A Simplified Guide to Large Antichains in the Partition Lattice

E. R. Canfield
University of Georgia

L. H. Harper
University of California, Riverside

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Abstract

Let Π_n denote the lattice of partitions of an n-set, ordered by refinement. We show that for all large n there exist antichains in Π_n whose size exceeds $n^{1/35}S(n, K_n)$. Here $S(n, K_n)$ is the largest Stirling number of the second kind for fixed n, which equals the largest rank within Π_n. Some of the more complicated aspects of our previous proof of this result are avoided, and the variance of a certain random variable Z which plays a key role in the construction is determined to within $O(1)$.

1 The problem.

Let S be a finite set with n elements. A *partition* of S is a collection of pairwise disjoint subsets of S, called *blocks*, whose union is S. We say that partition π_1 *refines* partition π_2, denoted $\pi_1 < \pi_2$, provided π_1 is obtained from π_2 by further partitioning one or more blocks of π_2. Thus, for example, when $n = 8$ and $S = \{1, 2, \ldots, 8\}$, we have $\pi_1 < \pi_2$ for

$$\pi_1 = \{\{1,3,8\},\{2\},\{4,6\},\{5,7\}\}$$
$$\pi_2 = \{\{1,3,8\}\{2,5,7\},\{4,6\}\}.$$

Under the refinement relation the set Π_n of all partitions of an n-set is a partially ordered set (poset), in fact, a lattice, and the problem is to find large antichains in this poset. An *antichain* in a poset is a collection of elements no two of which are related.

How shall we decide if an antichain in the partition lattice Π_n is "large" in the absence of a proven upper bound? Observe that when $\pi_1 < \pi_2$ the partition π_1 has more blocks than π_2. Thus the set of all partitions having a fixed number of blocks, say k blocks, is an antichain. The size of this antichain is $S(n, k)$, the Stirling number of the second kind; we let $S(n, K_n)$ denote the largest Stirling number of the second kind. It has been known since [1] that there exist antichains within Π_n which for all large n are strictly larger than $S(n, K_n)$. However, all such antichains constructed to date have cardinality asymptotically equal to $S(n, K_n)$. It is the purpose of this paper to construct antichains $A \subseteq \Pi_n$ such that $|A|/S(n, K_n) \to \infty$.

The important ideas for this construction can be traced to the 1985 work [4]. The latter gave a heuristic, though not rigorously established, argument that antichains

A existed for which $|A|/S(n, K_n) > 1.6$. It was in an effort to establish this latter inequality with full rigor that the even better result reported here was found.

Our main result, that for all large n there exist antichains satisfying

$$|A| \geq n^{1/35} S(n, K_n) \tag{1.1}$$

will appear in [2]. However, the present paper differs in three essential ways from the latter: (i) the argument is shorter and simpler; (ii) the especially complicated proof, involving characteristic functions, that a compact family of random variables Z (see Section 2) is uniformly asymptotically normal has been replaced by the very simple, though less precise, Lemma 3.1 which relies only on Chebyshev's inequality; (iii) a better bound for the error in our estimate of Var (Z) – see Lemma 3.2 – is obtained in a very direct way.

2 Construction of a family of antichains.

Let N_j be the integer-valued function defined on Π_n by the rule

$$N_j(\pi) = \#\text{ blocks of size } j \text{ in } \pi.$$

We shall regard N_j as a random variable by considering Π_n endowed with the uniform probability measure. This is the underlying probability space for all random variables discussed in this paper.

Now suppose $A_j, 1 \leq j \leq n$, is a sequence of real coefficients satisfying

$$\min_{i,j} \; A_i + A_j - A_{i+j} = \lambda > 0, \tag{2.1}$$

and consider the random variable Z defined by

$$Z = \sum_{j=1}^{n} A_j N_j. \tag{2.2}$$

When a partition π is refined two blocks of sizes i and j are created and one of size $i + j$ is lost; by (2.1) $Z(\pi)$ changes by at least λ. Since no two partitions related by refinement can have Z-values belonging to a half open interval of width λ, it follows that for any sequence A_j satisfying (2.1) and any κ the set

$$\left\{\pi : \kappa - \frac{\lambda}{2} < Z(\pi) \leq \kappa + \frac{\lambda}{2}\right\} \tag{2.3}$$

is an antichain.

3 The size of the constructed anti chains.

For each real sequence A_j such that $A_{i+j} - A_i - A_j > 0$ and each real κ we have an antichain (2.3). In this section we prove two lemmas about the size of these antichains.

Lemma 3.1 *Let Z be any random variable with variance $\sigma^2, \lambda > 0$ a real number, and assume $\lambda/\sigma \leq 1$. Then there exists κ such that*

$$\text{Prob}\left\{\kappa - \frac{\lambda}{2} < Z \leq \kappa + \frac{\lambda}{2}\right\} \geq \frac{3}{20}\frac{\lambda}{\sigma}.$$

Proof. By Chebyshev's inequality,

$$\text{Prob}\left\{-2 < \frac{Z - E(Z)}{\sigma} \leq +2\right\} \geq \frac{3}{4}.$$

The half open interval $(-2, +2]$ can be covered by disjoint half open intervals of length $\leq \lambda/\sigma$, using at most $4\sigma/\lambda + 1$ such intervals. Hence, for at least one of these, say $(L, U]$, we have

$$\text{Prob}\left\{\frac{Z - E(Z)}{\sigma} \in (L, U]\right\} \geq \frac{3/4}{4\sigma/\lambda + 1} = \frac{3/4}{4 + \frac{\lambda}{\sigma}}\,\frac{\lambda}{\sigma} \;\geq\; \frac{3}{20}\,\frac{\lambda}{\sigma}.$$

Hence we have the lemma by taking $\kappa = E(Z) + \frac{U-L}{2}\sigma$. □

¿From Lemma 3.1 it is clear that we wish to minimize Var (Z); the next lemma estimates the latter quantity in the special case where Z is given by (2.2).

Lemma 3.2 *Let $Z = \sum_{j=1}^{n} A_j N_j$ and let r be the real positive solution to the equation $re^r = n$. Then*

$$\text{Var }(Z) = b - \frac{c^2}{r(r+1)e^r} + O(1),$$

where

$$b = \sum_{j=1}^{n}(A_j)^2 r^j/j!$$

$$c = \sum_{j=1}^{n} jA_j r^j/j!,$$

uniformly over all coefficient sequences A_j satisfying $A_j = O(1)$.

Proof. We evaluate Var (Z) by the formula

$$\text{Var}(Z) = E(Z^2) - E(Z)^2 = \sum_{j=1}^{n}(A_j)^2\text{Var }(N_j) + \sum_{\substack{1 \leq j,k \leq n \\ j \neq k}} A_j A_k \text{ Cov }(N_j, N_k). \tag{3.1}$$

The variances and covariances appearing in (3.1) can be expressed exactly in terms of the Bell numbers. (The n-th Bell number B_n equals $|\Pi_n|$, the size of the partition lattice.) Namely,

$$E(N_j) = \binom{n}{j} B_{n-j}/B_n \tag{3.2}$$

$$E(N_j^2) = E(N_j) + \frac{(n)_{2j}}{(j!)^2} B_{n-2j}/B_n \tag{3.3}$$

$$E(N_jN_k) = \frac{(n)_{j+k}}{j!k!}B_{n-j-k}/B_n. \tag{3.4}$$

To illustrate we prove (3.3); the other two can be demonstrated similarly. The random variable $N_j(N_j-1)$ counts the ways to distinguish an ordered pair of distinct blocks of size j in a partition. Since $N_j^2 = N_j + N_j(N_j-1)$, formula (3.3) follows when we see that

$$E(N_j(N_j-1)) = \frac{(n)_{2j}}{(j!)^2}B_{n-2j}/B_n. \tag{3.5}$$

But a partition with two distinguished j-blocks can be created by choosing the first j-block, then the second, then an arbitrary partition on the remaining $n-2j$ elements. This can be done in $\binom{n}{j}\binom{n-j}{j}B_{n-2j}$ ways, yielding (3.5).

The next step in evaluating Var (Z) is to use the Moser Wyman [5] approximation of the Bell numbers. We need both upper bounds that hold uniformly for $1 \le j \le n$, as well as more exact asymptotic expansions for $j = O(r)$. The essential tool is the Moser Wyman formula which we state without proof:

$$B_{n+h} = \frac{(n+h)!}{r^{n+h}} \frac{\exp(e^r-1)}{(2\pi r(r+1)e^r)^{1/2}}\left(1 + \frac{P_0 + hP_1 + h^2P_2}{e^r} + O(e^{-2r})\right) \tag{3.6}$$

We have modified slightly the original formula found in [5] so that a set of numbers B_{n+h} may all be estimated in terms of the same parameter r. In (3.6) the big-oh term on the right is uniform for positive and negative integers h satisfying $h = O(r)$; P_0, P_1, and P_2 are rational functions of r which satisfy

$$P_0 = O(1), P_1 = O(r^{-1}), P_2 = \frac{-1/2}{r(r+1)}.$$

We find immediately from (3.6)

$$\frac{nB_{n-1}}{B_n} = r(1 + O(n^{-1})) \tag{3.7}$$

and

$$\frac{(n)_jB_{n-j}}{B_n} = r^j\left(1 + \frac{-jP_1 + j^2P_2}{e^r} + O(e^{-2r})\right), \tag{3.8}$$

the latter uniformly for $j = O(r)$. We also calculate from (3.6) that

$$\frac{(n+1)B_n}{B_{n+1}} \div \frac{nB_{n-1}}{B_n} = 1 + \frac{1}{r(r+1)e^r} + O(e^{-2r}),$$

which tells us that the sequence nB_{n-1}/B_n is ultimately increasing, say for $n \ge n_0$. We have

$$\frac{(n)_jB_{n-j}}{B_n} = \prod_{t=n-j}^{n-1} \frac{(t+1)B_t}{B_{t+1}},$$

and by monotonicity and (3.7) all but at most n_0 of the factors on the right are less than $r(1+O(n^{-1}))$. However, the finitely many factors not covered by monotonicity will nevertheless certainly be less than r for large enough n, and so we have the useful inequality

$$\frac{(n)_j B_{n-j}}{B_n} = O(r^j), \text{ uniformly } 1 \leq j \leq n. \tag{3.9}$$

Combining (3.2) - (3.4) and the bound (3.9) we find, uniformly for $1 \leq j \leq n$,

$$E(N_j) = O(r^j/j!) \tag{3.10}$$

$$\text{Var}\ (N_j) = O\left(\frac{r^j}{j!} + \left(\frac{r^j}{j!}\right)^2\right) \tag{3.11}$$

$$\text{Cov}\ (N_j, N_k) = O\left(\frac{r^{j+k}}{j!k!}\right). \tag{3.12}$$

If we restrict j, k to be, say, less than $10r$, then we can maintain greater precision by combining (3.2) - (3.4) with (3.8) to compute

$$E(N_j) = \frac{r^j}{j!}\left(1 + \frac{-jP_1 + j^2P_2}{e^r} + O(e^{-2r})\right) \tag{3.13}$$

$$\text{Var}\ (N_j) = E(N_j) + \left(\frac{r^j}{j!}\right)^2\left(\frac{2j^2P_2}{e^r} + O(e^{-2r})\right) \tag{3.14}$$

$$\text{Cov}\ (N_j, N_k) = \frac{r^{j+k}}{j!k!}\left(\frac{2jkP_2}{e^r} + O(e^{-2r})\right). \tag{3.15}$$

We are almost ready to complete the lemma but first we must bound the tails of several sums. The sum $\sum_{j\geq 10r} r^j/j!$ is less than a constant times its first term (using a geometric series), and the first term is by Stirling's formula $O(e^{-10r})$. Thus

$$\sum_{j\geq 10r}(A_j)^2 r^j/j! = O(1)\sum_{j\geq 10r} r^j/j! = O(e^{-10r}).$$

Further, using

$$\left(\frac{1}{j!}\right)^2 = \frac{1}{(2j)!}\binom{2j}{j} < \frac{2^{2j}}{(2j)!},$$

we find

$$\sum_{j\geq 10r}(A_j)^2\left(\frac{r^j}{j!}\right)^2 = O(1)\sum_{j\geq 10r}\frac{(2r)^{2j}}{(2j)!} = O(e^{-20r})$$

and

$$\sum_{j\geq 1}(A_j)^2\left(\frac{r^j}{j!}\right)^2 = O(e^{2r}).$$

Combining these observations we have

$$\begin{aligned}\sum_{j=1}^{n}(A_j)^2\text{Var}\ (N_j) &= \sum_{j<10r}(A_j)^2\frac{r^j}{j!}\left(1 + O(e^{-r})\right)\\ &+ \sum_{j<10r}(A_j)^2\left(\frac{r^j}{j!}\right)^2\left(\frac{2j^2P_2}{e^r} + O(e^{-2r})\right) + O(e^{-10r})\end{aligned}$$

$$= \sum_{j<10r} (A_j)^2 \frac{r^j}{j!} + \sum_{j<10r} (A_j)^2 \left(\frac{r^j}{j!}\right)^2 \frac{2j^2 P_2}{e^r} + O(1)$$
$$= \sum_{j=1}^{n} (A_j)^2 \frac{r^j}{j!} + \sum_{j=1}^{n} (A_j)^2 \left(\frac{r^j}{j!}\right)^2 \frac{2j^2 P_2}{e^r} + O(1). \qquad (3.16)$$

Because

$$\sum_{\substack{j \geq 10r \\ j \neq k}} A_j A_k \frac{r^{j+k}}{j!k!} = O(1) \cdot \sum_{j \geq 10r} \frac{r^j}{j!} \cdot \sum_{k \geq 1} \frac{r^k}{k!} = O(e^{-9r}),$$

and similarly for the summation where $k \geq 10r$ and $j \neq k$, we find also that

$$\begin{aligned} \sum_{\substack{1 \leq j,k \leq n \\ j \neq k}} A_j A_k \operatorname{Cov}(N_j, N_k) &= \sum_{\substack{1 \leq j,k < 10r \\ j \neq k}} A_j A_k \frac{r^{j+k}}{j!k!} \left(\frac{2P_2 jk}{e^r} + O(e^{-2r})\right) \\ &+ O(e^{-9r}) \\ &= \sum_{\substack{1 \leq j,k < 10r \\ j \neq k}} A_j A_k \frac{r^{j+k}}{j!k!} \frac{2P_2 jk}{e^r} + O(1) \\ &= \sum_{\substack{1 \leq j,k \leq n \\ j \neq k}} A_j A_k \frac{r^{j+k}}{j!k!} \frac{2P_2 jk}{e^r} + O(1). \qquad (3.17) \end{aligned}$$

Combining (3.1), (3.16), and (3.17) we have

$$\operatorname{Var}(Z) = \sum_{j=1}^{n} (A_j)^2 r^j / j! + \frac{2P_2}{e^r} \left(\sum_{j=1}^{n} j A_j r^j / j! \right)^2 + O(1),$$

which is the desired result. □

4 Choosing A_j so that Var(Z) is small.

We continue to let r be the unique real solution to $re^r = n$. Define $A_j, 1 \leq j \leq n$, as follows:

$$A_j = \begin{cases} j/r, & j \in (\tau r, 2\tau r] \\ 1, & \text{otherwise.} \end{cases}$$

Here τ is a real parameter, approximately 2/3, whose exact value will be revealed later. There is some natural motivation to let A_j be proportional to j when trying to minimize Var(Z): if $A_j = j$ then $\Sigma A_j N_j = n$, a constant, and Var $(Z) = 0$. However, letting $A_j = j$ violates condition (2.1). The largest interval on which we can let A_j be proportional to j without violating (2.1) is a half open interval of the form $(\tau r, 2\tau r]$.

By considering various cases we find that

$$\min_{i,j} A_i + A_j - A_{i+j} > 1 - \tau.$$

(For the latter it is necessary to know that $\tau > 2/3$, but this will be seen to be true shortly.) Let us denote by J and K the two integers such that the "otherwise" condition in the definition of A_j holds for $j \leq J$ or $j \geq K$. Then we have, in the notation of Lemma 3.2,

$$b = \Sigma (A_j)^2 r^j / j! = (1 + r^{-1}) e^r + O\left(\frac{r^J}{J!} + \frac{r^K}{K!}\right). \tag{4.1}$$

The first quantity on the right is the result of summing $(j/r)^2 r^j/j!$ for $j \geq 1$; the second term makes up for the error committed in doing so. For $j < \tau r$ and $j \geq 2\tau r$ the terms $r^j/j!$ are bounded by geometric series whose ratios are bounded away from 1. In a similar manner

$$c = \sum j A_j r^j / j! = (r+1) e^r + r\, O\left(\frac{r^J}{J!} + \frac{r^K}{K!}\right). \tag{4.2}$$

¿From (4.1) and (4.2), using Lemma (3.2), we find

$$\text{Var}\ (Z) = O\left(\frac{r^J}{J!} + \frac{r^K}{K!}\right) + O(1).$$

We now choose the parameter τ to make the quantities $r^J/J!$ and $r^K/K!$ be of the same order of magnitude; this leads to

$$\tau = e/4.$$

5 Conclusions.

When $J = \frac{e}{4} r$, to within 1, we find by Stirling's formula that

$$r^J / J! = O\left(4^{er/4} r^{-1/2}\right),$$

and likewise $r^K/K!$ where K is within 1 of $\frac{e}{2} r$. Let us express σ^2 as a power of n thus

$$\begin{aligned} \sigma^2 &= O\left(2^{er/2} r^{-1/2}\right) \\ &= O\left(e^{(e \log 2) r/2} r^{-1/2}\right) \\ &= O\left(n^{(e \log 2)/2} r^{-\beta}\right), \end{aligned}$$

where we obtain the latter by replacing e^r with n/r. Hence, $\beta = (1 + e \log 2)/2$. Let A be the antichain corresponding to the sequence A_j with κ chosen by Lemma 3.1. Invoking Lemma 3.2 and the well known [4]

$$B_n = \left(\sqrt{2\pi} + O(1)\right) S(n, K_n) n^{1/2} / r,$$

we find, for suitable $c > 0$,

$$\begin{aligned} |A| &\geq \frac{3}{20} \frac{1-\tau}{\sigma} B_n \\ &\geq c \frac{n^{1/2}/r}{n^{e \log 2/4} / r^{\beta/2}} S(n, K_n) \\ &> n^{1/35} S(n, K_n), \end{aligned}$$

as

$$\frac{1}{2} > \frac{1}{35} + \frac{e \log 2}{4}.$$

6 References

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