. \( q \) blocking \( B \) and then \( v \) gets \( A \)-adopted w.p. \( 1 - q \) and then \( v \) surely gets \( A \)-adopted. Replicating sufficiently many \( v \)'s, all connected to \( s_1 \) and \( w \), will lead to \( \sigma_A(T, S_B) < \sigma_A(S, S_B) \). The intuition is that the additional \( A \)-seed \( s_2 \) “blocks” \( B \)-propagation as \( A \) competes with \( B \) (\( q_{B|A} < q_{B|B} \)) but \( B \) complements \( A \) (\( q_{A|B} = q_{A|A} \)). Clearly \( \sigma_A \) is not monotonically decreasing in \( S_A \). Hence, \( \sigma_A \) is not monotone in \( S_A \).

EXAMPLE 2 (NON-CROS-MONOTONICITY). We use the example shown in Figure 9. All edges have probability 1. Nodes \( g_1 \) and \( g_2 \) are the two fixed \( A \) seeds, and we want to grow \( B \) seed set from \( S = \{s_1\} \) to \( T = \{s_1, s_2\} \). GAPs satisfy \( 0 < q_{A|B} < q_{A|B} < q_{A|B} < q_{B|A} < 1 \), which means that \( A \) complements \( B \) but \( B \) competes with \( A \). Since \( B \) competes with \( A \), it is straightforward to have examples in which the \( \sigma_A \) decreases when \( B \) seed set grows. We use Figure 9 to show a possible world in which the growth of \( B \) seed sets leads to \( v \) adopting \( A \), indicating that \( A \) spread may also increase. Even though we do not have a direct example showing that \( \sigma_A \) increases when \( B \) seed set grows, we believe the possible world example is a good indication that \( \sigma_A \) is not cross-monotone in \( S_A \).

Figure 9 uses Figure 8 as a gadget. Intuitively, when \( B \) seed set grows from \( S \) to \( T \), the probability that \( z \) adopts \( B \) decreases as
shown in Example 1. Then we utilize this and the fact that $B$ competes with $A$ to show that when $S$ is the $B$ seed set, due to $B$'s competition from $x$ node $v$ will not adopt $A$, but when $T$ is the $B$ seed set, there is no longer $B$'s competition from $x$ and thus $v$ will adopt $A$.

The node thresholds in the possible world are as follows: $q_{A|B} < \alpha_A \leq q_{A|0}$, $q_{B|0} < \alpha_B \leq q_{B|A}$, $q_{A|B} < \alpha_A \leq q_{A|0}$, and all other non-specified $\alpha$ values take value 0, meaning that they will not block diffusion.

Consider first that $S$ is the $B$ seed set. Since $q_{B|0} < \alpha_B$, $x$ is informed about $B$ from $s_1$ but will not adopt $B$ directly from $s_1$. From $y_1$, we can see that $A$ will pass through $u$, $w$ and reaches $x$. After $x$ adopts $A$, it reconsideres $B$, and since $\alpha_B \leq q_{B|A}$, $x$ adopts $B$. Node $x$ then informs $z$ about $A$ and $B$, in this order. However, since $q_{A|0} < \alpha_A$, $z$ does not adopt $A$, but it adopts $B$ ($\alpha_B = 0 \leq q_{B|0}$). Next $z$ informs $v$ about $B$, which adopts $B$ since $\alpha_B = 0 \leq q_{B|0}$. This happens one step earlier than $A$ reaches $v$ from $y_2$, but since $q_{A|B} < \alpha_A$, $v$ will not adopt $A$.

Now consider that $T$ is the $B$ seed set. In this case, $w$ definitely adopts $B$ from $s_2$. Since $q_{A|B} < \alpha_A$ and $w$ adopts $B$ first, $w$ will not adopt $A$ from $y_1$. Because $q_{B|0} < \alpha_B$ and $w$ blocks $A$ from reaching $x$, $x$ will not adopt $A$ or $B$. Then $z$ will not adopt $A$ or $B$ either. This allows $A$ to reach $v$ from $y_2$, and since $q_{A|B} < \alpha_A$, $v$ adopts $A$.

We can certainly duplicate $v$ enough times so that when $B$ seed set grows from $S$ to $T$, the $A$-spread in this possible world also increases, even though $B$ is competing with $A$.  

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**Figure 10:** The graph for Example 3 and 4

The following two examples show that even when two items are mutually complementary, self-submodularity and cross-submodularity in general may not hold.

**Example 3 (Non-Self-Submodularity).** Consider the possible world in Figure 10. All edges are live. The node thresholds are: for $w$: $\alpha_A \leq q_{A|0}$, $q_{B|0} < \alpha_B \leq q_{B|A}$; for $z$: $\alpha_A > q_{A|B}$, $\alpha_B < q_{B|0}$; for $v$: $q_{A|0} < \alpha_A \leq q_{A|B}$, $\alpha_B < q_{B|0}$. Then fix $S_B = \{y\}$. For $S_A$, let $S = \emptyset$, $T = \{x\}$, and $u$ is the additional seed. It can be verified that only when $S_A = T \cup \{u\}$, $v$ becomes $A$-adopted, violating self-submodularity.

A concrete example of $Q$ for which submodularity does not hold is as follows: $q_{A|0} = 0.0785432$; $q_{A|B} = 0.243932$; $q_{B|0} = 0.375556$; $q_{B|A} = 0.99545$. Seed sets are as above. We denote by $p_v(S_A)$ the probability that $v$ becomes $A$-adopted with $A$-seed set $S_A$. It can be verified that: $p_v(S_B) = 0$, $p_v(S \cup \{u\}) = 8.898 \cdot 10^{-3}$, $p_v(T) = 0.0273854$, and $p_v(T \cup \{u\}) = 0.027383$. Clearly, $p_v(T \cup \{u\}) > p_v(T) > p_v(S \cup \{u\}) > p_v(S)$. Hence, replicating $v$ sufficiently many times will lead to $\sigma_A(T \cup \{u\}, S_B) = \sigma_A(T, S_B) \geq \sigma_A(S \cup \{u\}, S_B) - \sigma_A(S, S_B)$, violating self-submodularity.

**Example 4 (Non-Cross-Submodularity).** Consider the possible world in Figure 10. The node thresholds are: for $w$: $q_{A|0} < \alpha_A \leq q_{A|B}$, $\alpha_B < q_{B|0}$; for $z$: $\alpha_A \leq q_{A|0}$, $\alpha_B > q_{B|A}$; for $v$: $q_{A|0} < \alpha_A \leq q_{A|B}$, $\alpha_B < q_{B|0}$. Fix $S_A = \{y\}$. For $S_B$, let $S = \emptyset$, $T = \{x\}$, and $u$ is the additional seed. It can be verified that only when $S_B = T \cup \{u\}$, $v$ becomes $A$-adopted, violating cross-submodularity.

The above example applies even when $q_{B|A} = q_{B|0} < 1$. The key is that $q_{B|A} < 1$ and $\alpha_B > q_{B|A}$, which prevents $B$ to pass through $z$, and thus another $B$-seed $u$ is needed to inform $v$ of $B$.

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**B. PSEUDO-CODE OF RR-SIM+**

Here we give the pseudo-code to RR-SIM+, one of the RR-set generation procedure we proposed for SELFINFMAX in §6.

Algorithm 4: RR-SIM+ ($G = (V, E)$, $v$, $S_B$)

1. create an FIFO queue $Q$ and empty sets $R$, $T_1$
2. $Q$ enqueue ($v$); // first backward BFS
3. while $Q$ is not empty do
   4. $u \leftarrow$ $Q$ dequeue();
   5. $T_1 \leftarrow T_1 \cup \{u\}$;
   6. foreach unvisited $w \in N^-(u)$ such that ($w, u$) is live do
      7. $Q$.enqueue($w$) and mark $w$ visited;
8. if $T_1 \cap S_B \neq \emptyset$ then
    // auxiliary forward pass to determine $B$ adoption
   9. clear $Q$, enqueue all nodes of $T_1 \cap S_B$ into $Q$, and execute line 3 to line 7 in Algorithm 2;
10. execute line 8 to line 16 in Algorithm 2 // second backward BFS

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**C. MORE EXPERIMENTAL RESULTS**

GAPs learned from Real Social Networks. Tables 4 – 6 demonstrate selected GAPs learned from Flixster, Douban-Book, and Douban-Movie datasets, using methods in §7. Here we not only show the estimated value of $q$, but also give 95% confidence intervals. By the definition of GAPs (§3), we can treat each GAP as the parameter of a Bernoulli distribution. Consider any GAP, denoted by $q$, and let $\hat{q}$ be its estimated value from action log data. The 95% confidence interval of $\hat{q}$ is given by

$$q \pm 1.96 \sqrt{q(1-q)/n_q},$$

where $n_q$ is the number of samples used for estimating $q$.

**Effectiveness of Sandwich Approximation (SA).** As discussed in §6.4, SA is likely to be effective for SELFINFMAX and COMPINFMAX when the adoption probabilities are close, which is indeed the case for the real world data we learn. We now measure the effectiveness of SA by comparing the spread achieved by seed sets $S_B$, $S_v$, $S_u$, obtained w.r.t. the original, upper bound, and lower bound functions respectively. Such spread must be computed using the original function $\sigma$, and in our case, the unaltered GAPs. More specifically, we calculate the relative error defined as follows (for COMPINFMAX, disregard $S_u$).

$$\text{SA_error} = \max \{|\sigma(S_v) - \sigma(S_u)|/|\sigma(S_u)| - \sigma(S_v)/|\sigma(S_v)|\} / |\sigma(S_v)|$$

In all four datasets, for both problems, the largest error is only 0.2%! To see if this is due to that $q_{B|0}$ and $q_{B|A}$ being close in the GAPs learned from action log data, we further “stress test” SA with a much more adversarial set-up. For SELFINFMAX, vary $q_{B|0}$ to be $\{1.5, 9\}$ and fix $q_{B|A} = 0.96$. For COMPINFMAX, vary $q_{B|A}$ to be $\{1.5, 9\}$ and fix $q_{B|0} = 1$ (to maintain complementarity). No change is made to $q_{A|0}$ and $q_{A|B}$.

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3See any standard textbooks on probability theory and statistics.
A      | B         | $q_A|\emptyset$ | $q_A|\emptyset$ | $q_B|\emptyset$ | $q_B|A$
---    |------------|--------------|--------------|--------------|--------------
Monster Inc. | Shrek      | .88 ± .01    | .92 ± .01    | .92 ± .01    | .96 ± .01    
Gone in 60 Seconds | Armageddon | .63 ± .02    | .77 ± .02    | .67 ± .02    | .82 ± .02    
Harry Potter: Prisoner of Azkaban | What a Girl Wants | .85 ± .01    | .84 ± .02    | .66 ± .02    | .67 ± .02    
Shrek | The Fast and The Furious | .92 ± .02    | .94 ± .01    | .80 ± .02    | .79 ± .02    

Table 4: Selected GAPs learned for movies from Flixster

| A            | B                   | $q_A|\emptyset$ | $q_A|\emptyset$ | $q_B|\emptyset$ | $q_B|A$
---    |---------------------|--------------|--------------|--------------|--------------
The Unbearable Lightness of Being | Norwegian Wood (Japanese) | .75 ± .01    | .85 ± .02    | .92 ± .01    | .97 ± .01    
Harry Potter and the Philosopher’s Stone | Harry Potter and the Half-Blood Prince | .99 ± 1 | 1.0 ± 0 | .97 ± 1 | .98 ± 1 | 
Stories of Ming Dynasty III (Chinese) | Stories of Ming Dynasty VI (Chinese) | .94 ± 1 | 1.0 ± 0 | .88 ± 0.3 | .98 ± 0.1 |
Fortress Besieged (Chinese) | Love Letter (Japanese) | .89 ± .01 | .91 ± 0.03 | .82 ± 0.02 | .83 ± 0.03 | 

Table 5: Selected GAPs learned for movies from Douban-Book

| A           | B        | $q_A|\emptyset$ | $q_A|\emptyset$ | $q_B|\emptyset$ | $q_B|A$
---    |----------|--------------|--------------|--------------|--------------
Up | 3 Idiots | .92 ± .01    | .94 ± .01    | .92 ± .01    | .93 ± .01    
Pulp Fiction | Leon | .81 ± .01    | .83 ± .01    | .95 ± .00    | .98 ± .01    
The Silence of the Lambs | Inception | .90 ± .01 | .86 ± .01 | .92 ± .01 | .98 ± .01 | 
Fight Club | Se7en | .84 ± .01 | .89 ± .01 | .89 ± .01 | .95 ± .01 | 

Table 6: Selected GAPs learned for movies from Douban-Movie

Figure 11 compares $\sigma(S_A)$, $\sigma(S_B)$, and $\sigma(S_C)$ on Flixster. As can be seen, even in this adversarial setting, SA is still highly effective for both SELF$\text{INFMAX}$ and COM$\text{INFMAX}$. Amongst all test cases, the largest error is 0.4%. The results on other datasets are very similar and hence omitted. We further stress test more by separating $q_A|\emptyset$ and $q_A|\emptyset$ to a wider margin, and still achieve similar results (hence omitted). This means the SA strategy is highly robust and useful for our problems.

By the principle of deferred decisions and the fact that each edge is only tested once in one diffusion, edge transition processes are equivalent. To generate a possible world, the live/blocked status of an edge is pre-determined and revealed when needed, while in a Com-IC process, the status is determined on-the-fly.

Figure 11: Sandwich Approximation on Flixster

D. PROOFS AND ADDITIONAL THEORETICAL RESULTS

D.1 Proofs for Results in Section 5

LMMA 1 (re-stated). For any fixed $A$-seed set $S_A$ and $B$-seed set $S_B$, the joint distributions of the sets of $A$-adopted nodes and $B$-adopted nodes obtained (i) by running a Com-IC diffusion from $S_A$ and $S_B$ and (ii) by randomly sampling a possible world $W$ and running a deterministic cascade from $S_A$ and $S_B$ in $W$, are the same.

Proof. The proof is based on establishing equivalence on all edge-level and node-level behaviors in Com-IC and the PW model.

THEOREM 2 (re-stated). For any fixed $B$-seed set $S_B$, $\sigma_A(S_A, S_B)$ is monotonically increasing in $S_A$ for any set of GAPs in $Q^+$ and $Q^-$. Also, $\sigma_A(S_A, S_B)$ is monotonically increasing in $S_B$ for any GAPs in $Q^+$, and monotonically decreasing in $S_B$ for any $Q^-$. For ease of exposition, we also state a symmetric version of Theorem 2 w.r.t. $\sigma_B$. That is, given any fixed $A$-seed set, $\sigma_B(S_A, S_B)$ is monotonically increasing in $S_B$ for any set of GAPs in $Q^+$ and $Q^-$. Also, $\sigma_B(S_A, S_B)$ is monotonically increasing in $S_A$ for any GAPs in $Q^+$, and monotonically decreasing in $S_A$ for any $Q^-$. For
is always fixed, in the remaining proof we ignore technical reasons and notational convenience, in the proof of Theorem 2 presented below, we “concurrently” prove both Theorem 2 and this symmetric version, without loss of generality.

Proof of Theorem 2. We first fix a B-seed set \( S_B \). Since \( S_B \) is always fixed, in the remaining proof we ignore \( S_B \) from the notations whenever it is clear from context. It suffices to show that monotonicity holds in an arbitrary, fixed possible world, which implies monotonicity holds for the diffusion model. Let \( W \) be an arbitrary possible world generated according to §5.1.

Define \( \Phi^+_W(S_A) \) (resp. \( \Phi^+_W(S_A) \)) to be the set of \( A \)-adopted (resp. \( B \)-adopted) nodes in possible world \( W \) with \( S_A \) being the \( A \)-seed set (and \( S_B \) being the fixed \( B \)-seed set). Furthermore, for any time step \( t \geq 0 \), define \( \Phi^+_W(S_A,t) \) (resp. \( \Phi^+_W(S_A,t) \)) to be the set of \( A \)-adopted (resp. \( B \)-adopted) nodes in \( W \) by the end of step \( t \), given \( A \)-seed set \( S_A \). Clearly, \( \Phi^+_W(S_A) = \cup_{t \geq 0} \Phi^+_W(S_A,t) \) and \( \Phi^+_W(S_A) = \cup_{t \geq 0} \Phi^+_W(S_A,t) \). Let \( S \) and \( T \) be two sets, with \( S \subseteq T \subseteq V \).

| Mutual Competition \( Q^- \). Our goal is to prove that for any \( v \in V \), (a) if \( v \in \Phi^+_W(S_A) \), then \( v \in \Phi^+_W(T) \); and (b) if \( v \in \Phi^+_W(T) \), then \( v \in \Phi^+_W(S_A) \). Item (a) implies self-monotonicity increasing property while item (b) implies cross-monotonic decreasing property. We use an inductive proof to combine the proof of above two results together, as follows. For every \( t \geq 0 \), we inductively show that (i) if \( v \in \Phi^+_W(S_A,t) \), then \( v \in \Phi^+_W(T,t) \); and (ii) if \( v \in \Phi^+_W(T,t) \), then \( v \in \Phi^+_W(S_A,t) \).

Consider the base case of \( t = 0 \). If \( v \in \Phi^+_W(S_A,0) \), then it means \( v \in S \), and thus \( v \in T = \Phi^+_W(T,0) \). If \( v \in \Phi^+_W(T,0) \), it means \( v \in S_B \), and thus \( v \in \Phi^+_W(S_A,0) \).

For the induction step, suppose that for all \( t < t' \), (i) and (ii) hold, and we show (i) and (ii) also hold for \( t = t' \). For (i), we only need to consider \( v \in \Phi^+_W(S_A,t') \), then \( \Phi^+_W(S_A,t' - 1) \), i.e. \( v \) adopts \( A \) at step \( t' \) when \( S \) is the \( A \)-seed set. Since \( v \) adopts \( A \), we know that \( \alpha_A^t \leq q_{A|B} \). Let \( U \) be the set of in-neighbors of \( v \) in the possible world \( W \). Use \( U_A(S_A) = U \cap \Phi^+_W(S_A,t' - 1) \) and \( U_B(S_A) = U \cap \Phi^+_W(S_A,t' - 1) \). If \( U_A(S_A) \) is the set of in-neighbors of \( v \) in \( W \) that adopted \( A \) (resp. \( B \)) by time \( t' - 1 \), then \( S_A \) is the \( A \)-seed set. Since \( v \in \Phi^+_W(S_A,t') \), we know that \( U_A(S_A) \neq \emptyset \). By induction hypothesis, we have \( U_A(S) \subseteq U_A(T) \) and \( U_B(S) \subseteq U_B(T) \).

Thus, \( U_A(T) \neq \emptyset \), which implies that by step \( t' \), \( v \) must have been informed of \( A \) at \( T \) if \( \alpha_A^t \leq q_{A|B} \). This completes the proof of the mutual competition case. For the induction step, we inductively show that \( \alpha_A^t \leq \Psi^+_A(S,t') \) and \( \alpha_B^t \leq \Psi^+_B(S,t') \).

The statement of (ii) is symmetric to (i): if we exchange \( A \) and \( B \) and exchange \( S \) and \( T \), (ii) becomes (i). In fact, one can check that we can literally translate the induction step proof for (i) into the proof for (ii) by exchanging pair \( A \) and \( B \) and pair \( S \) and \( T \) (except that (a) we keep the definitions of \( U_A(S_A) \) and \( U_B(S_A) \), and (b) whenever we say some set is the \( A \)-seed set, we keep this \( A \)). This concludes the proof of the mutual competition case.

| Mutual Complementarity \( Q^+ \). The proof structure is very similar to that of the mutual competition case. Our goal is to prove that for any \( v \in V \), (a) if \( v \in \Phi^+_W(S_A) \), then \( v \in \Phi^+_W(T) \); and (b) if \( v \in \Phi^+_W(T) \), then \( v \in \Phi^+_W(S_A) \). To show this, we inductively prove the following: For every \( t \geq 0 \), (i) if \( v \in \Phi^+_W(S,t) \), then \( v \in \Phi^+_W(T,t) \); and (ii) if \( v \in \Phi^+_W(T,t) \), then \( v \in \Phi^+_W(S,t) \). The base case is trivially true.

For the induction step, suppose (i) and (ii) hold for all \( t < t' \), and we show that (i) and (ii) also hold for \( t = t' \). For (i), we only need to consider \( v \) belongs to \( \Phi^+_W(S,t') \), which means that \( v \) adopts \( A \) at step \( t' \) when \( S \) is the \( A \)-seed set. Since \( v \) makes \( A \) at \( t' \) when \( T \) is the \( A \)-seed set, \( v \) makes \( A \) at step \( t' \) when \( T \) is the \( A \)-seed set, i.e. \( v \in \Phi^+_W(T,t') \).

Now suppose \( q_{A|B} < \alpha_A^t \leq q_{A|B} \). Since \( v \in \Phi^+_W(S,t') \), the only possibility is that \( v \) adopts \( B \) first by time \( t' \), but after re-consideration, \( v \) adopts \( A \) due to condition \( \alpha_A^t \leq q_{A|B} \). Thus we have \( v \in \Phi^+_W(S,t') \), and \( \alpha_B^t \leq q_{B|A} \).

If \( v \in \Phi^+_W(S,t' - 1) \), by induction hypothesis \( v \in \Phi^+_W(T,t' - 1) \), which means that \( v \) makes \( B \) at time \( t' - 1 \) when \( T \) is \( A \)-seed set. Since \( v \) makes \( B \) at \( t' - 1 \), \( v \) must have been informed of \( B \) at \( T \) if \( \alpha_B^t \leq q_{B|A} \). This implies that when \( T \) is the \( A \)-seed set, \( v \) must have been informed of \( B \) by time \( t' \). Since \( \alpha_B^t \leq q_{B|A} \), \( v \) makes \( B \) at time \( t' \) when \( T \) is \( A \)-seed set. Then the condition \( \alpha_A^t \leq q_{A|B} \) implies that \( v \) makes \( A \) by time \( t' \) when \( T \) is the \( A \)-seed set, i.e. \( v \in \Phi^+_W(T,t') \).

This concludes the inductive step for item (i) in the mutual complementarity case. The induction step for item (ii) is completely symmetric to the inductive step for item (i). Therefore, we complete the proof for the mutual complementarity case. As a result, the whole theorem holds.

Lemma 2 (re-stated). Consider any Com-IC instance with \( Q^+ \). Given fixed \( A \)- and \( B \)-seed sets, for all nodes \( v \in V \), all permutations of \( v \)'s in-neighbors are equivalent in determining if \( v \) becomes \( A \)-adopted and \( B \)-adopted, and thus the tie-breaking rule is not needed for mutual complementarity case.

Proof. Without loss of generality, we only need to consider a node \( v \) and two of its in-neighbors \( u_A \) and \( u_B \) which become \( A \)-adopted and \( B \)-adopted at \( t' - 1 \) respectively. In a possible world, there are nine possible combinations of the values of \( \alpha_A^t \) and \( \alpha_B^t \). We show that in all such combinations, the ordering \( \pi_1 = \langle u_A,u_B \rangle \) and \( \pi_2 = \langle u_B,u_A \rangle \) produce the same outcome for \( v \).

(1) \( \alpha_A^t \leq q_{A|B} \land \alpha_B^t \leq q_{B|A} \). Both \( \pi_1 \) and \( \pi_2 \) make \( v \) \( A \)-adopted and \( B \)-adopted.

(2) \( \alpha_A^t \leq q_{A|B} \land q_{B|A} < \alpha_B^t \leq q_{B|A} \). Both \( \pi_1 \) and \( \pi_2 \) make \( v \) \( A \)-adopted and \( B \)-adopted. With \( \pi_2 \), \( v \) first becomes \( B \)-suspended,
then $A$-adopted, and finally $B$-adopted due to re-consideration.

(3. $\alpha_A' \leq q_{A|B} \land \alpha_B' \leq q_{B|A}$. Both $\pi_1$ and $\pi_2$ makes $v$ $A$-adopted only.

(4. $q_{A|B} < \alpha_A' \leq q_{A|B} \land \alpha_B' \leq q_{B|B}$. Symmetric to (2).

(5. $q_{A|B} < \alpha_A' \leq q_{A|B} \land \alpha_B' \leq q_{B|A}$. In this case, $v$ does not adopt any item.

(6. $q_{A|B} \leq \alpha_A'$ if $q_{A|B} > q_{A|B} \land \alpha_B' \leq q_{B|A}$. In this case, $v$ does not adopt any item.

(7. $\alpha_A' > q_{A|B} \land \alpha_B' \leq q_{B|A}$. Symmetric to (3); $v$ is $B$-adopted only.

(8. $\alpha_A' > q_{A|B} \land \alpha_B' \leq q_{B|A}$. Symmetric to (6).

(9. $\alpha_A' > q_{A|B} \land \alpha_B' \leq q_{B|A}$. In this case, $v$ does not adopt any item.

Since the possible world model is equivalent to Com-IC (Theorem 1), the lemma holds as a result. 

**Lemma 3** (re-stated). In the Com-IC model, if $B$ is indifferent to $A$ (i.e., $q_{B|A} = q_{B|B}$), then for any fixed $B$ seed set $S_B$, the probability distribution over sets of $B$-adopted nodes is independent of $A$-seed set. Symmetrically, the probability distribution over sets of $A$-adopted nodes is independent of $B$-seed if $A$ is indifferent to $B$.

**Proof.** Consider an arbitrary possible world $W$. Let $q := q_{B|B} = q_{B|A}$. A node $v$ becomes $B$-adopted in $W$ as long as $\alpha_B' \leq q$ and there is a live-edge path $P_{B}$ from $S_B$ to $v$ such that for all nodes $w$ on $P_{B}$ (excluding seeds), $\alpha_B' \leq q$. Since $q_{B|B} = q_{B|A}$, this condition under which $v$ becomes $B$-adopted in $W$ is completely independent of any node’s state w.r.t. $A$. Thus, the propagation of $B$-adoption is completely independent of the actual $A$-seed set (even empty). Due to the equivalence of the possible world model and Com-IC, the lemma holds.

**Claim 1** (re-stated). On any $A$-path $P_A$, if some node $w$ adopts $B$ and all nodes before $w$ on $P_A$ are $A$-ready, then every node following $w$ on $P_A$ adopts both $A$ and $B$, regardless of the actual $B$-seed set.

**Proof of Claim 1.** Since $P_A$ is an $A$-path (where all nodes are $A$-adopted), every node $w$ on $P_A$ (except the starting node) has $\alpha_A' \leq q_{A|B}$. For every such node $w$, if $q_{A|B} < \alpha_A' \leq q_{A|B}$, then $w$ adopting $A$ implies that $w$ must have adopted $B$ first and $\alpha_B' \leq q_{B|B}$. Now suppose a node $w$ on the path adopts $B$ (under some $B$-seed set), and all nodes before $w$ on path $P_A$ are $A$-ready. Then all nodes before $w$ on this path adopts $A$ regardless of $B$-seed set. Thus $w$ is informed of $A$. Since $\alpha_A' \leq q_{A|B}$, $w$ adopts $A$.

Then consider the node $w'$ after $w$ on the path $P_A$. Node $w'$ must be informed by both $A$ and $B$ since $w$ adopts both $A$ and $B$ and the edge $(w, w')$ is live. If $\alpha_A' \leq q_{A|B}$, $w'$ will adopt $A$, and then since $q_{B|B} = 1$, $w'$ will then adopt $B$ – this is where we use the key assumption that $q_{B|B} = 1$. If $q_{A|B} < \alpha_A' \leq q_{A|B}$, then we have argued that in this case $\alpha_B' \leq q_{B|B}$, so $w'$ would adopt $B$, followed by adopting $A$. We can then inductively use the above argument along the path to show that every node after $w$ will adopt both $A$ and $B$, which proves the claim.

**Claim 2** (re-stated). There is a $B$-path $P_B$ from some $B$-seed set $x_0 \in T \cup \{u\}$ to $w$, such that even if $x_0$ is the only $B$-seed, $w$ still adopts $B$.

**Proof of Claim 2.** We construct such a path $P_B$ backwards from $w$ as follows.

Since $w$ adopts $B$, if $w \in T \cup \{u\}$, then we are done; otherwise, there must be some $B$-path $P_B'$ from $B$-seed set $T \cup \{u\}$ to $w$. If $P_B'$ is a $B$-ready path, then we have found $P_B$ to be $P_B'$, and its starting node is $x_0$. This is because even if $x_0$ are only $B$-seed, $B$ can still pass through all nodes on $P_B$ to reach $w$, without the need of any node on the path to adopt $A$, because by definition, all nodes $v$ on a $B$-ready path satisfies that $\alpha_B' \leq q_{B|B}$.

Now suppose $P_B'$ is not $B$-ready. It is clear that every non-$B$-ready node on path $P_B'$ must be $A$-ready (otherwise, it would not be possible for such a node to adopt $B$). If every such non-$B$-ready node has an $A$-ready path from $S_A$, then we still find $P_B' = P_B$ with its starting node is $x_0$. This is because whenever $B$ reaches a non-$B$-ready node on the path, the node has an $A$-ready path from $S_A$ and thus the node always adopts $A$, which means it will also adopt $B$ (for $q_{B|A} = 1$).

Now consider the case where there exists some non-$B$-ready node on path $P_B'$ that does not have any $A$-ready path from $S_A$. Let $x_1$ be the first such node (we count backwards from $w$). By the definition of $x_1$, we know that as long as $x_1$ adopts $B$ (regardless what is the actual $B$-seed set), then $x_1$ would pass $B$ along path $P_B'$ to $w$. Thus our backward construction has found the last piece of the path $P_B$ as the path segment of $P_B'$ from $x_1$ to $w$, now we move the construction backward starting from $x_1$.

At $x_1$, find a $A$-path $P_A'$ from $S_A$ to $x_1$. By the definition of $x_1$, we know that $P_A'$ is not $A$-ready. Let $x_2$ be the first non-$A$-ready node on path $P_A'$ counting forward from the starting $A$-seed. Then $x_2$ must have adopted $B$. By Claim 1, we know that as long as $x_2$ adopts $B$, all nodes after $x_2$ on path $P_A'$ would adopt both $A$ and $B$, regardless of the actual $A$-seed set. Applying this to $x_1$, we know that $x_1$ is both $A$- and $B$-adopted. Then our backward construction has found the last piece of $P_B'$, which is the path segment of $P_A'$ from $x_2$ to $x_1$, which guarantees that if $x_2$ adopts $B$, then $w$ must eventually adopt $B$.

Now if $x_2$ is a $B$-seed, we are done. If not, we will repeat the above “zig-zag constructions”: there must be a $B$-path $P_B'''$ from $B$-seed set $T \cup \{u\}$ to $x_2$. The argument on path $P_B'''$ is exactly the same as the argument on $P_B'$, and if the construction still cannot stop, we will find path $P_A'''$ similar to $P_A'$, and so on.

The construction keeps going backwards, and since the construction actually follows the strict adoption time line and going backward in time, in the diffusion process when $T \cup \{u\}$ is the $B$-seed set, the construction must eventually reach a $B$-seed $x_2$ and stops. Then we have found the desired path $P_B$ and the desired starting node $x_0$.

**D.2 Proofs for Results in Section 6**

**Lemma 5** (re-stated). Let $M$ be a stochastic diffusion model and $M'$ be its equivalent possible world model. If $M'$ satisfies Properties (P1) and (P2), then the RR-sets as defined in Definition 1 have the activation equivalence property as in Definition 2.

**Proof:** It is sufficient to prove that in every possible world $W \in \mathcal{W}$, $S$ activates $v$ if and only if $S$ intersects with $v$’s RR set in $W$, denoted by $R_W(v)$.

Suppose $R_W(v) \cap S \neq \emptyset$. Without loss of generality, we assume a node $u$ is in the intersection. By the definition of RR set, set $\{u\}$ can activate $v$ in $W$. Per Property (P1), $S$ can also activate $v$ in $W$.

Now suppose $S$ activates $v$ in $W$. Per Property (P2), there exists $u \in S$ such that $\{u\}$ can also activate $v$ in $W$. Then by the RR-set definition, $u \in R_W(v)$. Therefore, $S \cap R_W(v) \neq \emptyset$.

**Lemma 4** (re-stated). Let $W$ be a fixed possible world. Let $f_{w,W}(S)$ be an indicator function that takes on 1 if $S$ can activate

3Recall that it is $A$-ready, on a $B$-path, and there is no $A$-ready path from $S_A$ to $x_1$
$v$ in $W$, and 0 otherwise. Then, $f_{x,W}(\cdot)$ is monotone and submodular for all $v \in V$ if and only if both $(P1)$ and $(P2)$ are satisfied in $W$.

**Proof.** First consider “if”. Suppose both properties hold in $W$. Monotonicity directly follows from Property $(P1)$. For submodularity, suppose $v$ can be activated by set $T \cup \{x\}$ but not by $T$, where $x \notin T$. By Property $(P2)$, there exists some $u \in T \cup \{x\}$ such that $\{u\}$ can activate $v$ in $W$. If $u \in T$, then $T$ can also activate $v$ by Property $(P1)$, a contradiction. Hence we have $u = x$. Then, consider any subset $S \subseteq T$. Note that by Property $(P1)$, $S$ cannot activate $v$ (otherwise so could $T$), while $S \cup \{x\}$ can. Thus, $f_{x,W}(\cdot)$ is submodular.

Next we consider “only if”. Suppose $f_{x,W}(\cdot)$ is monotone and submodular for every $v \in V$. Property $(P1)$ directly follows from monotonicity. For Property $(P2)$, suppose for a contradiction that there exists a seed set $S$ that can activate $v$ in $W$, but there is no $u \in S$ so that $\{u\}$ activates $v$ alone. We repeatedly remove elements from $S$ until the remaining set is the minimal set that can still activate $v$. Let the remaining set be $S'$. Note that $S'$ contains at least two elements. Let $u \in S'$, and then we have $f_{x,W}(\emptyset) = f_{x,W}(\{u\}) = f_{x,W}(S' \setminus \{u\}) = 0$, but $f_{x,W}(S') = 1$, which violates submodularity, a contradiction. This completes the proof.

**Theorem 6** (re-stated). Under one-way complementarity ($q_{A|0} \leq q_{A|1}$ and $q_{B|0} = q_{B|A}$), the RR-sets generated by the RR-SIM algorithm satisfy Definition 1 for the SELFINFMAX problem. As a result, Theorem 3 applies to GeneralTim with RR-SIM in this case.

**Proof.** Suppose that, given a fixed possible world $W$, a fixed $B$-seed set $S_B$, and a certain node $u \in V$, for any node $v \notin \Phi_A^W(\emptyset, S_B)$ with $\alpha^A_A \leq q_{A|B}$, have: $v \in \Phi_A^W(\{u\}, S_B)$ if and only if there exists a live-edge path $P$ from $u$ to $v$ such that for all nodes $w \in P$, excluding $u$, $w$ satisfies $\alpha^A_A \leq q_{A|B}$, and in case $\alpha^A_A > q_{A|0}$, then $v$ must be $A$-adopted.

The “if” direction is straightforward as $P$ will propagate the adoption information of $A$ all the way to $v$. If $\alpha^A_A \leq q_{A|0}$, it adopts $A$ without question. If $\alpha^A_A \in (q_{A|0}, q_{A|B}]$, then $v$ must be $B$-adopted by the definition of $P$, which makes it $A$-adopted.

For the “only if” part, suppose no such $P$ exists for $u$. This leads to a direct contradiction since $u$ is the only $A$-seed, and $u$ lacks a live-edge path to $v$, it is impossible for $v$ to get informed of $A$, let alone adopting $A$. Next, suppose there is a live-edge path $P$ from $u$ to $v$, but there is a certain node $w \in P$ which violates the conditions set out in the lemma. First, $w$ could be have a “bad” threshold: $\alpha^A_A > q_{A|B}$. In this case, $w$ will not adopt $A$ regardless of its status w.r.t. $B$, and hence the propagation of $A$ will not reach $v$. Second, $w$ could have a threshold such that $\alpha^A_A \in (q_{A|0}, q_{A|B}]$ but it does not adopt $B$ under the influence of the given $S_B$. Similar to the previous case, $w$ will not adopt $A$ and the propagation of $A$ will not reach $v$. This completes the “only if” part.

Then by Definition 1, the theorem follows.

**Lemma 6** In expectation, GeneralTIM with RR-SIM runs in $O((k + \ell)(|V| + |E|) \log |V| (1 + EPT_F / EPT_B))$ time. (re-stated).

**Proof.** Given a fixed RR-set $R \subseteq V$, let $\omega(R)$ be the number of edges in $G$ that point to nodes in $R$. Since in RR-SIM, it is possible that we do not examine incoming edges to a node added to the RR-set (cf. Cases 1(iii) and 2(ii) in the backward BFS), we have:

$$EPT_B \leq \mathbb{E}[\omega(R)],$$

where the expectation is taken over the random choices of $R$. By Lemma 4 in [25] (note that this lemma only relies on the activation equivalence property of RR-sets, which holds true in our current one-way complementarity setting),

$$\frac{|V|}{|E|} \cdot \mathbb{E}[\omega(R)] \leq OPT_k.$$

This gives

$$\frac{|V|}{|E|} \cdot EPT_B \leq OPT_k.$$

Following the same analysis as in [25] one can check that the lower bound $LB$ of $OPT_k$ obtained by the estimation method in [25] guarantees that $LB \geq EPT_B \cdot |V|/|E|$. Since in our algorithm we set $\theta = \lambda/LB$, where (following Eq.(1))

$$\lambda = \epsilon^{-2} \left( 8 + 2e \right) |V| \left( \ell \log |V| + \log \left( \frac{|V|}{k} \right) + \log 2 \right),$$

then we have that the expected running time of generating all RR-sets is:

$$O(\theta \cdot EPT) = O\left( \frac{\lambda}{LB} \cdot (EPT_F + EPT_B) \right) = O\left( \frac{\lambda |E|}{|V| EPT_B} (EPT_F + EPT_B) \right) = O\left( \frac{\lambda |E|}{|V|} \left( 1 + \frac{EPT_F}{EPT_B} \right) \right) = O\left( (k + \ell)(|V| + |E|) \log |V| \left( 1 + \frac{EPT_F}{EPT_B} \right) \right).$$

The time complexity for estimating $LB$ and for calculating the final seed set given RR-sets are the same as in [25], and thus the final complexity is

$$O\left( (k + \ell)(|V| + |E|) \log |V| \left( 1 + \frac{EPT_F}{EPT_B} \right) \right).$$

**Theorem 7** (re-stated). Suppose that $q_{A|0} \leq q_{A|1} = q_{A|B}$ and $q_{B|0} \leq q_{B|A} = 1$. The RR-sets generated by the RR-CIM algorithm satisfies Definition 1 for the COMFINFMAX problem. As a result, Theorem 3 applies to GeneralTIM with RR-CIM in this case.

**Proof.** Suppose that, given a fixed possible world $W$, a fixed $A$-seed set $S_A$, and a certain node $u \in V$. Then, for any node $v \notin \Phi_A^W(S_A, \emptyset)$ with $\alpha^A_A \leq q_{A|1}$, have: $v \in \Phi_A^W(S_A, \emptyset)$ if and only if there exists a live-edge path $P(u, v)$ from $u$ to $v$ such that one of the following holds:

(i) $P(u, v)$ consists entirely of $A$-adopted or diffusible $A$-suspended/potential nodes, and $u$ must be $A$-suspended, or

(ii) There exists an $A$-suspended node $u' \neq u$ on $P(u, v)$, such that all nodes on the sub-path of $P(u, u')$ (excluding $u'$) are $B$-diffusible and the remaining ones (excluding $u'$ and $v$) are either $A$-adopted or diffusible $A$-suspended/potential.

**PROOF OF CLAIM 3.** (ii). Suppose (i) holds. Clearly, $u$ will adopt $B$ first as a seed and then adopt $A$ by reconsideration. Then all nodes on $P(u, v)$ including $v$ will adopt both $A$ and $B$ since they are either $A$-adopted or diffusible $A$-suspended/potential. Consider any node $w \neq v$ on $P(u, v)$: if $w$ is already $A$-adopted, it will adopt $B$ since $q_{B|A} = 1$; otherwise, $w$ adopts $B$ first since $\alpha^A_A \leq q_{B|0}$ (diffusible) and then $A$. Next, suppose (ii) holds. Since
Suppose otherwise. That is, there exists a node $z \in P$, such that $z$ cannot be explored by the first round backward BFS from $v$. We have established that in the complete possible world $W$, there is a live-edge path from $z$ to $u$ and from $u$ to $v$ respectively. Thus, connecting the two paths at node $u$ gives a single live-edge path $P_z$ from $z$ to $v$. Now recall that the continuation of the first backward BFS phase in RR-SIM+ relies solely on edge status (as long as an edge $(w, u)$ is determined live, $w$ will be visited by the BFS). This means that $z$ must have been explored in the first backward BFS. To show the backward path from $v$ to $u$ and then along the path $P$, which is a contradiction.  

**Theorem 8 (re-stated). Sandwich Approximation solution gives:**

$$
\sigma(S_{\text{sand}}) \geq \max \left\{ \frac{\sigma(S_u)}{\nu(S_u)}, \frac{\mu(S^*_u)}{\sigma(S^*_u)} \right\} \cdot (1 - 1/e) \cdot \sigma(S^*_u),
$$

where $S^*_u$ is the optimal solution maximizing $\sigma$ (subject to cardinality constraint $k$).

**Proof.** Let $S^*_u$ and $S^*_z$ be the optimal solution to maximizing $\mu$ and $\nu$ respectively. We have

$$
\sigma(S_u) = \frac{\sigma(S_u)}{\nu(S_u)} \cdot \nu(S_u) \geq \frac{\sigma(S_u)}{\nu(S_u)} \cdot (1 - 1/e) \cdot \nu(S^*_u) \geq \frac{\sigma(S_u)}{\nu(S_u)} \cdot (1 - 1/e) \cdot \sigma(S^*_u);
$$

$$
\sigma(S_z) \geq \mu(S^*_z) \geq (1 - 1/e) \cdot \mu(S^*_z) \geq (1 - 1/e) \cdot \mu(S^*_u) \geq (1 - 1/e) \cdot \sigma(S^*_u).
$$

The theorem follows by applying Eq. (3), the definition of $S_{\text{sand}}$.  

**Theorem 9 (re-stated). Suppose $q_{A|B} \leq q_{A|B}$ and $q_{A|B} \leq q_{B|A}$. Then, under the Com-IC model, for any fixed $A$ and $B$ seed sets $S_A$ and $S_B$, $\sigma_A(S_A, S_B)$ is monotonically increasing w.r.t. any one of $\{q_{A|B}, q_{A|B}, q_{B|A}, q_{B|A}\}$ with other three GAPs fixed, as long as after the increase the parameters are still in $Q^+$.**

**Proof (Sketch).** The detailed proof would follow the similar induction proof structure for each possible world as in the proof of Theorem 2. Intuitively, we would prove inductively that at every step increasing $q_{A|B}$ or $q_{A|B}$ would increase both $A$-adopted and $B$-adopted nodes.  

### D.3 Submodularity Analysis for Competitive Cases of Com-IC

First, we address cross-submodularity. Note that by Theorem 2, $\sigma_A$ is monotonically decreasing w.r.t. $S_B$ (with any fixed $S_A$). Intuitively, cross-submodularity for competitive products means adding an additional $B$-seed to a smaller $B$-seed set yields a larger decrease in the spread of $A$: for any $S \subseteq T \subseteq V$ and any $u \not\in T$, $\sigma_A(S_A, S) - \sigma_A(S_A, S \cup \{u\}) \geq \sigma_A(S_A, T) - \sigma_A(S_A, T \cup \{u\})$. This notion is relevant and useful to the problem of influence blocking maximization, where one party wants to find the best seed set to block the spread of competitors [6, 14]. Since influence blocking is not the focus of this work, from here on, we focus on self-submodularity and self-monotonicity only, and hereafter we drop “self-” in the terminologies.
When \( q_{A|\emptyset} = q_{B|\emptyset} = 1 \) and \( q_{A|B} = q_{B|A} = 0 \), the Com-IC model degenerates to the homogeneous CiC model, for which submodularity and monotonicity both hold [9]. In the general \( Q^- \) setting, monotonicity holds (Theorem 2). However, the following counter-example shows the opposite for submodularity.

**Example 5.** Consider the graph in Figure 12, where all edges have an influence probability of 1. Values of \( Q \) are: \( q_{A|\emptyset} = q \in (0, 1) \), \( q_{A|B} = q_{B|A} = 0 \), \( q_{B|\emptyset} = 1 \). The \( B \)-seed set is \( S_B = \{y\} \). For \( A \)-seed set \( S_A \), let \( S = \{s_1, s_2\} \), and \( u = s_3 \).

We consider the probability \( v \) becomes adopted by \( S_A \), denoted by \( a_{AP}(v, S_A)(S_B \) is omitted since it is clear from context): 

\[
\begin{align*}
a_{AP}(v, S) &= 0, \\
a_{AP}(v, S \cup \{u\}) &= q^2, \\
a_{AP}(v, T) &= 0, \\
a_{AP}(v, T \cup \{u\}) &= q^2 + (1 - q) \cdot q^6.
\end{align*}
\]

Hence, we have 

\[
(a_{AP}(v, T \cup \{u\}) - a_{AP}(v, T)) - (a_{AP}(v, S \cup \{u\}) - a_{AP}(v, S)) = (1 - q) \cdot q^6 > 0.
\]

Non-submodularity occurs for the entire graph if \( v \) is replicated sufficiently many times.

The intuition of the above counter-example is that \( A \)-seeds \( s_2 \) and \( s_3 \) together can block \( B \) completely, and thus even if they cannot successfully activate \( v \) to adopt \( A \), \( s_1 \) could later activate \( v \) (with the additional probability \( (1 - q) \cdot q^6 \)), but when \( s_2 \) or \( s_3 \) acts alone, it cannot block \( B \), so the influence of \( A \) from \( s_1 \) will not reach \( v \).

Next, we show a positive result which says that submodularity is satisfied as long as \( q_{A|\emptyset} = q_{B|\emptyset} = 1 \).

**Theorem 10.** In the Com-IC model, when \( q_{A|\emptyset} = q_{B|\emptyset} = 1 \), the influence spread function \( \sigma_{AP}(S_A, S_B) \) is submodular w.r.t. \( S_A \) for any given \( S_B \).

**Proof.** We fix a \( B \)-seed set \( S_B \) and consider an arbitrary possible world \( W \) to show that submodularity is satisfied in \( W \).

In the possible world \( W \), for an \( A \)-adopted node \( v \), there must exist a live-edge path from \( A \)-seed set \( S_A \) to \( v \) with all nodes on the path adopting \( A \), and we call this path an \( A \)-path (\( B \)-path is defined symmetrically). With mutual competition, the length of the shortest \( A \)-path from \( S_A \) to \( v \) is the same as the time step at which \( v \) adopts \( A \). This is because in competitions when a node is informed about an item, it only has one chance at the same time step to decide about the adoption (reconsideration in later steps is not possible).

**Claim 4.** If a node \( v \) is reachable from \( S_A \) or \( S_B \) in the possible world \( W \), then \( v \) must adopt at least one of \( A \) and \( B \).

**Proof of Claim 4.** This is due to the fact that \( q_{A|\emptyset} = q_{B|\emptyset} = 1 \). In this case, suppose a node \( v \) is reachable from \( S_A \), i.e., there is a path from \( S_A \) to \( v \) in the possible world \( W \). The claim can be proven by an induction on the path length. Essentially, no matter which item an earlier node on the path adopts, it would inform the next node on the path, and since \( q_{A|\emptyset} = q_{B|\emptyset} = 1 \), the node would adopt the first item it gets informed.

Let \( S \subseteq T \subseteq V \) be two sets of nodes, and \( u \in V \setminus T \) be an additional \( A \)-seed. For a fixed \( B \)-seed set \( S_B \) (omitted from the following notations), let \( \Phi_{A|B}^y(S_A) \) denote the set of \( A \)-adopted nodes in the possible world \( W \) with \( A \)-seed set \( S_A \) after the diffusion ends. Consider a node \( v \in \Phi_{A|B}^y(T \cup \{u\}) \setminus \Phi_{A|B}^y(T) \). We need to show \( v \in \Phi_{A|B}^y(S \cup \{u\}) \setminus \Phi_{A|B}^y(S) \).

We now construct a shortest \( A \)-path, denoted \( P_A \), from some node in \( w_0 \in T \cup \{u\} \) to \( v \) when \( S_A = T \cup \{u\} \). The path construction is done backwards. Starting from \( v \), if \( v \) has only one in-neighbor \( w \) attempting to inform \( v \) through the live edge (open information channel) from \( w \) to \( v \), then in the previous time step, \( w \) must have adopted \( A \), and we select \( w \) as the predecessor of \( v \) on path \( P_A \). If there are multiple in-neighbors that have live edges pointing to \( v \), we examine the permutation \( \pi_v \) and pick \( w \) as \( v \)’s predecessor on this path, such that \( w \) ranks first over all \( v \)’s in-neighbors that adopted \( A \) in the previous time step. We trace back and stop once we reach an \( A \)-seed in \( T \cup \{u\} \). It is clear that \( P_A \) is a shortest \( A \)-path from \( T \cup \{u\} \) to \( v \), since in every construction step we move back one time step.

**Claim 5.** Let \( w_0 \in T \cup \{u\} \) be the starting point of \( P_A \). Then, even when \( S_A = \{w_0\} \), all nodes on \( P_A \) would still adopt \( A \).

**Proof of Claim 5.** Suppose, for a contradiction, that some node on \( P_A \) does not adopt \( A \) when \( S_A = \{w_0\} \). Let \( w \) be the first of such nodes on \( P_A \) (counting from \( w_0 \)). By Claim 4, \( w \) must adopt \( B \) as it does not adopt \( A \). This means that there must exist a \( B \)-path from \( S_A \) to \( w \). We construct a shortest \( B \)-path \( P_B \) from \( S_B \) to \( w \) in the same way as we constructed \( P_A \); start from \( w' = w \) backwards and always select the in-neighbor of the current node \( w' \) that adopts \( B \) and is ordered first in \( \pi_w \). Moreover, since \( w \) is informed of \( A \) (by its predecessor on path \( P_B \)) but does not adopt \( A \), it holds that \( q_{A|B} < q_{A|\emptyset} \leq q_{A|\emptyset} \) in \( W \).

We now compare the length of \( P_B \), denoted by \( \ell_B \), to the length of the segment of \( P_A \) from \( w_0 \) to \( w \), denoted by \( \ell_A \). Suppose \( \ell_B > \ell_A \). Then \( w \) would still adopt \( A \) when \( S_A = \{w_0\} \) (regardless of whether it will adopt \( B \) or not), a contradiction. Next consider \( \ell_B < \ell_A \). For \( T \cup \{u\} \) is the \( A \)-seed set, from \( B \)-seed set \( S_B \) and through the path \( P_B \), every time step one more node on path \( P_B \) has to adopt either \( A \) or \( B \). By Claim 4. Thus by time step \( \ell_B \), \( w \) adopts either \( A \) or \( B \). But since \( P_A \) is the shortest \( A \)-path from \( T \cup \{u\} \) to \( w \) and \( \ell_B > \ell_B \), \( w \) cannot adopt \( A \) at step \( \ell_B \), so \( w \) must adopt \( B \) at step \( \ell_B \). Since \( q_{A|B} > q_{A|\emptyset} \), it means that \( w \) would not adopt \( A \) after adopting \( B \), also a contradiction.

We are left with the case \( \ell_B = \ell_A \). Let \( w_A \) and \( w_B \) be the predecessor of \( w \) on \( P_A \) and \( P_B \), respectively. Assume \( w_A \neq w_B \) (we
deal with \( w_A = w_B \) later). If \( w_A \) is ordered ahead of \( w_B \) in \( \pi_w \), then according to the tie-breaking rule, \( w \) would be informed of \( A \) first and adopt \( A \) (since \( \alpha_A^w \leq q_{A|B} \)) when \( w_B \) is the only \( A \)-seed, contradicting the definition of \( w \).

If \( w_B \) is ordered ahead of \( w_A \) in \( \pi_w \), then consider again the scenario when \( S_A = T \cup \{ u \} \). By Claim 4, through the path \( P_B \), \( w_B \) adopts either \( A \) or \( B \) by step \( \ell_B - 1 \). If \( w_B \) adopts \( A \), it contradicts the construction of \( P_A \) since we would have chosen \( w_B \) instead of \( w_A \) in the backward construction. If \( w_B \) adopts \( B \), then \( w \) would be informed of \( B \) from \( w_B \) first, and then due to \( \alpha_A^w > q_{A|B} \), \( w \) would not adopt \( A \), again a contradiction.

Finally, we also need to consider the case of \( w_A = w_B \). Since \( w \) does not adopt \( A \) when \( w_0 \) is the only \( A \)-seed, according to our tie-breaking rule, node \( w_A \) must adopt \( B \) first and then adopt \( A \) at the same step. We then trace back the predecessor of \( w_A \) on path \( P_A \) and \( P_B \) respectively. If these two paths never branch backward, i.e. \( P_A = P_B \), then we know that \( w_0 \) is both \( A \) and \( B \)-seed. According to the possible world model we would use \( \tau_{w_0} \) to decide whether \( w_0 \) adopts \( A \) or \( B \) first, and then all nodes on the path \( P_A = P_B \) would follow the same order. Since \( w_A \) adopts \( B \) first before \( A \), we know that \( \tau_{w_0} \) is such that \( B \) is ranked ahead of \( A \).

Now we consider the scenario when \( S_A = T \cup \{ u \} \). By the construction of path \( P_A \), the successor of \( w_0 \) on this path, \( w_1 \), orders \( w_0 \) first among all \( A \)-adopted in-neighbors, so \( w_1 \) would take \( w_0 \) to get informed of \( A \), but since \( w_0 \) adopts \( B \) first, this implies that \( w_1 \) also adopts \( B \) first. We can apply the same argument along the path \( P_A \) to see that \( w \) adopts \( B \) first before \( A \). However, since \( \alpha_A^w > q_{A|B} \), we know that \( w \) would not adopt \( A \), a contradiction.

If paths \( P_A \) and \( P_B \) branch at some node \( x \) when we trace backward, then let \( x_A \) and \( x_B \) are the two predecessors of \( x \) on paths \( P_A \) and \( P_B \) respectively, and \( x_A \neq x_B \). When \( w_0 \) is the only \( A \)-seed, we know from above that \( w_A \) adopts \( B \) first before adopting \( A \). By the construction of \( P_B \), \( w_A \) must be informed of \( B \) from its predecessor on \( P_B \), so if this predecessor has not branched yet, it must also adopt \( B \) before \( A \). Using the same argument, we know that \( x \) must adopt \( B \) first before adopting \( A \). Then we know that \( x_B \) must be ordered before \( x_A \) in \( \pi_w \). Now consider the scenario when \( S_A = T \cup \{ u \} \). The argument is exactly the same as the previous case of \( w_B \) ordered before \( w_A \), and we also reach a contradiction.

We have exhausted all cases and Claim 5 is proven. \( \square \)

Claim 5 immediately implies that \( w_0 = u \), since otherwise, we have \( w_0 \in T \), and by the proof of monotonicity (Theorem 2) we know that \( v \in \Phi_A^w (\{ w_0 \}) \subseteq \Phi_A^w (T) \). This contradicts with the definition of \( v \in \Phi_A^w (T \cup \{ u \}) \setminus \Phi_A^w (T) \).

Again by monotonicity, we have \( v \in \Phi_A^w (\{ u \}) \subseteq \Phi_A^w (S \cup \{ u \}) \). Since \( S \subseteq T \) and \( v \notin \Phi_A^w (T) \), then \( v \notin \Phi_A^w (S) \). This gives \( v \in \Phi_A^w (S \cup \{ u \}) \setminus \Phi_A^w (S) \), which was to be shown. \( \square \)