Trajectory planning in robotic systems: a continuation method approach

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Abstract—This paper addresses the trajectory planning problem in robotic systems that can be represented by an affine control system with output. Using as a guideline the continuation method, a partial differential equation has been associated with the control system and its variational system, whose solution yields a 1–parameter family of control functions. After passing with the parameter to infinity a solution to the trajectory planning problem is obtained. The approach developed in the paper has been illustrated with a trajectory planning problem for the kinematics of the rolling ball.

I. INTRODUCTION

Given a robotic system represented by a control system, the trajectory planning problem consists in defining a control function that will steer the system along a prescribed trajectory [1]. Expressed in control theoretic terminology, a solution to the trajectory planning problem requires a sort of inversion of the control system. More formally, the system inversion means inversion of its input–output map at a given system initial state. When this map is injective, the system is called left–invertible; in the case when this map is surjective, the system is referred to as right–invertible. Classically, the last property is also called functional reproducibility [2]. There exists a rich literature concerned with inversion of control systems, both linear and nonlinear. Specifically, it should be noted that, if the system is differentially flat [3], and we know the trajectory of its flat outputs, then the trajectory planning problem is easily solvable. A comprehensive overview of theory and algorithms of inversion can be found in [4]. The relevance of system inversion to the trajectory planning problem in robotic systems is well established; think e.g. of the method of input-output decoupling of mobile robots by static or dynamic feedback [5]. Again, differential flatness of many robotic systems appears to be crucial.

The system inverters, presented in the control literature, have the form of dynamic systems performing a number of differentiations of the system trajectory. Differently to these existing solutions, a core of our approach is a 2nd order partial differential equation for the control, called trajectory reproduction equation being the core of the trajectory planning problem. Section III presents our main result: a trajectory reproduction equation being the core of the trajectory planning algorithm. Section IV demonstrates performance of the algorithm applied a trajectory planning problem of the rolling ball. The papers concludes with section V.

II. BASIC CONCEPTS

We shall study the kinematics of a nonholonomic mobile robot or the dynamics of a manipulation or a mobile robot represented in the form of an affine control system with output

\[
\begin{cases}
  \dot{x} = f(x) + G(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i \\
  y = k(x) = (k_1(x), k_2(x), \ldots, k_r(x)),
\end{cases}
\]

with state variables \(x \in \mathbb{R}^n\), control variables \(u \in \mathbb{R}^m\) and output variables \(y \in \mathbb{R}^r\). System (1) will be assumed square, in the sense that \(r = m\). We let admissible control functions belong to a function space \(\mathcal{U}\) of time functions defined on a given interval \([0, T]\), with values in \(\mathbb{R}^m\). All the vector fields and functions appearing in (1) are assumed smooth. Given an initial state \(x_0\) and a control function \(u(t)\), we compute the state trajectory \(x(t) = \varphi_{x_0,t}(u(\cdot))\) and the output trajectory \(y(t) = k(x(t))\) of (1).

For a given \(x_0\), the input–output map of system (1), transforming admissible control functions into output trajectories, can be defined as

\[
\mathcal{F}_{x_0} : \mathcal{U} \rightarrow \mathcal{Y}, \quad \mathcal{F}_{x_0}(u(\cdot))(t) = y(t).
\]

An output trajectory \(y(\cdot) \in \mathcal{Y}\) will be called reproducible at \(x_0\), provided that there exists a control function \(u(\cdot) \in \mathcal{U}\),

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such that \( y(\cdot) = \mathcal{F}_{x_0}(u(\cdot)) \). When the time instant \( t \) is fixed, the map
\[
\mathcal{F}_{x_0,t} : \mathcal{U} \rightarrow \mathcal{R}, \quad \mathcal{F}_{x_0,t}(u(\cdot)) = y(t),
\]
is referred to as the end–point map of system (1). For essentially bounded measurable control functions equipped with \( L_\infty \) topology for \( p \geq 2 \), the end–point map is of class \( C^0 \) [10]. Its derivative can be computed on the basis of the linear approximation to system (1) along the control–trajectory pair \( (u(t), x(t)) \) in the following way
\[
D\mathcal{F}_{x_0,t}(u(\cdot))v(\cdot) = C(t) \int_0^t \Phi(t,s)B(s)v(s)ds,
\]
where \( C(t) = \frac{\partial^2 x(s(t))}{\partial t^2}, \ B(t) = G(x(t)), \) and the matrix \( \Phi(t,s) \) satisfies the evolution equation \( \frac{\partial}{\partial t} \Phi(t,s) = A(t)\Phi(t,s), \) with \( A(t) = \frac{\partial f(x(t)) + G(x(t))u(t)}{\partial x} \), for initial condition \( \Phi(s,s) = I_n \).

For fixed control functions \( u(\cdot) \) and \( v(\cdot) \) we shall introduce an auxiliary variable
\[
\xi(t) = D\varphi_{x_0,t}(u(\cdot))v(\cdot) = \int_0^t \Phi(t,s)B(s)v(s)ds
\]
obeying a differential equation
\[
\xi(t) = A(t)\xi(t) + B(t)v(t).
\]
System (6) represents the variational system [10] associated with (1). By definition, \( \xi(0) = 0 \). The pair of systems (1) and (6) constitutes a lift of the original system to the tangent bundle, and is referred to as the prolongation of (1).

Having defined the control system representation (1) and the input–output map (2) of a robotic system, we shall address the following trajectory tracking problem: given a demanded output trajectory \( y_d(t) \) reproducible at \( x_0 \), find a control function \( u_0(t) \), such that \( \mathcal{F}_{x_0}(u_0(\cdot)) = y_d(\cdot) \).

III. MAIN RESULT

Suppose that the admissible control functions are smooth. In this case, a solution of the trajectory tracking problem can be found using a reasoning borrowed from the continuation method. To be more specific, we choose a smooth curve \( u_\theta(\cdot) \in \mathcal{U}, \ \theta \in \mathcal{R}, \) of admissible control functions, and let \( y_\theta(t) \) denote the output trajectory corresponding to the control function \( u_\theta(t) \). Then, given a demanded trajectory \( y_d(t) \), we define an error function
\[
e(t,\theta) = y_\theta(t) - y_d(t) = k(\varphi_{x_0,t}(u_\theta(\cdot))) - y_d(t).
\]
Now, we want to determine a control \( u_\theta(t) \) that will make \( e(t,\theta) \) decreasing to zero with a prescribed rate \( \alpha > 0 \).

Following the idea of the continuation method, we introduce a homotopy map
\[
H(t,\theta) = \frac{\partial e(t,\theta)}{\partial t} + \alpha e(t,\theta) - \left( \frac{\partial e(t,0)}{\partial t} + \alpha e(t,0) \right) \exp(-\gamma \theta),
\]
where \( \gamma > 0, \ t \in [0,T], \ \theta \in \mathcal{R}, \) and \( e(t,0) \) denotes the error corresponding to the initial control function \( u_0(t) \). After equating the map (8) to zero,
\[
H(t,\theta) = 0,
\]
it is easily seen that for \( \theta = 0 \) this identity is satisfied trivially, while at \( \theta \) approaching \( +\infty \) the identity defines the error equation
\[
\frac{\partial e(t,\theta)}{\partial t} + \alpha e(t,\theta) = 0,
\]
that yields exponential decrease of the error, \( e(t,\theta) = e(0,\theta) \exp(-\alpha t) \). The identity \( H(t,\theta) = 0 \) corresponds to a family of tracking problems, parameterized by \( \theta \). If we are able to solve the problem labeled with \( \theta \) then, by passing to the limit \( \theta \rightarrow +\infty \), we shall also solve the original problem.

Now, by differentiation of the identity \( H(t,\theta) = 0 \) with respect to \( \theta \), we get
\[
\frac{\partial^2 e(t,\theta)}{\partial t \partial \theta} + \alpha \frac{\partial e(t,\theta)}{\partial \theta} + \gamma \left( \frac{\partial e(t,0)}{\partial t} + \alpha e(t,0) \right) \exp(-\gamma \theta) = 0.
\]
Furthermore, since the component on left hand side multiplied by \( \gamma \) is equal to
\[
\frac{\partial e(t,\theta)}{\partial t} + \alpha e(t,\theta),
\]
we get the following error equation
\[
\frac{\partial^2 e(t,\theta)}{\partial t \partial \theta} + \alpha \frac{\partial e(t,\theta)}{\partial \theta} + \gamma \frac{\partial e(t,\theta)}{\partial t} + \alpha y(t,\theta) = 0. \quad (9)
\]
Now, in order to determine the control \( u_\theta(t) \), we compute
\[
\frac{\partial e(t,\theta)}{\partial t} = C_\theta(t)(f(x_\theta(t)) + G(x_\theta(t))u_\theta(t)) - y_d(t),
\]
and
\[
\frac{\partial e(t,\theta)}{\partial \theta} = C_\theta(t) \frac{\partial \varphi_{x_\theta,t}(u_\theta(\cdot))}{\partial \theta} = C_\theta(t) \xi_\theta(t),
\]
where matrix \( C_\theta(t) \) is computed along trajectory \( x_\theta(t) \), while
\[
\xi_\theta(t) = \frac{\partial \varphi_{x_\theta,t}(u_\theta(\cdot))}{\partial \theta} = D\varphi_{x_\theta,t}(u_\theta(\cdot)) \frac{du_\theta(\cdot)}{d\theta}.
\]
Finally, using (6), we obtain
\[
\frac{\partial^2 e(t,\theta)}{\partial t \partial \theta} = C_\theta(t) \xi_\theta(t) + C_\theta(t) \dot{\xi}_\theta(t) = C_\theta(t) \xi_\theta(t) + C_\theta(t)A_\theta(t) \xi_\theta(t) + C_\theta(t)B_\theta(t) \frac{du_\theta(t)}{d\theta},
\]
where \( A_\theta(t) \) and \( B_\theta(t) \) need to be computed along \( x_\theta(t) \).
After a substitution into (9), we get the following differential–integral equation for the control \( u(t,\theta) \)
\[
N_\theta(t) \frac{du(t,\theta)}{d\theta} + M_\theta(t) \xi_\theta(t) + \gamma(C_\theta(t)(f(x_\theta(t)) + G(x_\theta(t))u(t,\theta)) + \gamma(\dot{y}_d - \alpha e(t,\theta)), \quad (11)
\]
where

\[ M_0(t) = C_0(t) + C_0(t)(\alpha I_n + A_0(t)), \]

\[ N_0(t) = C_0(t)B_0(t). \]

Finally, a differentiation of the left hand side of (11) with respect to \( t \), and invoking (6), yield the following partial differential equation for the control function

\[ N_0(t) \frac{\partial^2 u(t, \theta)}{\partial t \partial \theta} + (N_0(t) + M_0(t)B_0(t)) \frac{\partial u(t, \theta)}{\partial \theta} + \gamma N_0(t) \frac{\partial u(t, \theta)}{\partial t} + \gamma M_0(t)B_0(t)u(t, \theta) + M_0(t)A_0(t)\xi_0(t) + \gamma (M_0(t)f(x_0(t)) + \gamma \dot{y}_d + \alpha \ddot{y}_d). \quad (12) \]

The partial differential equation (12) will be called the trajectory reproduction equation of system (1). Together with the prolongation

\[
\begin{align*}
\dot{x}_0(t) &= f(x_0(t)) + G(x_0(t))u_0(t) \\
\dot{\xi}_0(t) &= A_0(t)\xi_0(t) + B_0(t)\frac{\partial u(t, \theta)}{\partial \theta},
\end{align*}
\]  

(13)

the reproduction equation forms a reproduction system. Initial and boundary conditions for the reproduction system are

\[ x_0(0) = x_0, \quad \xi_0(0) = 0, \quad u(t, 0) = u_0(t), \]

where \( u(0, \theta) \) is the solution of the linear differential equation resulting from (11)

\[ N(0) \frac{d u(0, \theta)}{d \theta} + \gamma N(0)u(0, \theta) = \gamma (C(0)f(x_0) + \dot{y}_d(0) - \alpha (k(x_0) - y_d(0))) \]

where \( C(0) = \frac{\partial f(x_0)}{\partial \theta}, \quad N(0) = C(0)G(x_0). \) The reproduction system (12) and (13) will be solved under assumption that the matrix \( N_0(t) \) is invertible.

IV. ROLLING BALL

The kinematics of the ball rolling on a plane have been studied in [11], [12]. In coordinates, the kinematics are represented by the following driftless control system

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
y
\end{bmatrix} =
\begin{bmatrix}
\sin x_4 \sin x_5 & \cos x_5 \\
-\sin x_4 \cos x_5 & \sin x_5 \\
1 & 0 \\
0 & 1 \\
-\cos x_4 & 0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix},
\]  

(14)

with computation of matrices defining the variational system associated with (14)

\[
A(t) =
\begin{bmatrix}
u_1 \cos x_4 \sin x_5 & u_1 \sin x_4 \cos x_5 - u_2 \sin x_5 \\
0 & 0 & -u_1 \cos x_4 \sin x_5 & u_1 \sin x_4 \sin x_5 + u_2 \cos x_5 \\
0 & 0 & 0 & 0 \\
0 & 0 & u_1 \sin x_4 & 0
\end{bmatrix}
\]

where, for the sake of space, we have omitted the dependence of matrix entries on time, and

\[
B(t) =
\begin{bmatrix}
\sin x_4(t) \sin x_5(t) & \cos x_5(t) \\
-\sin x_4(t) \cos x_5(t) & \sin x_5(t) \\
1 & 0 \\
0 & 1 \\
-\cos x_4(t) & 0
\end{bmatrix}
\]

Clearly, the output matrix \( C(t) = C = [I_2 \quad 0_{2 \times 3}] \). It is easily checked that the matrix \( N(t) = CB(t) \) is invertible for every control, provided that \( 0 < x_4 < \pi \).

In the following subsections we shall study the following trajectory tracking problem of the rolling ball: given a demanded trajectory \( y_d(t) \) in the plane, find control \( u(t) \) that would reproduce this trajectory, starting from a certain \( x_0 \).

A. Trajectory reproduction equation

From (12), using invertibility of \( N_0(t) \), we derive the following trajectory reproduction equation for the rolling ball kinematics

\[
\begin{align*}
\frac{\partial^2 u(t, \theta)}{\partial t \partial \theta} + \gamma \frac{\partial u(t, \theta)}{\partial t} + \gamma (\alpha + 2 \cot x_4 u_2, 2 \cot x_4 u_1) \frac{\partial u(t, \theta)}{\partial \theta} + \\
\gamma (\alpha + \gamma \cot x_4 u_2, -\gamma \cos x_4 \sin x_4 u_1) \frac{\partial u(t, \theta)}{\partial \theta} + \\
0 & 0 & u_1(\alpha \cot x_4 - 2 u_2) + \cot x_4 u_1 \\
0 & 0 & u_2(-\cos^2 x_4 + \sin^2 x_4) \\
\frac{1}{\sin x_4}(-\cos x_4 \sin x_4^2 + \alpha u_2 + \alpha u_1) & 2 \cos x_4 \alpha x_4 + u_1 (\alpha + u_1)
\end{align*}
\]

\[
\frac{\partial \xi_0(t)}{\partial \theta} =
\begin{bmatrix}
\sin x_4 & -\cos x_4 \\
\sin x_5 & \cos x_5
\end{bmatrix}
\begin{bmatrix}
\dot{y}_d + \alpha \ddot{y}_d
\end{bmatrix},
\]  

(15)
where, for the sake of space, dependence of entries of matrices on \( t \) and \( \theta \) is not shown, as well as the notation \( \dot{u} \) abbreviates \( \frac{\partial u(t, \theta)}{\partial t} \). Equation (15) along with the original kinematics equation (14) and the variational equation

\[
\dot{\xi}_\theta(t) = A_\theta(t)\xi_\theta(t) + B_\theta(t)\frac{\partial u(t, \theta)}{\partial \theta}
\]

defines the trajectory reproduction system for the rolling ball. This system should be solved for the control assuming the initial and boundary conditions \( x_\theta(0) = x_0, \xi_\theta(0) = 0, u(t, 0) = u_0(t) \), and \( u(0, \theta) \) obtained by solving a linear differential equation

\[
\frac{du(0, \theta)}{d\theta} + \gamma u(0, \theta) = \gamma \begin{bmatrix}
\sin x_s(0) \\
\cos x_s(0)
\end{bmatrix} - \alpha \begin{bmatrix}
\cos x_s(0) \\
\sin x_s(0)
\end{bmatrix}\begin{bmatrix}
y_d(0) - \alpha(x_0) - y_d(0)\end{bmatrix}.
\]

**B. Trajectory planning**

We shall solve the trajectory tracking problem for the rolling ball, characterized by the following data: a circular demanded trajectory

\[
y_{1d}(t) = \tan 1 - \cos(t \cos 1) \tan 1, \quad y_{2d}(t) = \sin(t \cos 1) \tan 1,
\]

the initial state of the ball \( x(0) = (0, 0, 0, -1, 0) \), the control horizon \( T = 20 \), homotopy coefficient \( \gamma = 1 \), the initial control \( u_0(t) = (1, 1) \). Numerical solution of this problem, for homotopy coefficient \( \alpha = 50 \) and \( \alpha = 200 \), and control parameter \( \theta = 5 \cdot 10^3 \) and \( \theta = 10^5 \) are displayed below in figs. 2–5 and 6–9. It can be noticed that, as might be expected, for bigger values of \( \theta \) demanded trajectory is reproduced more efficiently.
Fig. 5. Solution of reproduction equation for $\alpha = 50$, $\theta = 10^5$

Fig. 6. Controls for $\alpha = 200$, $\theta = 5 \cdot 10^3$

Fig. 7. Contact point trajectory for $\alpha = 200$, $\theta = 5 \cdot 10^3$

Fig. 8. Contact point trajectory and controls for $\alpha = 200$, $\theta = 10^5$
V. CONCLUSIONS AND FUTURE WORKS

Relying on the continuation method paradigm, we have derived a trajectory reproduction equation for affine control systems with output. This equation, in connection with the system prolongation equations, allows one to compute the control function steering the system along a prescribed output trajectory. Theoretically, the control is associated with a parameter going to $+\infty$, however, in practice, finite and not excessively big values of the parameter provide satisfactory solution. The algorithm based on this equation has been dedicated to trajectory planning problems in robotic systems. Performance of the algorithm has been illustrated by solving an example planning problem involving kinematics equations of the rolling ball. The computations confirm applicability of the algorithm.

By design, the presented reproduction system yields a $C^0$ reproduction. In this context, it would be instructive to extend the system toward providing a sort of $C^k$ reproduction for $k \geq 1$. Furthermore, we conjecture that the reproduction equation contains some important structural information about the robotic system. Examination of these two questions will be a subject of our future research.

REFERENCES