Fractional models for modeling complex linear systems under poor frequency resolution measurements

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ABSTRACT

When modeling a linear system in a parametric way, one needs to deal with (i) model structure selection, (ii) model order selection as well as (iii) an accurate fit of the model. The most popular model structure for linear systems has a rational form which reveals crucial physical information and insight due to the accessibility of poles and zeros. In the model order selection step, one needs to specify the number of poles and zeros in the model. Automated model order selectors like Akaike’s Information Criterion (AIC) and the Minimum Description Length (MDL) are popular choices. A large model order in combination with poles and zeros lying closer to each other in frequency than the frequency resolution indicates that the modeled system exhibits some fractional behavior. Classical integer order techniques cannot handle this fractional behavior due to the fact that the poles and zeros are lying close to each other to be resolvable and not enough data is available for the classical integer order identification procedure. In this paper, we study the use of fractional order poles and zeros and introduce a fully automated algorithm which (i) estimates a large integer order model, (ii) detects the fractional behavior, and (iii) identifies a fractional order system.

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1. Introduction

Modeling the transfer function of a linear system based on noisy output observations with an error-free input remains an important issue of statistical signal processing applications. One of the key model structures to obtain a parametric transfer function model is the rational form. This is due to the fact that the pole/zero configuration of the rational form provides the user some physical insight (e.g. damping of the system). Thus, for linear systems the model structure selection is often a problem but rather a matter of preference or application. Indeed, modal forms or partial fraction decompositions are popular for mechanical applications [1], analogue filter design for instance often relies on the use of pole-zero-gain models [2], improving the numerical stability and accuracy advocates the use of orthogonal polynomials and state-space models are efficient for control engineering applications [3].

Once the model structure is chosen a model order selection problem needs to be solved together with the model parameter estimation. Two popular model selection criteria are AIC [4] and MDL [5]. For a sufficiently good frequency resolution, the MDL tends to reveal the true model order with probability one given the correct choice of model structure. The AIC is known to select a model order that is larger than the true one [6]. For problems linear in the parameters, the Mallows criterion [7] is frequently used which is shown to be unbiased but not necessarily consistent for improving frequency resolutions [8].

The parameter estimation problem can be solved with various fairly-related methods: a maximum likelihood approach when the probability density function of the disturbing noise sources is known; the (nonlinear) least squares approach when the spectral density of the noise signals is assumed to be flat; or a weighted (nonlinear) least squares approach for noise signals with a colored spectral density.

It is observed in [13] that when model order selection methods suggest a large model complexity, these systems can be modeled by a fractional order system which reduces the model order significantly. This observation implies that large integer order systems may hold a fractional behavior. In this paper, we show that a group of poles and zeros which cannot be completely resolved due to the frequency resolution, can be replaced by a fractional order pole or zero. This observation bears an important consequence: if such a system is modeled by an integer order system, the poles and zeros are so close to each other that the frequency resolution is too poor to resolve them. This implies that the condition number of the regression matrices is very high which may result in an increased lack of fit. By replacing such a group of poles and zeros by a fractional order pole or zero, the frequency resolution becomes sufficiently good such that the condition number of regression matrices
is reasonable and the asymptotic theory of statistical system modeling holds.

Fractional order modeling is a fast emerging engineering field due to its enhanced frequency-domain flexibility to model the data. A general introduction can be found in [9] and [10]. Visionary publications [11] and [12] postulates fractional order models as the systems/models of the 21st century. A small selection of the literature in the two recent years gives the following overview in this emerging field. We discriminate between practical solutions and fundamental contributions.

In [13] high order integer systems were compressed into a lower order fractional system. However, the compression was only valid for systems with real poles and zeros and in the absence of noise. In [14], we extended this idea to support complex conjugate poles and zeros but the method was still not performing sufficiently well under noisy observations. Identification techniques for fractional order systems are presented in [15] and [16]. Statistical properties of the noise propagation and modeling errors in fractional order systems was investigated in [17] and [18]. The control community investigates the stability properties of fractional order systems for instance in [19–23]. Even a fractional order chaotic system was simulated and studied in [24].

These fractional systems have been applied in various fields. For instance, fractional order filters, differentiators and integrators have been designed and studied in [25–28]. Fractional order approximations have been applied in wireless communication systems [29] and [30]. Fractional dynamics have been discovered in hexapod locomotion in robotics [31]. Fractional dynamics are explored in particle physics in [32] and in graphene research in [33]. Modeling techniques for the fractional behavior in biological systems is investigated in [34] and [35]. In functional magnetic resonance imaging the study regarding fractional order properties is related to the diffusive processes in the brain, see for instance in [36–39].

The main advantages of the proposed method in this paper are three fold: (i) high order models can be replaced by a low order fractional model which enhances the frequency resolution, (ii) the method is fully automatic which detects possible fractional behavior, compresses a large integer order model to a low fractional order model and finally identifies the fractional order model parameters, and (iii) the method can be used for any kind of system fractional or integer.

2. Assumptions and preliminary example

In this section, we formalize the assumptions required to derive the theory and illustrate the problem by means of a clarifying example.

2.1. Assumptions and set-up

The modeled system \( G(s) \) is assumed to be causal, time invariant and continuous time. In the remainder of this paper, we treat the system as being stable and linear. However, the mathematics and methodology allow:

(i) Unstable systems as long as the system is captured in a stabilizing feedback loop (e.g. control applications).

(ii) Weakly nonlinear systems describable by a Volterra series. The parametric model identifies the linear dynamics of the system valid under the properties of the excitation signal.

Please note that discrete-time systems are not supported by the methodology. The system is excited by a band-limited multi-frequency signal \( u_0(t) \) with a finite root-mean-square (RMS) value. Further, we assume that the sampling frequency \( f_s \) is at least twice the highest frequency of the excitation signal \( u_0(t) \) such that no aliasing is considered. The true response of the system \( G(s) \) to the input signal \( u_0(t) \) is given by \( y_0(t) \).

Further, we allow the output signal \( y_0(t) \) to be disturbed by noise. The output noise process \( n_y(t) \) is considered to have an integrable spectral density.

2.2. Preliminary example

Consider the continuous-time transfer function,

\[
G(s) = \frac{1}{(s + 45 + 480j)^{0.75}(s + 45 - 480j)^{0.75}}
\]

where \( s \in \mathbb{C} \) is the Laplace variable and \( j \) the imaginary unit. The system is simulated with a sampling frequency of 3450 Hz. The system was excited by a band-limited white Gaussian process with a unit spectral density in the frequency band up to 1725 Hz such that aliasing errors are not present. Additional white noise in the same frequency band is added to the system's response to simulate measurement noise with a Signal-to-Noise Ratio (SNR) at the output of 80 dB. A model order selection is performed using both Akaike’s Information Criterion (AIC) and the Minimum Description Length (MDL). The probability that a specific number of poles and zeros is selected by the criteria is shown in Fig. 1 on a simulation consisting of 1000 runs. The AIC selects with highest probability (40%) 9 zeros and 13 poles. This results in 22 parameters to be estimated whereas the MDL selects with a probability of 85% 5 zeros and 11 poles resulting in 17 estimated parameters. The true transfer function only contains 4 parameters. This implies an over-modeling by a factor 5.5 for the AIC and 4 by the MDL. The average pole–zero separation in frequency is approximately 2 Hz. The frequency resolution used is 5 Hz. Using classical integer-order modeling techniques results in a poor frequency resolution estimation with the drawbacks of ill-conditioned regression matrices.

In Fig. 2, we show the estimated transfer functions for the AIC and MDL-selected model order, the Frequency Response Function (FRF) of the measurements and the difference between the FRF and the estimated transfer functions. Due to the modeling error, the difference between the FRF and the estimated model is significantly larger than the noise floor at –80 dB. The models provide...
3. Fractional order models

3.1. Fractional pole and zero models

Fractional poles and zeros are an immediate extension to Gain-Zero-Pole (GZP) models. As a result, the transfer function of the system \( G(s) \) can be expressed by its fractional GZP representation,

\[
G(s) = K \prod_{i=0}^{nB} \frac{(s - z_i)^{\beta_i}}{(s - p_i)^{\alpha_i}}
\]

where \( n_B, n_A \) the number of zeros and poles respectively, \( K \) denotes the gain, \( z_i, p_i \) are the \( i \)th zero and pole respectively with multiplicities \( \beta_i, \alpha_i \). To ensure a real valued impulse response the following constraints need to be satisfied,

\[
\begin{align*}
\bar{z}_i &= \bar{z}_{-i} \\
\bar{p}_i &= \bar{p}_{-i} \\
\alpha_i &= \alpha_{-i} \\
\beta_i &= \beta_{-i} \\
\beta_0 &= \alpha_0 = 0
\end{align*}
\]

with \( \bar{a} \) the complex conjugate of \( a \).

Clearly the fractional order representation in (1) is an immediate extension of the classical GZP representation for which the coefficients \( \alpha_i = \beta_i = 1 \) for all \( i \neq 0 \).

3.2. Physical insight

The generalization is clear but its physical implications do not immediately follow from the model (1). To understand the enrichment of the classical GZP model by allowing general multiplicities, we study the following simple example,

\[
G(s) = K \frac{(s - z)^{\beta}}{(s - p)^{\alpha}}
\]

where \( z, p \in \mathbb{R} \).

To compute the impulse response function by means of the inverse Laplace transform, we recall the definition of the fractional order derivative for a function \( f(t) \),

\[
\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{-\infty}^{t} \frac{f(u)}{(t-u)^\alpha} du
\]

where \( \Gamma(.) \) denotes the gamma function.

Note that Eq. (3) is hardly ever used in practice due to its difficulty to compute the integral. However for monomials, exponentials and trigonometric functions the definition (3) boils down to

\[
\frac{d^\alpha}{dt^\alpha} t^\beta = \begin{cases} 
\Gamma(\beta+1), & \alpha = 1 \\
\Gamma(\beta+1) t^{\beta-\alpha}, & \alpha \neq 1 
\end{cases}
\]

\[
\frac{d^\alpha}{dt^\alpha} \sin(\omega t) = \sin \left( \omega t + \frac{\pi}{2} \right)
\]

\[
\frac{d^\alpha}{dt^\alpha} e^{mt} = m^\alpha e^{mt}
\]

(4)

By virtue of Taylor’s theorem, fractional derivatives can be computed with Eqs. (4) in most engineering problems.

Now, we can compute the inverse Laplace transform of the example in (2). The impulse response becomes

\[
g(t) = \begin{cases} 
K \frac{t^{\alpha-1} e^{pt}}{\Gamma(\alpha)} & \text{for } \beta = 0 \\
K \sum_{k=0}^{\infty} \frac{\Gamma(\beta+1)}{\Gamma(\alpha-\beta+k+1) \Gamma(k+1)} (p-z)^{k} t^{\alpha-1-\beta+k} e^{pt} & \text{for } \beta \neq 0
\end{cases}
\]

The proof is found in Appendix A.

The impulse response is a product of an exponentially decaying part and a polynomial part. This observation implies that fractional order models can explain impulse responses with a slowly varying and a fast varying part. Note that such systems cannot be described by ordinary differential equations. From an analytical point of view, fractional order differential equations can, however, be arbitrarily well approximated by ordinary differential equations with a high order and this is where the statistical problem resides: high model orders.

4. Integer order model compression

In the previous section, it was shown that the impulse response of a fractional order GZP is more involved than an integer order GZP. We will show that a group of poles and zeros can be replaced by a fractional order pole and zero and construct the compression formulas. The compression formulas can be seen as extension of the compression technique in [13] and [14] such that complex conjugate poles/zeros are allowed and no user-interaction is required.

4.1. Bode plot

The main principle is to replace a group of (complex conjugate) poles and zeros by one pole or zero with fractional multiplicity such that the respective straight line Bode diagrams of the original GZP and the fractional GZP describe the same features in terms of important cut-off frequencies.

To derive the Bode plot, the transfer function \( G(\omega) \) needs to be decomposed in GZP representation such that

\[
G(\omega) = K \prod_{m=0}^{nB/2} \frac{(j\omega - z_m)^{\beta_m}}{(j\omega - p_m)^{\alpha_m}}
\]

(5)

where \( K \) denotes the system's gain, \( z_m \) the system zeros and \( p_m \) the poles of the system. Next, we write the poles and zeros in its real and imaginary parts,
The construction operates under the condition that the poles and zeros are ordered in such a way that their respective cut-off frequencies in the Bode-diagram, \( \omega_c = |z_m| \) for a zero and \( \omega_c = |p_m| \) are increasing. The idea of the approximation method is to “compress” the two poles and the zero into one pole of a fractional order. The transfer function of the fractional order system \( G_{\text{comp}}(\omega) \) satisfies the following equation,

\[
G_{\text{comp}}(\omega) = \frac{\alpha K_{\text{comp}}}{(j\omega - \hat{p}_c)\alpha (j\omega - \hat{p}_{\text{comp}})\alpha}
\]

where \( \alpha \) denotes the fractional order, \( K_{\text{comp}} \) is the compressed gain, and \( p_{\text{comp}} \) is the compressed pole.

The magnitude of (7) in dB is approximately given by,

\[
dB(|G_{\text{comp}}(\omega)|) \approx 20\log(|K_{\text{comp}}|) - 20\alpha \log((\omega - p_{\text{comp}})^2 + (p_{\text{comp}})^2) - \alpha \log(|p_{\text{comp}}|^2) + \log(\frac{\omega}{\omega_{\text{comp}}})
\]

for \( \omega \leq \omega_{\text{comp}} \)

\[
dB(|G_{\text{comp}}(\omega)|) \approx 20\log(|K_{\text{comp}}|) - 20\alpha \log((\omega - p_{\text{comp}})^2 + (p_{\text{comp}})^2) - \alpha \log(|p_{\text{comp}}|^2) + \log(\frac{\omega_{\text{comp}}}{\omega})
\]

for \( \omega > \omega_{\text{comp}} \)

An illustration is provided in Fig. 4. The blue curve is the transfer function of the simulated system. The Bode diagram is given by the red curve. The Bode diagram of the compressed system \( G_{\text{comp}}(\omega) \) is shown by the dashed green curve. The objective is to derive the compressed system \( G_{\text{comp}}(\omega) \) such that the Bode diagram passes through the middle of the staircase Bode diagram of \( G(\omega) \) as seen in Fig. 4. The idea for chosen the middles is based on a regression point of view where the fractional slope is fitted through the cut-off frequencies of the alternating integer order poles and zeros exhibiting a staircase frequency characteristic.

It is quite straightforward to verify that the third and fourth green circles are given by the coordinates,

\[
\left( \log(\sqrt{\omega_{\text{c}}(1)\omega_{\text{c}}(2)}), dB(G(0)) - 40\left( \log\left(\frac{\omega_{\text{c}}(2)}{\omega_{\text{c}}(1)}\right)\right) \right)
\]

and

\[
\left( \log(\sqrt{\omega_{\text{c}}(2)\omega_{\text{c}}(3)}), dB(G(0)) - 40\log\left(\frac{\omega_{\text{c}}(2)}{\omega_{\text{c}}(3)}\right) \right)
\]

with \( \omega_{\text{c}}(1) = p_{\text{c}} - \hat{p}_c \), \( \omega_{\text{c}}(2) = z_{\text{c}} - \hat{z}_c \), and \( \omega_{\text{c}}(3) = p_{\text{c}} - \hat{p}_c \).

The straight line through these points reveals the compression formulas,

\[
\alpha = \frac{\log(\omega_{\text{c}}(1))}{\log(\omega_{\text{c}}(3))}
\]

and

\[
\omega_{\text{comp}}^\alpha = \omega_{\text{c}}(1) \sqrt{\frac{\omega_{\text{c}}(2)}{\omega_{\text{c}}(3)}}
\]

\[
K_{\text{comp}} = G(0)\omega_{\text{c}}(1)^{\alpha + 1} \omega_{\text{c}}(1)^{-1}(2)
\]

The proof of the compression formulas is found in Appendix B. Note that the compression formulas (8) are improved with respect to the ones derived in [13] in the sense that the difference between the Bode diagram and the compressed Bode diagram is as small as possible by construction.
4.2.2. Compression of a group of poles and zeros

The main idea to compress the entire model (5) is to split (5) in groups of poles and zeros such that these satisfy an extension of Eq. (6) with

$$|p_1| \leq |z_1| \leq |p_2| \leq |z_2| \leq \cdots \leq |z_{m-1}| \leq |p_m|$$

(9)
or

$$|z_1| \leq |p_1| \leq |z_2| \leq \cdots \leq |p_{m-1}| \leq |z_m|$$

The first group is compressed in a fractional order pole; the second group of poles and zeros is compressed to a fractional order zero. As a result, the compression technique can be applied to different groups of poles and zeros satisfying the inequalities. To extend the compression formulas to this general case, we apply the notation: \(\omega_c(1) = |p_1|, \omega_c(2) = |z_1|, \omega_c(3) = |p_2|, \ldots, \omega_c(m - 1) = |p_m|\).

Hence, the compression formulas become

$$\alpha = \frac{1}{m - 1} \sum_{k=0}^{m-2} \log \left( \frac{\omega_c(2k + 1)}{\omega_c(2k)} \right)$$

$$\omega_{comp} = \frac{1}{m - 1} \sum_{k=0}^{m-2} \omega_c(2k + 1) / \omega_c(2k + 1)$$

$$K_{comp} = \frac{G(0)}{m - 1} \sum_{k=0}^{m-2} \omega_c^{2k+1} (2k + 1) \omega_c^{-1} (2k + 2)$$

(10)

The cut-off frequency \(\omega_{comp}\) in (10) does not give the actual position of the fractional order pole or zero but only its norm. We choose the angle of the fractional pole or zero equal to the closest pole or zero in the compressed group of poles and zeros.

5. Fractional order GZP identification

In the previous section, the compression formulas where derived under the assumption that the inequality (9) holds. In this section, we identify a fractional order GZP model from a measured data set. The method consists of 3 steps. The 3-step procedure and some sub-steps are depicted in Fig. 5:

Step 1. Identify an Integer Order model with a model selection criterion of choice

$$\arg\min_{\theta, A} \sum_{k=1}^{N} |Y(\omega_k)A(\omega_k) - U_0(\omega_k)B(\omega_k)|^2$$

Step 2a. Identify the roots of the polynomials \(A(\omega_k), B(\omega_k)\) as a function of their magnitudes.

Step 2b. Partition the poles and zeros in groups consisting of pole–zero–pole or zero–pole–zero sequences as a function of their magnitudes.

Step 2c. Apply the compression formulas (10) for each group. A group with a pole–zero–pole–zero–pole sequence is compressed into a fractional pole whereas a group with a zero–pole–zero–pole–zero sequence is compressed in a fractional zero.

Step 3. Join the fractional order models into one fractional order gain–pole–zero model serving as the initial guess of the fractional order model. Optimize this model by a local optimization algorithm like Levenberg–Marquardt.

Fig. 5. Algorithm flowchart.

5.1. Least squares integer order GZP identification

In a first step, we identify an integer order GZP model. We apply the AIC to obtain an integer model order which is then compressed to a low complexity fractional order in a next step. This integer order GZP model is estimated by a least squares (LS) estimator. Note that this estimator is not the most efficient one (i.e. the estimator reaches the Cramér–Rao lower bound) see [40–42] for more details. However, the integer order model is not the final objective such that we choose the least squares estimator for simplicity. The system \(G(\omega)\) is parameterized as an infinite impulse response filter such that,

$$G_{GR}(\omega) = \frac{B(\omega)}{A(\omega)} = \frac{\sum_{n=0}^{N} b_n(\omega)\omega^n}{\sum_{n=0}^{M} a_n(\omega)\omega^n}$$

with \(a_0 = 1\). Thus, we solve,

$$\arg\min_{\theta, A} \sum_{k=1}^{N} |Y(\omega_k)A(\omega_k) - U_0(\omega_k)B(\omega_k)|^2$$

(11)

where \(Y(\omega_k), U_0(\omega_k)\) are the discrete Fourier coefficients of the signals \(y(t), u_0(t)\) given by,

$$X(\omega_k) = \sum_{n=0}^{N} x(nT_s)e^{-j\omega_knT_s}$$

with \(\omega_k = \frac{2\pi}{N} f_s, T_s = 1/f_s \text{ and } x = \{y, u_0\}\).

5.2. Compression group selection

In a second step, we identify the different groups of poles and zeros satisfying the inequalities in (9) which can be compressed. We start by computing the poles and zeros from the model estimated in Section 5.1. Hence, we obtain the vector of cut-off frequencies \(\omega_k\) by computing and sorting the absolute values of the poles and zeros in an ascending order. To partition the different groups satisfying Eq. (9), we define an initial sequence of pole–zero–pole chain or a zero–pole–zero chain such that the consecutive poles and zeros have an increasing cut-off frequency. The lowest cut-off frequency among the poles and zeros specifies whether we define a pole–zero–pole chain or a zero–pole–zero chain.

This construction implies that the poles and zeros separately have an increasing cut-off frequency but the relationship (9) is not necessarily valid. The remainder of the algorithm extracts those sub-chains where the inequality (9) holds. The remaining poles and zeros which are not part of a sub-chain consisting of at least 3 elements cannot be compressed and are discarded.

Now that the different groups have been identified, the groups can be compressed accordingly to Eqs. (10). Finally, an initial estimate of the fractional order model \(G_{frac}\) is obtained,

$$G_{frac}(\omega) = K \left( \prod_{m=-M}^{M} \left( \frac{1}{(\omega - p_{frac}(m))^\alpha_m} \right) \right) G_R(\omega)$$

(12)

where \(M\) is the number of detected groups. The fractional order poles satisfy \(p_{frac}(-m) = p_{frac}(m)\) and \(\alpha_m = \alpha_m\). The multiplicities \(\alpha_m\) can be negative when a fractional order zero is detected. The transfer function \(G_R(\omega)\) captures the remaining poles and zeros which could not be compressed.

Due to the construction of the groups the most dominant features of the transfer function are captured in the fraction order part in (12). In all examples in which the compression formulas have been applied, the poles and zeros captured by \(G_R(\omega)\) did not add information to the transfer function. It is conjectured that \(G_R(\omega)\) holds the insignificant poles and zeros introduced by overfitting due to the AIC selection used in the first step (see Section 5.1).
5.3. Fractional model optimization

The final step in the identification method is the optimization of the compressed model (12). Note that this is only performed from the true fractional part of the model (12) defined as,

\[
\tilde{G}_{\text{frac}}(\omega) = K \prod_{m=M_0}^{M-1} \frac{1}{(j\omega - p_{\text{frac}}(m))^\alpha_m}
\]

Hence, a nonlinear least squares problem is solved such that the following cost function is minimized,

\[
\arg\min_{\alpha_m, \beta_m, \gamma_m, K} \sum_{k=1}^{N} \left| \frac{Y(\omega_k)}{U_0(\omega_k)} - \tilde{G}_{\text{frac}}(\omega_k) \right|^2
\]

Since, we have proper starting values, a Levenberg–Marquardt routine is used, [43]. Hence, we aim at obtaining the closest local optimum w.r.t. the initial solution (12).

6. Simulations

6.1. Fractional order pole system

In this first simulation, we use a fractional order pole system such that its continuous-time transfer function is given by

\[
G(s) = \frac{75}{(s + 45 + 480j)^{0.35}(s + 45 - 480j)^{0.35}}
\]

Thus, the system reveals a pole of multiplicity 0.35 and a complex conjugate pole pair at \( s = 45 \pm 480j \). This implies a cut-off frequency at \( f_c \approx 77 \) Hz and a resonance of approximately 5 dB. To model the system, we excite the system by a random phase multisine \( u_0(t) \) in the frequency band of 10 to 1725 Hz with steps of 5 Hz. The Root Mean Square (RMS) of the signal \( u_0(t) \) is chosen equal to 1. Hence, the signal \( u_0(t) \) is mathematically given by,

\[
u_0(t) = \frac{1}{\sqrt{2F}} \sum_{k=1}^{F} \sin(2\pi(k + 1) f_0 t + \phi_k)
\]

where \( F = 344 \) is the number of excited frequencies, \( f_0 = 5 \) Hz is the fundamental frequency, and the phases \( \phi_k \) are randomly drawn from the interval \([0, 2\pi)\). The response is sampled at a sampling frequency \( f_s = 3450 \) such that one period consists of 690 samples. The simulation is performed in the presence of Gaussian white noise at the output with different signal-to-noise settings: ranging from 140 dB to 40 dB.

6.1.1. Robustness of the model selection criterion

In Fig. 6, the 3-step identification procedure is shown for an SNR of 100 dB. The FRF is given by the black cross markers, the green solid curve is the fractional order model, the MDL and AIC selected integer order models as revealed by Fig. 1 are given by the blue curves. The error curves between the models and the FRF are given by the dashed curves. The error of the fractional order models reaches the noise floor at 100 dB both implying convergence of the algorithm to the global minimum. Although the initial MDL and AIC integer models show a small difference in performance, both converge to the global optimum. This illustrates that the procedure is robust to small changes of the initial model choice.

6.1.2. Mean squared error as a function of the SNR

In Fig. 7 the Mean Squared Error (MSE) is given for the fractional order model at the top and the AIC selected integer order model at the bottom. We see that the fractional order modeling reaches the noise level for every SNR used. The shape of the transfer function is still present for the integer order. The latter implies that significant modeling errors remain present although its large number of parameters.

6.2. Integer order system of order \((1, 2)\)

In this example, we study the method’s performance on a classical example. The transfer function of the chosen integer order
order the regression matrices (i.e. the Jacobian in the Levenberg-Marquardt algorithm) are poorly conditioned. This introduces systematic errors in the estimates leading to a larger error. The average condition number of the Jacobian matrices range from $10^4$ for a frequency resolution 5 Hz to $10^{14}$ for a frequency resolution of 30 Hz.

7. Electrochemical measurement example

The measurement deals with a biomedical engineering example in which the electrochemical impedance is modeled of a substance. The model's pole/zero configuration is in a post-processing step used to quantify the glucose concentration of the substance. Three solutions were mixed to identify their impedance spectra. Demineralized water (MiliPore MilliQ Element system), human albumin at 4 gr/dl (Baxter), a sodium-chloride solution at 350 mg/dl (VWR) and 70 mg/dl glucose.

The hardware used was the potentiostat Bank POS2 configured for three electrodes experimentation, with signals generated and recorded by a NI 4461 PCI DAQ card. The excitation signal is an odd random phase multisine such that only odd frequency lines are excited with a sine wave with a fixed amplitude but a random phase. The signal was applied by means of a voltage source by the actuator in the potentiostat at the port “Ext In”. The frequency range employed for the impedance evaluation was from 1 Hz to 30 kHz with a resolution of 1 Hz. The electrochemical cell consists of a working electrode of platinum (4 mm diameter), a counter electrode of platinum and a reference electrode of Silver (Dropsens). These sensors exhibit a high electrochemical activity and good repeatability [44]. The electrodes measure the voltage and the current. We measured the voltage to verify that the actuator was able to apply all frequencies with the specified amplitudes as given by the computer signal. All the experiments were performed at room temperature (approx. 21°C). A schematic set-up of the measurement is given in Fig. 11.

In Fig. 12, the modeling results are reported. The AIC selected a model order of 8/8 resulting in 17 parameters. The algorithm compressed this integer order model to a fraction order model with only 1 pair of complex conjugate poles and one pair of complex conjugate zeros resulting in 7 parameters. Thus, the number of parameters is reduced by a factor 2.5 without increasing the modeling error significantly. The pole revealed a multiplicity of 0.47 where the zero has a multiplicity of 0.42. Hence, the system acts as a diffusion process. The minimal separation of the integer order model order is given in red. The error curves between the models and the FRF is given by the dashed curves. Due to a sufficiently well frequency resolution (5 Hz), the different models perform similarly.

8. Conclusion

In this paper, we developed a fully automatic identification procedure to compress an integer order model into a fractional order model. The method consists of three steps: (i) an AIC selected integer order model identification, (ii) compression step to obtain an initial fractional order model and (iii) optimization of the fractional order model.

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Fig. 8. The probability that a specific model order was selected by the AIC for the example in Section 6.2.

Fig. 9. Illustration of the identification procedure for the example in Section 6.2 for an SNR of 100 dB. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

system is given by,

$$G(s) = \frac{150(s + 1571)}{(s + 45 + 480j)(s + 45 - 480j)}$$

To model the system, the same scenario as in the previous example was used. For the first step, the AIC criterion is used to select the model order of the initial integer order model exhibiting some over-modeling. The chosen order implies 6 zeros and 7 poles or 14 parameters, see Fig. 8. Hence, the AIC over-parameterizes by a factor 3.5.

6.2.1. Model performance for a good frequency resolution

In Fig. 9 the results of the different estimation procedures are shown for an SNR of 100 dB. The FRF is given by the black cross markers, the green solid curve is the fractional order model, the AIC selected integer order model is given by the blue curve and the estimated integer order model corresponding to the correct model order is given in red. The error curves between the models and the FRF is given by the dashed curves. Due to a sufficiently well frequency resolution (5 Hz), the different models perform similarly. Since the AIC selected model order is an over-parameterization the residual error is lower due to slight noise modeling.

6.2.2. Model performance for poorer frequency resolutions

In this simulation, the frequency resolution of the excitation signal was varied from 30 Hz to 5 Hz for the same sampling frequency. The model order was (6, 7) for all frequency resolutions. The Mean Squared Errors are computed over a Monte Carlo simulation of 100 runs. This change in frequency resolution implies that the number of excited lines in the band from $[0, \frac{f_s}{3}]$ varies from 173 to 680 lines. The strong law of large numbers implies that the MSE should improve by a factor 4 or 6 dB. In Fig. 10, we see that the fractional models comply with the asymptotic law whereas the integer order modeling error drops from $-75$ dB to $-100$ dB.

This is a finite sample effect since the strong law of large numbers becomes active for a sufficiently good frequency resolution. Indeed, due to the model complexity of the AIC selected model.
The methodology was tested under different simulation scenarios where a purely fractional system was used and an integer order system. Finally the algorithm was evaluated on a real electrochemical impedance measurement.

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Appendix A. The impulse response of one fractional order pole and zero

In this appendix, we compute the inverse Laplace transform of the fractional order transfer function,

\[ G(s) = K \frac{(s - z)^\beta}{(s - p)^\alpha} \]

An equivalent representation of \( G(s) \) is given by,

\[ G(s) = \mathcal{L} \left( \frac{K}{\Gamma(\alpha)} e^{\frac{\beta}{\alpha} \frac{\partial}{\partial t} (e^{(p-z)t} t^{\alpha-1})} \right) \]

This can be shown as follows,

\[ \mathcal{L} \left( \frac{K}{\Gamma(\alpha)} e^{\frac{\beta}{\alpha} \frac{\partial}{\partial t} (e^{(p-z)t} t^{\alpha-1})} \right) = K \frac{\partial^\beta}{\partial t^\beta} f(t) \]

To compute the fractional derivative of a product, we use the product rule,

\[ \frac{\partial^\beta}{\partial t^\beta} f(t) = \sum_{k=0}^{\infty} \binom{\beta}{k} f^{(\beta-k)}(t) g^{(k)}(t) \]

wherein \( f^{(\beta)}(t) = \frac{\partial^\beta}{\partial t^\beta} f(t) \). This completes the proof.

Appendix B. The compression formulas (8)

We start by constructing the straight line through the points:

\[ \left( \log(\sqrt{\omega_k(1)\omega_k(2)}), \text{dB}(G(0)) - 40\log \left( \frac{\omega_k(2)}{\omega_k(1)} \right) \right) \]

This straight is given by,

\[ \text{dB}(G(\omega)) - \text{dB}(G(0)) + 20 \log \left( \frac{\omega_k(2)}{\omega_k(1)} \right) = a \left( \log(\omega) - \log(\sqrt{\omega_k(1)\omega_k(2)}) \right) \]

where the tangent \( a \) is equal to,

\[ a = -40 \frac{\log(\frac{\omega_k(1)}{\omega_k(2)})}{\log(\frac{\omega_k(1)}{\omega_k(3)})} \]

The cut-off frequency is given by \( \omega_{\text{comp}} \) where the straight equals \( \text{dB}(G(0)) \), implying

\[ 20 \log \left( \frac{\omega_k(2)}{\omega_k(1)} \right) = a \log \left( \frac{\omega_{\text{comp}}}{\sqrt{\omega_k(1)\omega_k(2)}} \right) \]

Using the expression for \( a \) results in the following simplification

\[ \log \left( \frac{\omega_k(1)}{\omega_k(3)} \right) = 2 \log \left( \frac{\omega_{\text{comp}}}{\sqrt{\omega_k(1)\omega_k(2)}} \right) \]
Finally, we obtain
\[
\omega_{\text{comp}} = \omega_c (1 + \frac{\omega_c (2)}{\omega_c (3)} - 20 \log (\frac{\omega}{\omega_c (1)}) - 40 \alpha \log (\frac{\omega_c (2)}{\omega_c (1)})
\]

and further,
\[
dB(G(\omega)) = dB(G(0)) - 20 \log (\frac{\omega_c (2)}{\omega_c (1)})
+ 20 \alpha \log (\frac{\omega_c (1)}{\omega_c (2)}) - 40 \alpha \log (\omega_{\text{comp}})
- 40 \log (\frac{\omega}{\omega_{\text{comp}}})
\]

Hence,
\[
K_{\text{Comp}} = G(0) \omega_c (1)^{\alpha+1} \log^\alpha (2)
\]

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