On Max-SINR Receiver for Hexagonal Multicarrier Transmission Over Doubly Dispersive Channel

Kui Xu, Youyun Xu, Xiaochen Xia, Dongmei Zhang
Institute of Communications Engineering
PLA University of Science and Technology
Nanjing 210007, P. R. China.
Email: lgdxxukui@126.com, yyxu@vip.sina.com

Abstract—In this paper, a novel receiver for Hexagonal Multicarrier Transmission (HMT) system based on the maximizing Signal-to-Interference-plus-Noise Ratio (Max-SINR) criterion is proposed. Theoretical analysis shows that the prototype pulse of the proposed Max-SINR receiver should adapt to the root mean square (RMS) delay spread of the doubly dispersive (DD) channel with exponential power delay profile and U-shape Doppler spectrum. Simulation results show that the proposed Max-SINR receiver outperforms traditional projection scheme and obtains an approximation to the theoretical upper bound SINR performance within the full range of channel spread factor. Meanwhile, the SINR performance of the proposed prototype pulse is robust to the estimation error between the estimated value and the real value of time delay spread.

I. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM) systems with guard-time interval or cyclic prefix can prevent inter-symbol interference (ISI). OFDM has overlapping spectra and rectangular impulse responses. Consequently, each OFDM sub-channel exhibits a sinc-shape frequency response. Therefore, the time variations of the channel during one OFDM symbol duration destroys the orthogonality of different subcarriers, and results in power leakage among subcarriers, known as inter-carrier interference (ICI), which causes degradation in system performance. In order to overcome the above drawbacks of OFDM system, several pulse-shaping OFDM systems were proposed [1]–[4].

It is shown that signal transmission through a rectangular lattice is suboptimal for doubly dispersive (DD) channel [5]–[7]. By using results from sphere covering theory, the authors have demonstrated that lattice OFDM (LOFDM) system, which is OFDM system based on hexagonal-type lattice, providing better performance against ISI/ICI [5]. However, LOFDM confines the transmission pulses to an orthogonal set. As pointed out in [2], these orthogonalized pulses destroy the time-frequency (T-F) concentration of the initial pulses, hence lower the robustness to the time and frequency dispersion caused by the DD propagation channel.

In [6]–[8], the authors abandoned the orthogonality condition of the modulated pulses and proposed a multicarrier transmission scheme on hexagonal T-F lattice named as hexagonal multicarrier transmission (HMT). To optimally combat the impact of the DD propagation channels, the lattice parameters and the pulse shape of modulation waveform are jointly optimized to adapt to the channel scattering function from a minimum energy perturbation point of view. It is shown that the hexagonal multicarrier transmission systems obtain lower energy perturbation by incorporation the best T-F localized Gaussian pulses as the elementary modulation waveform, hence outperform OFDM and LOFDM systems from the robustness against channel dispersion point of view [9]–[11].

In HMT system, there is no cyclic prefix and the data symbols of HMT signal are transmitted at the hexagonal lattice point of T-F plane and subcarriers are interleaved in the T-F space, as is illustrated in Fig. 1. The basic mathematical operation on the received signal performed by the demodulator is a projection onto an identically structured function set generated by the prototype pulse function [10]–[12], i.e., an optimal match filter. It is shown in [13]–[15] that the optimum sampling time of wireless communication systems over DD channel depends on the power distribution of the channel profiles, and that zero timing offset does not always yield the best system performance. Max-SINR ISI/ICI-Shaping receiver for multicarrier modulation system is discussed in [2], the Max-SINR prototype pulse can be obtained by maximizing the generalized Rayleigh quotient.

We will present in this paper that receivers proposed in [6]–[8] are suboptimal approaches in the view of Signal-to-Interference-plus-Noise Ratio (SINR) performance. A novel receiver based on maximizing SINR (Max-SINR) criterion for HMT system is proposed in this paper. Theoretical analyses and simulation results show that the proposed optimal Max-SINR receiver outperforms the traditional projection receiver in [6]–[8] and obtains an approximation to the theoretical upper bound SINR performance. Meanwhile, the proposed scheme is robust to the estimation error between the estimated value and the real value of root mean square (RMS).

II. HEXAGONAL MULTICARRIER TRANSMISSION SYSTEM

In the view of signal transmission on lattice in the T-F plane, the system performance is mainly determined by two factors: a) energy concentration of the elementary modulation pulse, a better T-F concentrated pulse would lead to more robustness against the energy leakage and b) distance between the transmitted symbols in the T-F plane, it is obvious that the larger the distance, the less the perturbation among the...
transmitted symbols. As pointed out in [6]–[8], for a given signaling efficiency, the information-bearing pulses arranged on a hexagonal T-F lattice, as is illustrated in Fig. 1, can be separated as sufficiently as possible in the T-F plane.

In HMT systems, the transmitted baseband signal can be expressed as [6]

\[ x(t) = \sum_{m} \sum_{n} c_{m,n} g(t - mT) e^{j2\pi nFt} + \sum_{m} \sum_{n} c_{m,n+1} g(t - mT - \frac{T}{2}) e^{j2\pi (nF + \frac{\delta}{2})t} \]  

(1)

where \( T \) and \( F \) are the lattice parameters, which can be viewed as the symbol period and the subcarrier separation, respectively; \( c_{m,n} \) is the user data, which is assumed to be taken from a specific signal constellation and independent and identically distributed (i.i.d) with zero mean and average power \( \sigma_c^2 \); \( m \in \mathcal{M} \) and \( n \in \mathcal{N} \) are the position index in the T-F plane; \( \mathcal{M} \) and \( \mathcal{N} \) denote the sets from which \( m, n \) can be taken, with cardinalities \( \mathcal{M} \) and \( \mathcal{N} \), respectively. The prototype pulse \( g(t) \) is the Gaussian window

\[ g(t) = (2/\sigma)^{1/4} e^{-(\pi/\sigma)t^2} \]  

(2)

with \( \sigma \) being a parameter controlling the energy distribution in the time and frequency directions. The ambiguity function of Gaussian pulse is defined by

\[ A_g(\tau, v) = \int_{-\infty}^{\infty} g(t)g^*(t - \tau - v)e^{-j2\pi vt}dt = e^{-\frac{\delta}{4}(\pi^2 + v^2)}e^{-j\pi \tau v} \]  

(3)

where \( (\cdot)^* \) denotes the complex conjugate.

It is shown in Fig. 1 that the original hexagonal lattice can be expressed as the disjoint union of a rectangular sublattice \( \mathcal{V}_{\text{rect1}} \) and its coset \( \mathcal{V}_{\text{rect2}} \). The transmitted baseband signal in (1) can be rewritten as

\[ x(t) = \sum_{i=1}^{2} \sum_{m} \sum_{n} c_{m,n}^i g_{m,n}^i(t) \]  

(4)

where \( i \in \{1,2\} \), \( c_{m,n}^1 \) and \( c_{m,n}^2 \) represent the symbols coming from two sets \( \mathcal{V}_{\text{rect1}} \) and \( \mathcal{V}_{\text{rect2}} \), respectively. \( g_{m,n}^i(t) = g(t - mT - \frac{T}{2}) e^{j2\pi (nF + \frac{\delta}{2})t} \) is the transmitted pulse on \( m \)-th symbol and \( n \)-th subcarrier.

The baseband DD channel can be modeled as a random linear operator \( H \) [16]

\[ H[x(t)] = \int_{-\tau_{\text{max}}}^{\tau_{\text{max}}} \int_{-f_d}^{f_d} H(\tau, v)x(t - \tau)e^{j2\pi vt}d\tau dv \]  

(5)

where \( \tau_{\text{max}} \) and \( f_d \) are the maximum multipath delay spread and the maximum Doppler frequency, respectively [17]. \( H(\tau, v) \) is called the delay-Doppler spread function, which is the Fourier transform of the time-varying channel impulse response \( h(t, \tau) \) with respect to \( t \). The product \( \theta = \tau_{\text{max}}f_d \) is referred to as the channel spread factor (CSF). If \( \theta < 1 \), the channel is said to be underspread; otherwise, it is overspread. Practical wireless channels usually satisfy the assumption of wide-sense stationary uncorrelated scattering (WSSUS), and \( \theta \ll 1 \) [16].

In the WSSUS assumption the channel is characterized by the second order statistics

\[ E[H(\tau, v)H^*(\tau_1, v_1)] = S_H(\tau, v)\delta(\tau - \tau_1)\delta(v - v_1) \]  

(6)

where \( E[\cdot] \) denotes the expectation and \( S_H(\tau, v) \) is called the scattering function, which characterizes the statistics of the WSSUS channel. Without loss of generality, we use \( \int_{-\tau_{\text{max}}}^{\tau_{\text{max}}} \int_{-f_d}^{f_d} S_H(\tau, v)d\tau dv = 1 \).

As shown in (5), the propagation channel introduces energy perturbation among the transmitted symbols. It is shown in [5]–[7] that the symbol energy perturbation function is dependent on the channel scattering function and the pulse shape. The properly designed HMT system with Gaussian prototype pulse achieves minimum symbol energy perturbation over DD channel. The optimality of such system in combating the ISI/ICI caused by the DD channel is guaranteed by the Heisenberg uncertainty principle and the sphere-packing theory. The choice of the system parameters \( (\sigma, T, F) \) to minimize such undesired energy perturbation is extensively dealt with in [5]–[7]. In general, the pulse shape of the Gaussian window \( \sigma \) and the transmission pattern parameters \( T \) and \( F \) should be matched to the channel scattering function. The optimal system parameter for DD channels with exponential-U scattering function can be chosen as [6] \footnote{In the case of no time and frequency dispersion, \( S_H(\tau, v) = \delta(\tau)\delta(v) \), the matching criteriaes (7) and (8) are also reduced to \( (\sqrt{3}T/F) = (W_{1F})/(W_{1F}) \) and \( T/(\sqrt{3}F) = (W_{1F})/(W_{1F}) \), respectively. \( W_{1F}^2 \) and \( W_{1F}^2 \) are the centralized temporal and spectral second-order moments, respectively [17].}

\[ \sigma = \alpha \frac{\tau_{\text{max}}}{f_d} = \sqrt{3} \frac{T}{F} \]  

(7)
and
\[ \sigma = \alpha \frac{\tau_{\text{rms}}}{T} = \frac{1}{\sqrt{3}} \frac{T}{F} \]
where \( \tau_{\text{rms}} \) is the RMS delay of the DD channel and the coefficient \( \alpha \) for various signaling efficiency \( \rho \) is listed in Table I. The received signal can be expressed as
\[ r(t) = H[x(t)] + w(t) \]  
where \( w(t) \) is the AWGN with variance \( \sigma_w^2 \).

III. MAX-SINR RECEIVER FOR HMT SYSTEM

The basic mathematical operation of the received signal performed by the demodulator is a projection onto an identically structured function set generated by the prototype pulse function, i.e. an optimal match filter [7], [11]. To obtain the data symbol \( c_{m,n} \), the match filter receiver projects the received signal \( r(t) \) on the prototype pulse function set \( \psi_{m,n}^i(t) \), i=1, 2, i.e.,
\[ \hat{c}_{m,n}^i = \langle r(t), \psi_{m,n}^i(t) \rangle = \sum_j \sum_{m',n'} c_{m',n'}^j \langle H[g_{m',n'}(t)], \psi_{m,n}^i(t) \rangle + \langle w(t), \psi_{m,n}^i(t) \rangle \]
where \( \psi_{m,n}^i(t) = \psi(t - mT - \frac{l}{2}T)e^{j2\pi(nF + \frac{lf}{2})t} \), and \( \psi(t) \) is the prototype pulse at the receiver. The energy of the received symbol, after projection on the prototype pulse set \( \psi_{m,n}^i(t) \) can be expressed as
\[ E_s = E \left\{ \left| \sum_{m',n'} c_{m',n'}^j \langle H[g_{m',n'}(t)], \psi_{m,n}^i(t) \rangle + \langle w(t), \psi_{m,n}^i(t) \rangle \right|^2 \right\} \]
Under the assumptions of WSSUS channel and source symbols are statistically independent, (11) can be rewritten as
\[ E_s = \sigma_e^2 \int_\tau \int_v S_H(\tau, v) 
\cdot \left[ \sum_{m,n} \left( |A_{g,\psi}(mT + \tau, nF + v)|^2 \right. \right.
+ \left. |A_{g,\psi}(mT + T/2 + \tau, nF + F/2 + v)|^2 \right] d\tau dv 
+ \sigma_w^2 |A_{g,\psi}(0,0)| \]
where the interference-plus-noise energy
\[ E_{\text{IN}} = \sigma_e^2 \int_\tau \int_v S_H(\tau, v) 
\cdot \left[ \sum_{z=[m,n],T \neq (0,0)^T} \left( |A_{g,\psi}(mT + \tau, nF + v)|^2 
+ |A_{g,\psi}(mT + T/2 + \tau, nF + F/2 + v)|^2 \right) d\tau dv 
+ \sigma_w^2 |A_{g,\psi}(0,0)| \right] \]
Clearly, the interference-plus-noise energy function \( R_{\text{IN}} \) is dependent on the channel scattering function and the pulse shape (through its ambiguity function). According to the form of channel scattering functions, the Max-SINR receiver can be discussed in two cases [6]: Case A: DD channel with exponential power delay profile and U-shape Doppler spectrum. Case B: DD channel with uniform power delay profile and uniform Doppler spectrum. Due to the space limitations, this paper only to discuss the Case A, that is the Max-SINR receiver for HMT system over DD channel with exponential power delay profile and U-shape Doppler spectrum.

For the DD channel with exponential power delay profile and U-shape Doppler spectrum, the scattering function can be expressed as [18]
\[ S_H(\tau, v) = \frac{e^{-\frac{\tau^2}{\alpha^2}}}{\pi \tau_{\text{rms}} f_d \sqrt{1 - (v/f_d)^2}} \]
with \( \tau > 0, |v| < f_d \). We assume that \( \psi(t) = g(t-\Delta t) \), the theoretical SINR of the received signal over the DD channel with exponential power delay profile and U-shape Doppler spectrum can be expressed as
\[ R_{\text{SIN}} = \frac{\sigma_e^2}{\pi \tau_{\text{rms}} f_d E_{\text{IN}}} \int_0^{\infty} e^{-\frac{\tau^2}{\alpha^2}} e^{-\frac{1}{2}((\tau - \Delta t))^2} d\tau 
\cdot \int_{-f_d}^{f_d} e^{-\sigma \pi v^2} dv \]
Substituting (15) into (14), the interference-plus-noise energy

\[ \text{SINR} = \frac{\sigma_e^2}{E_{\text{IN}}} \int_\tau \int_v S_H(\tau, v) |A_{g,\psi}(\tau, v)|^2 d\tau dv \]

2It is shown in [13]–[15] that the optimum sampling time of wireless communication systems over DD channel depends on the power distribution of the channel profiles, and that zero timing offset does not always yield the best system performance. In other words, there is a timing offset between the prototype pulses at the transmitter and the receiver. In [2], the Max-SINR prototype pulse \( g \) of multicarrier transmission system with rectangular T-F lattice over DD channel is obtained by maximizing the generalized Rayleigh quotient \( g = \arg \max_{\phi} \frac{\phi^T R \phi}{\phi^T R \phi} \). The solution is the generalized eigenvector of the matrix pair \( (B, A) \) corresponding to the largest generalized eigenvalue.

It is shown that there is a delay between the transmitted Gaussian prototype pulse and the received Gaussian prototype pulse. In this paper, the close form time offset expressions between the transmitted and received prototype pulse is derived.

### TABLE I

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>0.5</th>
<th>1.0</th>
<th>2.0</th>
<th>4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>2.25</td>
<td>2.00</td>
<td>1.90</td>
<td>1.85</td>
</tr>
</tbody>
</table>
function $E_{IN}$ in (16) can be expressed as

$$
E_{IN} = \frac{\sigma_c^2}{\pi \tau_{rms} f_d} \left\{ \sum_{(m,n)\neq(0,0)} \int_{0}^{\infty} e^{-\frac{\tau}{\tau_{rms}}} e^{-\frac{\pi(T + \tau - \Delta\tau)^2}{\sigma^2 \tau_{rms}}} d\tau 
- \int_{-f_d}^{f_d} \frac{e^{-\sigma \pi (nF + v)^2}}{\sqrt{1 - (v/f_d)^2}} dv 
+ \sum_{(m,n)\neq(0,0)} \int_{0}^{\infty} e^{-\frac{\tau}{\tau_{rms}}} e^{-\frac{\pi(m + \frac{\tau}{\sigma})^2}{\sigma^2 \tau_{rms}}} d\tau 
- \int_{-f_d}^{f_d} \frac{e^{-\sigma \pi (nF + \frac{\tau}{\sigma} + v)^2}}{\sqrt{1 - (v/f_d)^2}} dv \right\} + \sigma_w^2 |A_\vartheta,\varphi(0,0)|
$$

(17)

The theoretical SINR upper bound of the received signal can be expressed as

$$
R_{UB} = \arg \max_{\Delta t} R_{SINR}
$$

(18)

Plugging (16) and (17) in (18), the Max-SINR prototype pulse can be expressed as $\psi(t) = g(t - \Delta t)$ and (see Appendix)

$$
\Delta t = \frac{\sigma}{2\pi \tau_{rms}} \left( \frac{3.28\sqrt{\sigma}}{\tau_{rms}} - \sqrt{\frac{3.28^2 \sigma}{\tau_{rms}^2} - 3.52\left( \frac{\sigma}{\tau_{rms}} - 1 \right)} \right)
$$

(19)

We can see from equation (19) that the prototype pulse of the proposed Max-SINR receiver is a function of RMS. In HMT system, the prototype pulse shape of the Gaussian window and the transmission pattern parameters $T$ and $F$ should be matched to the channel scattering function including RMS. In other word, the RMS delay spread and the maximum Doppler frequency are known at the HMT transceiver. Hence, it is reasonable to assume that the RMS delay spread priori known in HMT system.

### IV. Simulation and Discussion

In this section, we test the proposed Max-SINR receiver via computer simulations based on the discrete signal model. In the following simulations, the number of subcarriers for HMT system is chosen to $N=40$, and the length of prototype pulse $N_p=600$. The center carrier frequency is $f_c=5$GHz and the sampling interval is set to $T_s=10^{-6}$s. The system parameters of HMT system are $F=25$kHz, $T=1 \times 10^{-4}$s and $\sigma$ is set to $\sigma=7/3F$. Traditional projection receiver proposed in [6]–[8] is named as Traditional Projection Receiver (TPR) in the following simulation results. WSSUS channel is chosen as DD channel with exponential power delay profile and U-shape Doppler spectrum.

The prototype pulses of Max-SINR receiver at $\vartheta=0.1$ and $\vartheta=0.04$ are given in Fig. 2. The prototype pulse of TPR scheme is also depicted for comparison. We can see from Fig. 2 that there is a delay between the TPR prototype pulse and the Max-SINR prototype pulse. The delay is obtained according to equation (19), and increases with the increasing of CSF $\vartheta$.

The SINR performance of different receivers with the variety of $\sigma_c^2/\sigma_w^2$ for HMT system over DD channel is depicted in Fig. 3. The CSFs are set to $\vartheta=0.07$ and $\vartheta=0.2$, respectively. We can see from Fig. 3 that the SINR performance of the proposed Max-SINR receiver outperforms TPR scheme about 1~4dB at $\vartheta=0.07$ and 1.5~3.5dB at $\vartheta=0.2$, respectively. The SINR gap between the proposed Max-SINR receiver and the theoretical SINR upper bound is smaller than 0.5dB and 1.0dB at $\vartheta=0.07$, 0.2, respectively.

The SINR performance of the proposed Max-SINR receiver with the variety of $\sigma_c^2/\sigma_w^2$ is given in Fig. 4. We assume that there is an estimation error between the estimated value of RMS delay spread $\tau_{est}$ and the real value $\tau_{rms}$. We can conclude from Fig. 4 that the proposed Max-SINR receiver is robust to the estimation error of RMS delay spread.

The SINR performance with the variety of channel spread factors $\vartheta$ at $\sigma_c^2/\sigma_w^2=20$dB is depicted in Fig. 5. The SINR performance of the TPR scheme is depicted for comparison. It can be seen that there is a degradation of SINR with the increasing of channel spread factor. The proposed Max-SINR receiver obtains an approximation to the theoretical upper bound SINR performance within the full range of $\vartheta$. There is an about 2.5dB maximum SINR gap between the Max-SINR receiver and the TPR scheme at $\vartheta=0.35$, and the SINR gap decreases as the CSF $\vartheta$ decreases.

### V. Conclusion

The Max-SINR receiver is proposed for HMT system over DD channel in this paper. After a detailed analysis, we present that the prototype pulse of the Max-SINR receiver should adapt to the RMS delay spread. Theoretical analysis shows that the proposed Max-SINR receiver outperforms the TPR scheme in SINR and obtains an approximation to the theoretical upper bound SINR performance within the full range of CSF.
Simulation results show that the proposed scheme is robust to the estimation error of RMS and is suitable for the actual HMT system.

**APPENDIX A**

**PROOF OF EQUATION (19)**

Under the assumption that the prototype pulse at the receiver is \( \psi(t) = g(t - \Delta t) \), the SINR of the received signal can be expressed as

\[
R_{\text{SIN}} = \frac{\sigma_c^2}{\pi \tau_{\text{rms}} f_d \overline{E}_n} \int_{-f_d}^{f_d} e^{-\sigma^2 \tau^2} e^{-\tau/(\Delta t)^2} d\tau
\]

\[
\leq \frac{\sigma_c^2}{\pi \tau_{\text{rms}} f_d} \int_{-f_d}^{f_d} \frac{1}{\sqrt{1 - (v/f_d)^2}} e^{-\sigma \tau v^2} d\tau
\]

\[
\leq \frac{\sigma_c^2 \tau_{\text{rms}} f_d}{\pi \tau_{\text{rms}} f_d} \int_{-f_d}^{f_d} e^{-\sigma \tau v^2} d\tau
\]

\[
\int_{-f_d}^{f_d} e^{-\tau/(\Delta t)^2} d\tau = b(\Delta t)
\]

where \( b(\Delta t) \) in (20) can be rewritten as

\[
b(\Delta t) = \sqrt{\frac{\sigma}{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx
\]

\[
= \frac{\sqrt{\pi}}{2} \text{erfc} \left( \sqrt{\frac{\sigma}{2 \pi \tau_{\text{rms}}} - \Delta t} \right)
\]

where \( \text{erfc}(\cdot) \) is the complementary error function. If \( x > 0 \), we may obtain an approximate solution of the complementary error function \( \text{erfc}(\cdot) \) by [19]

\[
\text{erfc} \left( \frac{x}{\sqrt{2}} \right) \approx \frac{2e^{-x^2}}{\sqrt{\pi} \left( 1.64x + 0.76x^2 + 4 \right)}
\]

The Max-SINR receiver can be obtained by solving the gradient of \( a(\Delta t)b(\Delta t) \) with respect to \( \Delta t \)

\[
\frac{da(\Delta t)}{d\Delta t} - \frac{db(\Delta t)}{d\Delta t} = 0
\]

where \( \frac{da(\Delta t)}{d\Delta t} \) and \( \frac{db(\Delta t)}{d\Delta t} \) can be expressed as

\[
\frac{da(\Delta t)}{d\Delta t} = -\frac{a(\Delta t)}{\tau_{\text{rms}}}
\]

\[
\frac{db(\Delta t)}{d\Delta t} = e^{-\frac{x}{2\Delta t}} \left( \frac{\sigma}{2 \pi \tau_{\text{rms}}} - \Delta t \right)^2
\]
hence, equation (23) can be rewritten as
\[
\frac{b(\Delta t)}{t_{\text{rms}}} = e^{-\frac{\Delta t}{2t_{\text{rms}}}} \left( \frac{\sigma}{2\pi t_{\text{rms}}} - \Delta t \right)^2
\]
\[
= \frac{\sqrt{\sigma}}{2t_{\text{rms}}} \text{erfc} \left( \sqrt{\frac{\pi}{\sigma}} \left( \frac{\sigma}{2\pi t_{\text{rms}}} - \Delta t \right) \right)
\]
\[
\approx \frac{\sqrt{\sigma}}{t_{\text{rms}}} e^{-\frac{\Delta t}{2t_{\text{rms}}}} \left( \frac{\sigma}{2\pi t_{\text{rms}}} - \Delta t \right)^2 \left( 1.64 \sqrt{\frac{2\pi}{\sigma}} \left( \frac{\sigma}{2\pi t_{\text{rms}}} \right) - \Delta t + \sqrt{\frac{1.52\pi}{\sigma} \left( \frac{\sigma}{2\pi t_{\text{rms}}} - \Delta t \right)^2 + 4 \right)^{-1}
\]
Equation (25) can be simplified to a quadratic equation. Under the constraint of \( \Delta t > 0 \), the solution of the quadratic equation can be expressed as
\[
\Delta t = \frac{\sigma}{2\pi t_{\text{rms}}} \left( \frac{3.28\sqrt{\sigma}}{t_{\text{rms}}} - \frac{3.28\pi^2/\sigma}{t_{\text{rms}}} - 3.52 \right) \left( \frac{\sigma}{2\pi t_{\text{rms}}} - \Delta t \right)^2
\]
\[
\approx \frac{\sigma}{2\pi} \left( \frac{3.28\sqrt{\sigma}}{t_{\text{rms}}} - \frac{3.52}{1.76} \right)
\]

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of China (No. 60972050) and Major Special Project of China under Grant (2010ZX03003-003-01) and the Jiangsu Province National Science Foundation under Grant (BK20111002) and the Young Scientists Pre-research Fund of PLAUST under Grant (No. KYTVLX1211).

REFERENCES