Robust Helicopter Stabilization in the Face of Wind Disturbance

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Abstract—When a helicopter is required to hover with minimum deviations from a desired position without measurements of a persistent wind disturbance, a robustly stabilizing control action is vital. In this paper, the stabilization of the position and translational velocity of a nonlinear helicopter model affected by a wind disturbance is addressed. The wind disturbance is assumed to be a sum of a fixed number of sinusoids with unknown amplitudes, frequencies and phases. An estimate of the disturbance is introduced to be adapted using state measurements for control purposes. A nonlinear controller is then designed based on nonlinear adaptive output regulation and robust stabilization of a chain of integrators by a saturated feedback. Simulation results show the effectiveness of the control design in the stabilization of helicopter motion and the built-in robustness of the controller in handling parameter and model uncertainties.

I. INTRODUCTION

Autonomous helicopters are highly agile and have six degrees of freedom maneuverability making them a favourite candidate for a wide range of practical applications including agriculture, cinematography, inspection, search and rescue, reconnaissance, etc. For certain tasks the ability of a helicopter to follow a given state reference is crucial for a successful outcome; for instance when hovering over a ship for rescue operations or when flying close to power lines or wind turbines for inspections. In windy conditions this becomes a significant challenge for any pilot and hence an autopilot capable of accounting effectively for the wind disturbance is a realistic alternative. In this work, the authors present a control design for longitudinal, lateral, and vertical helicopter stabilization in the presence of a wind disturbance, with intrinsic robustness property in handling parameter and model uncertainties.

Firstly, some previous works are reviewed. In [11], a feedback-feedforward proportional differential (PD) controller is developed for heave motion control. With the assistance of a gust estimator, the controller is reported to be able to handle the influence from horizontal gusts. The effects of rotary gusts in forward and downward velocity of a helicopter is addressed in [7]. It is shown that via a state feedback law, the rotary gust rejection problem is always solvable. In a work by Wang et al., a multi-mode linear feedback law, the rotary gust rejection problem is always addressed in [7]. It is shown that via a state controller is developed for heave motion control. With the overall control inputs given by main rotor thrust $T_M$, tail rotor thrust $T_T$, longitudinal main rotor tip path plane tilt angle $a$ and lateral main rotor tip path plane tilt angle $b$, a simplified resultant external force $f^b$ in a body-fixed coordinate frame $F_b$ is taken as

$$M \ddot p^b = R f^b,$$

where $M$ is the mass and $p^b = [x \ y \ z]^\top \in \mathbb{R}^3$ is the position of the center of mass of the helicopter with respect to the origin of $F_b$. The rotation matrix $R := R(q)$ is parametrized in terms of unit quaternions $q = (q_0, q) \in S_4$ where $q_0$ and $q = [q_1 \ q_2 \ q_3]^\top$ denote the scalar and the vector parts of the quaternion respectively (see [4, Appendix A.2]). With the overall control inputs given by main rotor thrust $T_M$, tail rotor thrust $T_T$, longitudinal main rotor tip path plane tilt angle $a$ and lateral main rotor tip path plane tilt angle $b$, a simplified resultant external force $f^b$ in a body-fixed coordinate frame $F_b$ is taken as

$$f^b = \begin{bmatrix} 0 \\ 0 \\ -T_M \end{bmatrix} + R^\top \begin{bmatrix} 0 \\ 0 \\ Mg \end{bmatrix} + R^\top \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix},$$

(1)

where $d_x$, $d_y$ and $d_z$ are wind disturbances that affect the helicopter motion in $x$, $y$ and $z$ axis respectively. Subsequently the following equations of translational motion can be derived,

$$\ddot x = \frac{-(2q_1 q_3 + 2q_0 q_2)T_M}{M} + \frac{d_x}{M},$$

(2)
\[ \dot{y} = \frac{-2(2q_2q_3 - 2q_0q_1)T_M}{2g} + \frac{d_y}{M}, \]
\[ \ddot{z} = \frac{-2(1 - 2q_1^2 + 2q_2^2)T_M}{M} + g + \frac{d_z}{M}. \]

Now for the angular dynamics,
\[ J\dot{\omega}^b = -S(\omega^b)J\omega^b + \tau^b, \quad \dot{q} = \frac{1}{2} \left[ \begin{array}{c} -q^T \\ q_0 I + S(q) \end{array} \right] \omega^b, \]
where \( S(\cdot) \) is a skew symmetric matrix, \( \omega^b \in \mathbb{R}^3 \) represents the angular velocity in \( F_0 \) and \( J \) is the inertia matrix. The external torques \( \tau^b \) expressed in \( F_b \) are given by the following equation,
\[ \tau^b = \begin{bmatrix} \tau_{f_1} \\ \tau_{f_2} \\ \tau_{f_3} \end{bmatrix} = \begin{bmatrix} R_M \\ M_M + M_T \end{bmatrix}, \]
where \( \tau_{f_1} \), \( \tau_{f_2} \), \( \tau_{f_3} \) are moments generated by the main and tail rotors and \( R_M, M_M, N_M, M_T \) are moments of the aerodynamic forces [6]. With the approximations a compact torque equation is obtained,
\[ \tau^b = A(T_M)v + B(T_M), \quad v = [a \ b \ T_T]^T, \]
where \( A(T_M) \) and \( B(T_M) \) are a matrix and a vector whose entries are functions of \( T_M \) with dependence on the helicopter parameters.

One of the objectives of the controller to be designed is to handle parameter uncertainties. Taking \( \mu \) as a vector of all uncertain parameters with a nominal value \( \mu_0 \), the additive uncertainty is given by \( \mu_\Delta = (\mu - \mu_0) \in \mathcal{P} \), where \( \mathcal{P} \) is a given compact set. Correspondingly, \( M_0 = M + M_\Delta, J = J_0 + J_\Delta, A(T_M) = A_0(T_M) + A_\Delta(T_M) \) and \( B(T_M) = B_0(T_M) + B_\Delta(T_M) \).

**B. Problem Statement**

The objective of the controller is to stabilize a helicopter affected by a disturbance \( d = [d_x \ d_y \ d_z]^T \) in the presence of parameter and model uncertainties. The disturbance that affects the acceleration of the helicopter can be written as a linear combination of \( N \) (possibly \( \infty \)) sinusoidal functions of time modeled in the following form,
\[ d_j = \sum_{i=1}^{N} A_{ji} \cos(\Omega_i t + \varphi_{ji}), \]
with unknown amplitudes \( A_{ji} \), phases \( \varphi_{ji} \) and frequencies \( \Omega_i \), for \( j = x, y, z \). In our setup however, it is assumed that \( d \) can be approximated with a small \( N \). It can be shown that the disturbance is generated by the following linear time-invariant exosystem,
\[ \dot{w}_j = S(q)w_j, \]
\[ d_j = RS^2(q)w_j, \quad j = x, y, z, \]
where \( w_j \in \mathbb{R}^{2N}, q = [\Omega_1 \ \Omega_2 \ \ldots \ \Omega_N]^T, S(q) = \text{diag}(H(\Omega_1), \ldots, H(\Omega_N)) \) with
\[ H(\Omega_i) = \begin{bmatrix} 0 & \Omega_i \\ -\Omega_i & 0 \end{bmatrix}, \quad i = 1, \ldots, N, \]
and \( R = [1 \ 0 \ 1 \ 0 \ \ldots \ 1 \ 0] \ a 1 \times 2N \text{ matrix (see [4, pp. 89-90])}. \) We remark that the initial condition \( w_j(0) \) of the exosystem represents the amplitudes \( A_{ji} \) and phases \( \varphi_{ji} \) of the disturbance. In the next section, it will be clear how the representation of the disturbance in such a form can be advantageous in the development of an internal model for stabilizing control input generation.

**III. CONTROLLER DESIGN**

In this section, a controller for helicopter stabilization with wind disturbance elimination is designed in two stages. Firstly in Section III-A, a control law is constructed to achieve vertical (motion in \( z \) direction) stabilization. Next in Section III-B, it is shown that the stabilizing \( v \) can be tuned separately (without jeopardizing the vertical stabilization) to render the horizontal (\( x \) and \( y \) direction) motion stable.

**A. Stabilization of Vertical Dynamics**

With reference to the vertical dynamics (4) given above, to counter the nominal effect of the gravity, the following preliminary control law is chosen,
\[ T_M = \frac{gM_0 + u}{1 - \text{sat}_c(2q_1^2 + 2q_2^2)}, \]
where \( \text{sat}_c(s) := \text{sign}(s) \min\{|s|, c\} \) with \( 0 < c < 1 \) is a saturation function introduced to avoid singularities and \( u \) is an input to be designed. This yields
\[ \begin{align*}
\dot{z} &= -\psi_z(q)u + g(1 - \frac{M_0}{M}\psi_z(q)) + \frac{d_z}{M}, \\
\psi_z(q) &= \frac{1 - 2q_1^2 - 2q_2^2}{1 - \text{sat}_c(2q_1^2 + 2q_2^2)}. 
\end{align*} \]

If \( q \) is small enough such that \( \psi_z(q) = 1 \), then a feedforward \( u \) needed for vertical stabilization is
\[ c_u(w_z, \mu, \varphi) = gM_\Delta + d_z, \]
provided that the initial conditions are set to zero. Since the steady-state control \( c_u(w_z, \mu, \varphi) \) depends on unknown \( M_\Delta \in \mu \) and \( d_z \) which is in turn influenced by unknown \( q \) and the unmeasurable state \( w_z \), it cannot be produced directly. However, it can be asymptotically generated by the use of an internal model. As a first step in designing a controller with an internal model, the desired control (9) is rewritten as an output of a linear system (recall that \( d_z \) is given by (7))
\[ \frac{\partial \tau_z}{\partial w_z} S(q)w_z = \Phi(q)\tau_z(w_z, \mu), \]
\[ c_u(w_z, \mu, \varphi) = \Gamma(q)\tau_z(w_z, \mu), \]
where
\[ \tau_z(w_z, \mu) = \begin{bmatrix} gM_\Delta \\ \Phi(q) \end{bmatrix}, \quad \Gamma(q) = \begin{bmatrix} 0 & 1 \\ 0 & S(q) \end{bmatrix}, \quad \Phi(q) = [1 \ 0 \ 1 \ 0 \ \ldots \ 1 \ 0] \ (2N + 1) \times 1 \text{ vector}. \]

It is shown in [4, Lemma 3.3.1] that, by constructing a \((2N + 1) \times (2N + 1)\) Hurwitz matrix \( F \) and a \((2N + 1) \times 1 \text{ vector} \)

where $\tilde{\tau}(w_z, \mu, \varrho)$ is $T_0\tilde{\tau}(w_z, \mu)$. Consequently for any initial condition of exosystem (7) $w_z(0)$, the control $c_u(w_z, \mu, \varrho)$ can be viewed as an output of the linear system

$$
\dot{\xi}_z = (F + G\hat{\Psi})\xi_z + g_{st},
$$

if it is initialized as $\xi_z(0) = \tilde{\tau}(w_z(0), \mu, \varrho)$, where

$$
\xi_z = \begin{bmatrix} \xi_z1 \\ \xi_z2 \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^{2N}.
$$

Note that even if $\Psi_\theta$ and $\tilde{\tau}(w_z(0), \mu, \varrho)$ are known and hence enabling the generation of the control $c_u(w_z, \mu, \varrho)$ by (10), vertical stabilization can only be achieved if the initial conditions of (4) are set correctly (i.e., $z = \hat{z} = 0$). To deal with the unknowns in (10) and to obtain vertical stabilization given nonzero initial conditions, choose $u$ in (8) as an output of the system

$$
\dot{\xi}_z = (F + G\hat{\Psi})\xi_z + g_{st},
$$

where $u_{st} = k_2(\tilde{z} + k_1z)$ and $g_{st} = G\hat{u}_{st} + FGM\hat{z}$ for $k_1, k_2 > 0$. The vector $\Psi = [1 \tilde{\Psi}_2]$ with a $1 \times 2N$ row vector $\tilde{\Psi}_2$ is an estimate of $\Psi_\theta$ and is to be tuned by the adaptation law

$$
\dot{\tilde{\Psi}}_2 = \gamma \tilde{\Psi}_2^T(\tilde{z} + k_1z) - \text{tas}_d(\tilde{\Psi}_2),
$$

where $\gamma > 0$ is a design parameter and $\text{tas}_d(\hat{\psi}_i)$ := $\hat{\psi}_i$ - $\text{sat}_d(\hat{\psi}_i)$ is a dead zone function in which $\hat{\psi}_i$ is the $i$th component of $\hat{\Psi}_2$ and $d > \max_{i=1}^{2N} \{||\{\Psi_{\theta, i}\}||\}$. Note that the term $\text{tas}_d(\hat{\psi}_i)$ is zero as long as $|\hat{\psi}_i| \leq d$ and has a stabilizing effect on $\hat{\psi}_i$ if the fixed bound is exceeded. As demonstrated next, the inclusion of the stabilizing terms $g_{st}$ and $u_{st}$ in (11) enables the convergence of $\xi_z$ to the desired function $\tilde{\tau}(w_z, \mu, \varrho)$ and renders $(z, \tilde{z})$ globally asymptotically and locally exponentially stable under appropriate conditions.

To prove that the control defined by (8) and (11) stabilizes (4), let

$$
\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \tilde{\psi}_2^T \end{bmatrix},
$$

$$
\eta_1 = \begin{bmatrix} \chi \\ z \end{bmatrix},
$$

where

$$
\chi := \xi_z - \tilde{\tau}(w_z, \mu, \varrho) + GM\hat{z},
$$

$$
\hat{\Psi}_2 := \tilde{\Psi}_2 - \Psi_{\theta, \varrho},
$$

$$
\xi := \dot{z} + k_1z.$$

The time derivative of $\eta$ can be derived as,

$$
\dot{\eta}_1 = \left( A + A_1(\psi^*_i(q) - 1) \right) \eta_1
- \left( 1 - \frac{1}{\lambda} \eta_2 \right) \eta_2 - B\rho,
$$

$$
\dot{\eta}_2 = \gamma \xi_2 \tilde{\eta}_1 - \text{tas}_d(\eta_2 + \Psi_{\theta, \varrho}^T),
$$

where $\rho = \left( \psi^*_i(q) - 1 \right) \left( gM_\theta + \Psi_\theta \tilde{\tau}(w, \mu, \varrho) \right)$.

Using the same arguments as in [4, Proposition 3.3.2], it can be shown that there exists $k_2 > 0$ such that if $k_2 > k_2^*$, then $A$ is a Hurwitz matrix (see [1, Appendix A] for the proof). Subsequently when $q$ is small enough such that $\psi^*_i(q) = 1$, [4, Proposition 5.4.1] guarantees that if the initial state $w_z(0)$ of the exosystem belongs to the compact set $W$ defined therein, then system (15) is globally asymptotically and locally exponentially stable. Hence, $\eta$ is bounded and $\lim_{t \to \infty} \eta(t) = 0$. Note that even though system equation (15) is different than that of in [4, Eq. (5.26)] due to the addition of a disturbance and therefore a different vertical control, the two propositions still apply.

It is only natural now to ensure that the condition $\psi^*_i(q) = 1$ can be achieved in finite time. Setting

$$
v = A_0(T_M)^{-1}(\tilde{\psi} - B_0(T_M))
$$

with the assumption that $A_0(T_M(t))$ is nonsingular for all $t \geq 0$ and substituting in (6), yields the torque equation

$$
\tau^b(\tilde{\psi}) = L(T_M)\tilde{\psi} + \Delta(T_M),
$$

where $\tilde{\psi}$ is an additional control input to be determined,

$$
L(T_M) = I + A_\Delta(T_M)A_0^{-1}(T_M) \text{ and } \Delta(T_M) = B_\Delta(T_M) - A_\Delta(T_M)A_0^{-1}(T_M)B_0(T_M).
$$

Dropping the superscript $b$ in $\omega^b$, choose

$$
\tilde{\psi} = K_P(\eta_1 - K_D(\omega - \omega_d)),
$$

where $K_P, K_D > 0$ are design parameters,

$$
\eta_1 := q_r - q \text{ and } q_r = q^* + q_d.
$$

To follow the notations in [4] closely, $\eta_1$ is redefined in (19) to represent a different quantity than (14). While the expressions for the reference angular velocity $\omega_d$ and reference quaternion $q_d$ are given in the next subsection (see (28), (22), (24)), it is important to mention here that
\[\|q^*(t)\| \leq \sqrt{3} \lambda_3 \\text{ and it is assumed that } \|q_d(t)\| \leq K_d m_{q_d},\]
\[\|\omega_d(t)\| \leq K_d m_{\omega_d}, \|\dot{\omega}_d(t)\| \leq m_{\omega_d} \text{ for all } t \geq 0,\]
where \(K_d > 0, \lambda_3 > 0\) are design parameters and \(m_{q_d}, m_{\omega_d}, m_{\omega_d}\) are fixed positive numbers. Next, for the problem in hand the following proposition is stated.

**Proposition 1:** Suppose there exists \(l_1^* > 0\) such that
\[0 < 2l_1^* \leq L(T_M) + L^\top(T_M)\]
and let \(l_2^*, \delta^* > 0\) satisfy
\[\|L(T_M)\| \leq l_2^*, \|\Delta(T_M)\| \leq \delta^*.\]
Choose \(0 < \varepsilon < 1\) arbitrarily and fix compact sets \(Q, \Omega\) of initial conditions for \(q(t)\) and \(\omega(t)\) respectively, with \(Q\) contained in the set
\[\{q \in \mathbb{R}^3 : \|q\| < \sqrt{1 - \varepsilon^2}\}.\]
Then for any \(T^* > 0\), there exist \(K_p^*(K_D) > 0, K_D^* > 0, K_D^*(K_D) > 0\) and \(\lambda_3^*(K_D) > 0\), such that for all \(K_p \geq K_p^*(K_D), K_D \leq K_D^*, K_D \leq K_D^*(K_D)\) and \(\lambda_3 \geq \lambda_3^*(K_D),\)
1) the trajectory of attitude dynamical system (5) given (17) and (18) with initial conditions \((q_0(0), q(0), \omega(0)) \in (0, 1] \times Q \times \Omega\) is bounded and \(q_0(t) > 0, \forall t > 0,\)
2) \(\psi^*_3(q(t)) = 1, \forall t \geq T^*\).

**Proof:** See [1].

B. Stabilization of Longitudinal and Lateral Dynamics

It will be shown now how \(q_1\) and \(q_2\) can be manipulated to produce horizontal stability. The control law (8) that has been designed for vertical stabilization also appears in the longitudinal (2) and lateral (3) dynamics. By expanding the numerator of (8) as
\[gM_0 + u = gM + dz + y_p(\eta, w),\]
where \(\bar{\Psi} = [0 \bar{\Psi}^2]\) and
\[y_p(\eta, w) = \bar{\Psi} \tilde{r}_w(\varepsilon, \mu, \varphi) + (\bar{\Psi} + \bar{\Psi}_\varphi)(\chi - GM \tilde{z}) + k_2(\tilde{z} + k_1 \tilde{z}),\]
(2) and (3) can be rewritten as
\[\dot{x} = -\tilde{d}(q, t)q_d + m(q, t)q_1 + n_x(q)y_p(\eta, w) + \frac{dz}{M},\]
\[\dot{y} = \tilde{d}(q, t)q_1 + m(q, t)q_2 + n_y(q)y_p(\eta, w) + \frac{dy}{M},\]
where
\[\begin{align*}
\tilde{d}(q, t) &= \frac{2(gM + dz)q_0}{1 - \text{sat}_c(2q_1^2 + 2q_3^2)}, \\
m(q, t) &= \frac{2(gM + dz)}{1 - \text{sat}_c(2q_1^2 + 2q_3^2)}, \\
n_x(q) &= -(2q_1 q_3 + 2q_0 q_2) \\
n_y(q) &= -(2q_3 q_2 - 2q_0 q_1).
\end{align*}\]
Recall that from the previous analysis on vertical stabilization, with an appropriate selection of the design parameters, \(y_p(\eta, w)\) is an asymptotically diminishing signal.

Next by introducing a group of integrators \(\tilde{\eta}_d = x, \tilde{\eta}_p = y\) and \(\tilde{\eta} = q_3\), the following new state variables are defined,
\[\begin{align*}
\zeta_1 &= [\eta_p \eta_2]^\top, \\
\zeta_2 &= [y x]^\top + \lambda_1 \sigma(K_1 \lambda_1 \zeta_1), \\
\zeta_3 &= [\dot{y} \eta 0 \cdots 0]^\top + P_1 \lambda_2 \sigma(K_2 \lambda_2 \zeta_2),
\end{align*}\]
where
\[P_1 = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0
\end{bmatrix}^{(4N+2) \times 2}.\]

By adopting the following nested saturated control law
\[\begin{align*}
\bar{q}^* &= -P_2 \lambda_3 \sigma(K_3 \lambda_3 \zeta_3),
\end{align*}\]
where \(P_2\) is a matrix, \(\sigma(\cdot)\) is a vector-valued saturation function of suitable dimension (see [1]), \(K_i, \lambda_i, i = 1, 2, 3\) are design parameters, the time derivatives can be written as
\[\begin{align*}
\dot{\zeta}_1 &= -\lambda_1 \sigma(K_1 \lambda_1 \zeta_1) + \zeta_2, \\
\dot{\zeta}_2 &= -\lambda_2 \sigma(K_2 \lambda_2 \zeta_2) + P_0 \zeta_3 + K_1 \sigma'(K_1 \lambda_1 \zeta_1) \dot{\zeta}_1, \\
M \dot{\zeta}_3 &= -\tilde{D}(t)P_2 \lambda_3 \sigma(K_3 \lambda_3 \zeta_3) + \tilde{D}(t)q_d, \\
&\quad + MK_2 P_1 \sigma'(K_2 \lambda_2 \zeta_2) \dot{\zeta}_2 - \tilde{D}(t)\hat{\eta}_1 + p + d_h,
\end{align*}\]
in which
\[\begin{align*}
D_1(t) &= \begin{bmatrix}
\tilde{d}(q, t) & m(q, t)q_3 & 0 \\
0 & -\tilde{d}(q, t) & 0 \\
0 & 0 & M
\end{bmatrix}, \\
\tilde{D}(t) &= \begin{bmatrix}
\bar{D}(t) & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}^{(4N+2) \times (4N+2)}, \\
\begin{bmatrix}
n_{\eta_1}(q)q_0(q, w) \\
n_{\eta_2}(q)q_0(q, w)
\end{bmatrix}, \\
p &= \begin{bmatrix}
d_y & d_x \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}, \\
\tilde{\eta}_1 &= \begin{bmatrix}
0 \\
\cdots \\
0 \\
\cdots \\
0
\end{bmatrix}, \\
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0
\end{bmatrix}^{(2 \times (4N+2))}.
\end{align*}\]

In a generic notation, for \(\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n, \sigma'(v) = d\sigma(v)/dv\), where \(\nu \in \mathbb{R}^n\). Note that if one can set \(q_d = -D_0(t)d_h\), where
\[D_0(t) = \begin{bmatrix}
D_0^{-1}(t) & 0 \\
0 & 0
\end{bmatrix},\]
the disturbance \(d_h\) can be completely eliminated from subsystem (23). In this case, (23) can be shown to be input-to-state stable (ISS) with restrictions on the inputs \((\tilde{\eta}_1, p)\) and linear asymptotic gains [4, Lemma 5.7.4].

Since \(q_d\) is a function of uncertain \(M\) and unknown \(d_h\), it cannot be generated directly. Thus the following is proposed,
where \( \hat{d} = [\hat{d}_y \hat{d}_x 0 \cdots 0]^T \) is a disturbance estimate to be adapted and

\[
\begin{align*}
\hat{D}_1(t) &= \begin{bmatrix} \hat{d}_0(q, t) & m_0(q, t) \phi & 0 \\ 0 & -\hat{d}_0(q, t) \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
\hat{D}_0(t) &= \begin{bmatrix} \hat{D}_1(t) \phi & 0 \\ 0 & 0 \end{bmatrix}, \\
\end{align*}
\]

\[ (4N+2) \times (4N+2) \]

(25)

with \( u \) given by (11) (see [1, Appendix C] for the expression of \( d \)). It is important to notice at this point that a constraint on \( d_z \) should be imposed. Because \( \lim_{t \to -\infty} (gM_0 + u(t)) = gM + d_z \), it is required that \( d_z(t) \neq -gM \) for all \( t \geq 0 \) to avoid singularities in (25). In this paper we assume that \( |d_z(t)| < gM \) for all \( t \geq 0 \). As a result, \( T_M > 0 \) for all \( t \geq 0 \). Next, the following state variable are defined,

\[
\begin{align*}
\eta_2 &= \tilde{\omega} - \omega_d - \frac{1}{K_D} \tilde{\eta}_1, \\
\eta_3 &= \tilde{J} \eta_2 - K_d \tilde{e}_\xi, \\
\end{align*}
\]

(26)

(27)

where \( e_\xi \) is the internal model error, and \( \tilde{\omega} \) and \( \tilde{J} \) is a vector and a matrix respectively (see [1]). Note that in (26), \( \eta_2 \) is redefined to represent a different variable than that of in (13). From (24), we may write \( q_d = [q_d q_{d2} 0 \cdots 0]^T \) and the desired angular velocity is defined as

\[
\omega_d = Q_d q_d.
\]

(28)

Taking the time derivatives,

\[
\begin{align*}
\dot{\eta}_1 &= -\frac{1}{2} (q_0 I + \tilde{S}(q_i)) (\eta_2 + \frac{1}{K_D} \tilde{\eta}_1 + \omega_d) + \dot{q}_r, \\
\dot{\eta}_2 &= -\tilde{S}(\omega) \tilde{J} (\eta_2 + \frac{1}{K_D} \tilde{\eta}_1 + \omega_d) - K_P K_D \tilde{L}(T_M) \eta_2 \\
&\quad + \Delta(T_M) - \tilde{J} \omega_d - \frac{1}{K_D} \tilde{\eta}_1, \\
\dot{\eta}_3 &= \tilde{J} \eta_2 - K_d \tilde{e}_\xi.
\end{align*}
\]

(29)

See [1, Appendix C] for a complete expression of (28) and (29).

We will now study the stability of feedback interconnection of subsystems (23) and (29). Subsystem (23) is a system with state \((\zeta_1, \zeta_2, \zeta_3)\) and input \((\dot{\eta}_1, \eta_2, \eta_3, p_0)\), while subsystem (29) has \((\tilde{\eta}_1, \eta_2, \eta_3)\) and \((y_{c0}, y_{c1}, I_{\tilde{\eta}_1}, I_{\eta_2}, I_{\eta_3})\) as its state and input respectively (see [1, Appendix C]). Since the vertical stability analysis in Section III.A guarantees that \( q_0(t) > \epsilon > 0 \) for all \( t \geq 0 \) for an allowed range of the design parameters and initial conditions, it is assumed so in the next proposition. Moreover, let \( M^L, M^U, d^L \) and \( d^U \) be such that \( M^L \leq M \leq M^U \) and \( 0 < d^L \leq \hat{d}(q, t) \leq d^U \) for all \( t \geq 0 \). With that, the following proposition is presented.

**Proposition 2:** Let \( K_P \) be fixed and let \( K_i^* \) and \( \lambda_i^*, i = 1, 2, 3 \), be such that the following inequalities are satisfied

\[
\begin{align*}
\frac{\lambda_2^*}{K_2^*} < \frac{\lambda_3^*}{K_3^*} < \frac{\lambda_2^*}{K_2^*} < \frac{\lambda_3^*}{K_3^*} < \frac{\lambda_2^*}{K_2^*}, \\
4K_2^* \lambda_2^* < \frac{d^L}{M^U} \frac{\lambda_3^*}{K_3^*} < \frac{d^L}{M^U} \frac{\lambda_2^*}{K_2^*} < \frac{d^L}{M^U} \frac{\lambda_3^*}{K_3^*} < \frac{d^L}{M^U} \frac{\lambda_2^*}{K_2^*},
\end{align*}
\]

and

\[
\frac{24K_2^* \lambda_2^*}{K_3^*} < \frac{1}{6} \frac{d^L}{M^U} \frac{\lambda_3^*}{K_3^*}.
\]

Then, there exist positive numbers \( \gamma_1^*, \gamma_2^*, \epsilon_1^*, \epsilon_2^*, \epsilon_3^*, R_1, R_2, R_3 \) such that, taking \( \lambda_i = \epsilon_i^{-1} \lambda_i^* \) and \( K_i = \epsilon K_i^* \), \( i = 1, 2, 3 \), for all \( K_P \geq K_1^* \), \( K_d \leq K_3^* \) and \( \epsilon_i^* \leq \epsilon \leq \epsilon_i^* \), the feedback interconnection of subsystems (23) and (29) is ISS

1) without restrictions on the initial state;
2) with restrictions \((\hat{x}(x, y, \tilde{x}(t), \tilde{y}(t)) \neq 0 \) on input \((p_0, I_{\tilde{\eta}_1}, I_{\eta_2}, I_{\eta_3}), \) where \( R_{\eta_1}, R_{\eta_2} \) and \( R_{\eta_3} \) are arbitrary positive numbers;
3) with linear asymptotic gains.

Therefore, if \( \|p_0\|_{\infty} < \epsilon^2 R_1, \|I_{\tilde{\eta}_1}\|_{\infty} < R_1, \|I_{\eta_2}\|_{\infty} < R_2 \) and \( \|I_{\eta_3}\|_{\infty} < R_3 \), then \( (\zeta_1(t), \zeta_2(t), \zeta_3(t), \eta_1(t), \eta_2(t), \eta_3(t)) \) satisfies the asymptotic bound

\[
\|((\zeta_1, \zeta_2, \zeta_3, \eta_1, \eta_2, \eta_3))\|_{a} \leq \max \{\gamma_1^* \|p_0\|_{a}, K_D \gamma_{\eta_1} \|I_{\tilde{\eta}_1}\|_{a} \gamma_2^* \|I_{\eta_2}\|_{a}, \gamma_3^* \|I_{\eta_3}\|_{a} \},
\]

where \( \| \cdot \|_{\infty} \) and \( \| \cdot \|_{a} \) denote the \( \mathcal{L}_{\infty} \) norm and asymptotic \( \mathcal{L}_{\infty} \) norm respectively [9].

**Proof:** The proof of Proposition 2 involves showing that subsystems (23) and (29) are ISS separately and that the composed system satisfies the small gain theorem (see [1]).

Note that by choosing large enough \( K_P \) and sufficiently small \( \lambda_3 \) and \( K_d \), requirements for vertical, longitudinal and lateral dynamics stabilization as dictated by Proposition 1 and 2 can be simultaneously satisfied. To conclude the control design, consider the controller given by (8), (11), (16), (18), (19), (20), (28), (22) and (24). Choose the design parameters according to Proposition 1 and 2. Then for any initial conditions \( w(0) \in \mathcal{W}, \eta(0) \in \mathcal{Z}, (x(0), \dot{x}(0), y(0), \dot{y}(0)) \in \mathbb{R}^4, q_0(0) > 0, (q_0(t), \omega(t)) \in \mathcal{Q} \times \Omega \) where \( \mathcal{Z} \) is an arbitrary compact set, \((x(t), \dot{x}(t), y(t), \dot{y}(t))\) converges to a neighborhood of the origin which can be rendered arbitrarily small by choosing \( K_P \) and \( K_D \) sufficiently large and small respectively. In addition,

\[
\lim_{t \to \infty} \|z(t), \dot{z}(t)\| = 0.
\]

**IV. SIMULATION RESULTS**

Hover flight of an autonomous helicopter equipped with the proposed autopilot and influenced by a wind disturbance is simulated. The simulation results presented here are based on a model of a small autonomous helicopter from [8]. To test the robustness property of the controller, parameter
to stabilize the $x$ and $y$ positions, $z$ does converge fairly close to zero as could be seen in Fig. 2. Apparently, $T_{Ma}$ is still capable of acting as a vertical stabilizer to a certain degree although the disturbance adaptation is turned off due to the presence of other terms in (8). The importance of information on the disturbance to the longitudinal/lateral stabilizer is demonstrated in Fig. 3. Now that the disturbance adaptation is turned on, $z$ converges to zero and, $x$ and $y$ converge to a small neighbourhood of the origin as guaranteed by Proposition 2.

V. CONCLUSIONS AND FUTURE WORKS

A robust controller for helicopter stabilization to reject wind disturbance is presented. The wind disturbance affecting the helicopter is assumed to be a function of time of a fixed structure with unknown parameters. By designing an internal model that estimates the disturbance, a control design is carried out for longitudinal, lateral and vertical dynamics stabilization. Despite the presence of helicopter parameter and model uncertainties, simulation results clearly demonstrates the effectiveness of the control technique. As future works, indoor and outdoor flights are to be carried out to test the feasibility of the proposed controller. That gives an immediate challenge caused by the presence of servo dynamics and limitations on wind disturbance that could be handled.

REFERENCES