Plus–minus algorithm — A method for derivation of the Bäcklund transformations

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\textbf{1. Introduction}

The Bäcklund transformation was formulated in the 19th century and still remains the only hope to construct exact solutions of nonlinear partial differential equations. The existence of the Bäcklund transformations is usually taken as a criterion for a complete integrability. For the derivation of the Bäcklund transformations we refer to Rogers (1990), Harrison (1985) and Sokalski et al. (2001a, 2002, 2001b, 2005). In this paper we present a new way of deriving Bäcklund transformations for nonlinear partial differential evolution equations. The presented method is supported by the strong or semi-strong necessary condition concepts, recently derived in Sokalski (1979) and Sokalski et al. (2001a, 2002, 2001b, 2005). Therefore, in Section 2 we approach these concepts and in Section 3 we derive a method which we call \textit{The Plus–Minus Algorithm}. In further sections we present applications to the
following equations: the Korteweg–de Vries and the fifth order equations of the KdV hierarchies. In Section 4 we apply the method to a generalized form of such equations and we pose the question which equations possess the Bäcklund transformation in the frame of the plus–minus algorithm. As a result we derive the Gardner, the Burgers and two generalized KdV equations. In Section 5 we summarize these considerations by putting the method within the framework of an algorithm and present its implementation in MAPLE. Problems concerning the Computational Complexity and related questions are described in Section 6. In Summary (the last section) we give a general review of this paper.

2. Approach to strong and semi-strong necessary conditions concepts

Strong and semi-strong necessary conditions concepts are described in Sokalski et al. (2001a, 2002, 2001b). Here, we present a brief approach to both concepts which is necessary to derive the Plus–Minus Algorithm. Let us suppose that we wish to solve a given system of nonlinear PDEs

$$\mathcal{B}[\mathbf{w}] = 0,$$

where $\mathcal{B}[\cdot]$ denotes nonlinear partial differential operator and $\mathbf{w} = (u, v)$ is the unknown consisting of two functions. There are classes of such nonlinear problems that can be solved using techniques from nonlinear functional analysis. These are problems, where $1$ is the necessary condition for the extremum of a functional to exist. The point is that whereas it is usually difficult to solve (1) directly, it may be much easier to derive stationary points of this functional.

Let $\mathbf{L}$ be a particular functional on a set of smooth enough differentiable functions $\mathcal{F}$ which depend on smooth enough $u(x, t)$ and $v(x, t)$ as well as their derivatives:

$$\mathbf{L}[u, v] = \int_{t_1}^{t_2} \int_X F(u, v, u_t, v_t, u_{x}, v_{x}, u_{xx}, v_{xx}) \, dx \, dt,$$

where $X \subset \mathbb{R}, F \in \mathcal{F} : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}, u : X \times [t_1, t_2] \rightarrow \mathbb{R}, v : X \times [t_1, t_2] \rightarrow \mathbb{R}$. Accordingly, one can investigate the necessary conditions for the extremum of (2) to exist:

$$\int_{t_1}^{t_2} \int_X \left( \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u_t} \delta u_t + \frac{\partial F}{\partial u_{x}} \delta u_x + \frac{\partial F}{\partial u_{xx}} \delta u_{xx} \\
+ \frac{\partial F}{\partial v} \delta v + \frac{\partial F}{\partial v_t} \delta v_t + \frac{\partial F}{\partial v_{x}} \delta v_x + \frac{\partial F}{\partial v_{xx}} \delta v_{xx} \right) \, dx \, dt = 0,$$

where $\delta u, \delta v, \delta u_t, \delta v_t, \ldots, \delta u_{xx}, \delta v_{xx}$ are increments of $u$ and $v$ and their partial derivatives, respectively. Let the functions $u(x, t)$ and $v(x, t)$ satisfy the initial/boundary conditions:

$$u = u_x = 0 \quad \text{on } \partial X \times [t_1, t_2],$$
$$v = v_x = 0 \quad \text{on } \partial X \times [t_1, t_2],$$
$$u = g, \quad u_t = h \quad \text{on } X \times \{t = t_1\},$$
$$v = g, \quad v_t = h \quad \text{on } X \times \{t = t_1\}.$$

If $u$ and $v$ satisfy (3) and (4), then they automatically satisfy the Euler–Lagrange equations:

$$\frac{\partial F}{\partial u} - D_x F_{u,x} - D_t F_{u,t} + D_x^2 F_{u,xx} = 0,$$
$$\frac{\partial F}{\partial v} - D_x F_{v,x} - D_t F_{v,t} + D_x^2 F_{v,xx} = 0.$$

2.1. Strong necessary conditions

In order to derive the method we are talking about, for a moment we propose a naive method consisting in equating each term of (3) to zero:

$$F_{u} = 0, \quad F_{v} = 0,$$
which replace the Euler–Lagrange equations (5). (6)–(8) we call the strong necessary conditions.

Unfortunately, in most cases the set of solutions of (6)–(8) is trivial \((u = \text{const}, v = \text{const})\) or empty. In order to extend this set to a nontrivial one, we use the gauge transformation of (2)

\[
L^* = L + I
\]

(9)

and instead of (5) we apply the naive method to (9). The scaling functional \(I\) is invariant with respect to the local variation of \(u(x, t)\) and \(v(x, t)\): \(\delta I \equiv 0\). Therefore, the Euler–Lagrange equations resulting from the extremum of \(L\) and the extremum of \(L^*\) are equivalent. (6)–(8) are not invariant, however, with respect to (9), i.e. the gauge transformation contributes to the strong necessary conditions:

\[
F^*_{u,t} = 0, \quad F^*_{v,t} = 0,
\]

(10)

\[
F^*_{u,x} = 0, \quad F^*_{v,x} = 0,
\]

(11)

\[
F^*_{u,xx} = 0, \quad F^*_{v,xx} = 0,
\]

(12)

where an example of \(F^*\) is presented in (19). This contribution can extend the subset of solutions to the nontrivial one. In such a way, we derive simpler differential equations for extremals of \(L\), solutions which form a subset of solutions of the Euler–Lagrange equation. The described concept is presented in Fig. 1. The \(S_0\) and \(S_I\) represent the sets of solutions of (6), (7), (8) and (10), (11), (12), respectively. The \(S_0\) is trivial always. Similarly, the \(S\) represents the set of solutions of (5) being the necessary conditions for the extremum of (2) and (9) to exist. The dash-dotted arrows represent the Euler–Lagrange equations, whereas the continuous ones correspond to the strong necessary condition concept.

### 2.2. Semi-strong necessary conditions

Sometimes the strong necessary condition concept does not work even when we apply the gauge transformation to (2). Then we can try to set weaker conditions by replacing (8) with the following equations:

\[
F_{u,t} - D_x F_{u,xx} = 0,
\]

(13)

\[
F_{v,t} - D_x F_{v,xx} = 0.
\]

If (2) depends on the higher derivatives of \(u(x, t)\) until \(u(x, t),_{kx}\) then (13) takes the following extended form:

\[
F_{u,x} - D_x F_{u,xx} + D_x^2 F_{u,xxx} - D_x^3 F_{u,xxxx} + \cdots + (-1)^{k-1} D_x^{k-1} F_{u,kx} = 0.
\]

(14)

where \(u_{kx}\) means the derivative of the order \(k\).
The semi-strong necessary condition concept provides a helpful tool for the theory of nonlinear PDEs. We present its application to the Korteweg–de Vries equation. Therefore, we have to consider two independent fields \( \bar{u} \) and \( \bar{v} \) governed by twin KdV equations:

\[
\begin{align*}
\bar{u},_t - 6\bar{u},_{xx} + \bar{v},_{xxx} &= 0, \\
\bar{v},_t - 6\bar{v},_{xx} + \bar{v},_{xxx} &= 0.
\end{align*}
\]

(15)

It is easy to confirm that the action functional density

\[
F(u, v, u_x, v_x, u, _t, v, _t, u_{xx}, v_{xx}) = \frac{1}{2} u,_{xt} - u^3, - \frac{1}{2} u^2,_{xx} + \lambda \left( \frac{1}{2} v,_{xt} - v^3, - \frac{1}{2} v^2,_{xx} \right)
\]

(16)

generates (15) as the corresponding Euler–Lagrange equations, where \( \bar{u} = u,_{x} \) and \( \bar{v} = v,_{x} \) and \( \lambda \) is an arbitrary real constant (Drazin and Johnson, 1989). We attach three topological invariants (Sokalski et al., 2001b):

\[
\begin{align*}
I_1 &= \int_{E^2} G_1(u, v)(u,_{x}v,_{t} - u,_{t}v,_{x})dxdt, \\
I_2 &= \int_{E^2} D_1G_2(u, v)dxdt, \\
I_3 &= \int_{E^2} D_3G_3(u, v, u, _x, v, _x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx})dxdt.
\end{align*}
\]

Each topological invariant \( I[f] \), by the definition satisfies the following condition: \( \delta I[f] / \delta f \equiv 0 \) and therefore according to Section 2.1 the gauge transformation does not influence the Euler–Lagrange equations. The gauge transformed action functional takes the following form:

\[
L^*[u, v] = \int_{E^2} F(u, v, u, _x, v, _x, u, _t, v, _t, u_{xx}, v_{xx})dxdt + I_1 + I_2 + I_3.
\]

(18)

Therefore,

\[
F^* = F + G_1(u, _xv, _t - u, _tv, _x) + D_1G_2 + D_3G_3.
\]

(19)

Applying the concept of the semi-strong conditions to (18) we derive the following set of equations:

\[
\begin{align*}
\frac{\partial F^*}{\partial u} &= 0 \Rightarrow G_{1,u}(u,_{x}v,_{t} - u,_{t}v,_{x}) + (G_{2,u})_t + (G_{3,u})_x = 0, \\
\frac{\partial F^*}{\partial v} &= 0 \Rightarrow G_{1,v}(u,_{x}v,_{t} - u,_{t}v,_{x}) + (G_{2,v})_t + (G_{3,v})_x = 0, \\
\frac{\partial F^*}{\partial u, _t} &= 0 \Rightarrow \frac{1}{2} u,_{x} - G_{1}v,_{x} + G_{2,u} = 0, \\
\frac{\partial F^*}{\partial v, _t} &= 0 \Rightarrow \frac{\lambda}{2} v,_{x} + G_{1}u,_{x} + G_{2,v} = 0, \\
F^*_{u, x} &- D_xF^*_u,_{xx} + D_x^2F^*_u,_{xxx} + \cdots + (-1)^{n-1}D_x^{(n-1)}F^*_{u, xxx} = 0 \Rightarrow \frac{1}{2} u,_{t} - 3u^2,_{x} + u,_{xxx} + G_{1}v,_{t} + G_{3,u} = 0, \\
F^*_{v, x} &- D_xF^*_v,_{xx} + D_x^2F^*_v,_{xxx} + \cdots + (-1)^{n-1}D_x^{(n-1)}F^*_{v, xxx} = 0 \Rightarrow \lambda \left( \frac{1}{2} v,_{t} - 3v^2,_{x} + v,_{xxx} \right) - G_{1}u,_{t} + G_{3,v} = 0.
\end{align*}
\]

(20)

(21)

(22)

The application of partial derivatives with respect to \( x \) (first, second and third degrees) generates a lot of complicated terms. Therefore, when deriving (22) we have applied the following theorem:
**Theorem (Wietecha, 2003).** Let \( D_x G_3(u, v, u, v, u_{,xx}, v_{,xxx}) \) be a divergence of an arbitrary order \( n \). Then
\[
\frac{\delta(D_x G_3)}{\delta u_x} = \frac{\partial G_3}{\partial u},
\]
where
\[
\frac{\delta f(u, u_x, \ldots, u_{,xx})}{\delta u_x} = f_{,u_x} - D_x f_{,u_x} + D_x^2 f_{u_{,xx}} + \cdots + (-1)^{n-1}D_x^{(n-1)} f_{u_{,xx}},
\]
It can be proved that by the induction starting from \( n = 2 \): \( G_3(u, v, u, v, u_{,xx}, v_{,xxx}) \). Eqs. (20)–(22) form a system of the six simultaneous equations for the five unknown functions: \( u, v, G_1, G_2, G_3 \). It is necessary to explain that \( u(x, t) \) and \( v(x, t) \) are unknown functions of \( x \) and \( t \), whereas \( G_i \) are independent variables. The system (20)–(22) is over determined. Therefore, we reduce the number of equations by making some of them to be linear dependent as well as by choosing spatial forms for \( G_i \) functions. Assuming that (21) are linearly dependent we derive the following constrains:

\[
G_1^2 + \frac{\lambda}{4} = 0,
\]
\[
\frac{1}{2} G_{2,v} - G_{1} G_{2,u} = 0.
\]

Next, integrating (22) with respect to \( u \) and \( v \), respectively, we derive the following form for \( G_3 \):

\[
G_3(u, v, \ldots, u_{,xx}, v_{,xxx}) = \left[ -\frac{3}{2}(u_{,x}^2 + v_{,x}^2) + \frac{1}{2}(u_{,xxx} + v_{,xxx}) \right] (u - v).
\]

Substituting (25) and (26) to (20) we obtain \( G_2(u, v) \) function:

\[
G_2 = \frac{1}{12}(u - v)^3 + \gamma (u - v),
\]
where \( \gamma \) is an arbitrary real constant. For future convenience we choose the following particular values for: \( G_1 = -\frac{1}{2} \) and \( \lambda = -1 \). As a consequence, the two equations in (21) reduce to only one equation:

\[
(u + v),_x = 2\gamma + \frac{1}{2}(u - v)^2
\]
and (22) reduces to the following equation:

\[
(u - v),_t - 3(u_{,x}^2 - v_{,x}^2) + (u - v),_{xxx} = 0.
\]

Finally, using (27) we reduce (20) to

\[
(u - v) \left[ (u - v),_t - 3(u_{,x}^2 - v_{,x}^2) + (u - v),_{xxx} \right] = 0.
\]
(27)–(29) become the Bäcklund transformation for the Korteweg–de Vries equation.

### 3. Plus–minus algorithm

Looking through literature (Drazin and Johnson, 1989; Wadati et al., 1975; Wang, 2002; Leznov, 1992) concerning the integrable nonlinear PDEs we find out that the majority of the Bäcklund transformations have the following forms:

\[
\begin{align*}
    u_t \pm v_t &= H, \\
    u_x \pm v_x &= P,
\end{align*}
\]
where \( H \) and \( P \) are expressions depending on \( u, v \) and their partial derivatives. Different signs in both relations in (30) are relevant, however, it is sufficient to consider only one set of signs. For instance,

\[
\begin{align*}
    u_t - v_t &= H, \\
    u_x + v_x &= P.
\end{align*}
\]
The opposite set of signs in (30) which is equivalent to (31) can be obtained by the simple transformation: \( v \rightarrow -v \).

There are nonlinear PDEs for which the Bäcklund transformation takes a similar form to (30) but with both positive or both negative signs:

\[
\begin{align*}
  u_t &\mp v_t = H, \quad u_x \mp v_x = P. \quad (32)
\end{align*}
\]

For instance, the Bäcklund transformation (Drazin and Johnson, 1989) for the nonlinear Schrödinger equation takes the form of (32). In such cases we will be talking about the Plus–Plus Algorithm.

Let us consider two equations having conservation forms:

\[
\begin{align*}
  u_{,x} + \Phi (u)_{,x} &= 0, \quad (33) \\
  v_{,x} + \Psi (v)_{,x} &= 0. \quad (34)
\end{align*}
\]

Let us introduce new field variables by the following diffeomorphism:

\[
\begin{align*}
  U &= u + \alpha v, \quad (35) \\
  V &= \beta u + v, \quad (36) \\
  u &= \frac{U - \alpha V}{1 - \alpha \beta}, \quad (37) \\
  v &= \frac{V - \beta U}{1 - \alpha \beta}, \quad (38)
\end{align*}
\]

where \( \alpha \) and \( \beta \) are real and \( \alpha \beta \neq 1 \). We suggest the following form for the Bäcklund transformation expressed by \( U \) and \( V \) variables:

\[
\begin{align*}
  U_{,t} &= H(U, V, U_{,x}, V_{,x}, \ldots, U_{,kk}, V_{,kk}), \quad (39) \\
  V_{,x} &= P(U). \quad (40)
\end{align*}
\]

A motivation for this ansatz is the following: Multiplying (34) by \( \alpha \), adding to (33), and integrating the resulting equation with respect to \( x \), we derive the equation corresponding to (39):

\[
U_{,t} = - (\Phi (u) + \alpha \Psi (v)) = H. \quad (41)
\]

Substituting (37), (38) into (41) determines \( H \) in (39).

In order to justify (40), we return to Section 2. For systems possessing the Lagrangian density containing the following term: \( \frac{1}{2} u_{,x} u_{,t} + \frac{\lambda}{2} u_{,xx} v_{,t} \) (for instance, (16)) the semi-strong necessary condition leads to (21). According to the plus–minus algorithm, one of the Bäcklund transformation supplies a linear dependence between \( u_{,x} \) and \( v_{,x} \). The only candidates for such a relation are (21). In order to reduce them to the one linear dependence we set the following condition:

\[
\text{rank} \left[ \begin{array}{cc}
  \frac{1}{2} & -G_1 \\
  G_1 & \frac{1}{2} G_2, \ u \\
\end{array} \right] = 1. \quad (42)
\]

From (42) we derive

\[
\frac{\lambda}{4} + G_1^2 = 0, \quad G_2 = G_2 \left( u - \frac{\lambda}{2 G_1} \right). \quad (43)
\]

Substituting \( \alpha = - \frac{\lambda}{2 G_1}, \beta = - \frac{1}{2 G_1} \) into (37), (38) and (21), we obtain the equation corresponding to (40): \( V_{,x} = - G_2, u(U) \). In general case the derivation of \( P(U) \) in (40) follows the following procedure 1. Multiply (33) by \( \beta \) and add to (34) we get

\[
V_{,tx} + \beta \Phi (u)_{,x} + \Psi (v)_{,x} = 0. \quad (44)
\]

Taking into account (40) the mixed derivative \( V_{,xx} \) is eliminated from (44):

\[
P' H + \beta \Phi (u)_{,x} + \Psi (v)_{,x} = 0. \quad (45)
\]
2. Express \( V \) and all its derivatives by \( U \) in (45), where

\[
V_x = P, \\
V_{xx} = P'U_x, \\
V_{xxx} = P''U_x^2 + P'U_{xx}, \\
V_{4x} = P''U_x^3 + 3P''U_xU_{xx} + P'U_{xxx}, \\
V_{5x} = P''U_x^4 + 6P''U_x^2U_{xx} + 4P''U_xU_{xxx} + 3P''U_{xx}^2 + P'U_{4x}, \\
V_{6x} = P''U_x^5 + 10P''U_x^3U_{xx} + 10P''U_x^2U_{xxx} + 15P''U_xU_{xxx}^2 + 5P'U_xU_{4x} + 10P'U_xU_{xx}U_{xxx} + P'U_{5x},
\]

\[
\sum_{l_0,\ldots, l_k} W_{l_0,\ldots,l_k} U_{l_x}^{l_0} U_{l_xx}^{l_1} \cdots U_{l_kx}^{l_k} = 0,
\]  
(52)

where the polynomial coefficients \( W_{l_0,\ldots,l_k} \) depend only on \( P \) and its derivatives.

4. Reduce (52) to a tautology by equating all \( W_{l_0,\ldots,l_k} \) with zero. This gives

\[
W_{l_0,\ldots,l_k} (P', P'', \ldots, P^{(n)}) = 0.
\]  
(53)

If the simultaneous system (53) possesses a solution \( P(U) \) then (40) establishes the second Bäcklund relation. However, if (53) is inconsistent then we can only conclude that the possible Bäcklund transformation of (33) and (34) is not the form of (39) and (40).

4. Applications of the plus–minus algorithm

In order to illustrate the efficiency of the presented algorithm we demonstrate derivation of the Bäcklund transformations for the third and fifth order Korteweg–de Vries equations from the Lax (Lax, 1968) and the Caudrey hierarchies (Caudrey, 1978; Dodd, 1978; Caudrey et al., 1976). The fifth order equation of the Caudrey hierarchy is the Sawada–Kotera equation (Sawada and Kotera, 1974). It is presented as an example for which our algorithm does not work. More examples, the Gardner and the Burgers equations will be considered in the next section.

4.1. The Korteweg–de Vries Equation

Let us consider twin KdV equations for \( u \) and \( v \):

\[
u_{,xt} + (-3u_x^2 + u_{xxx})_x = 0, \\
v_{,xt} + (-3v_x^2 + v_{xxx})_x = 0.
\]  
(54)

(55)

Therefore, \( \Phi(u) \) and \( \Psi(v) \) in (33) and (34) are of the following forms:

\[
\Phi(u) = -3u_x^2 + u_{xxx}, \\
\Psi(v) = -3v_x^2 + v_{xxx}, \\
\Phi(u)_x = -6u_xu_{xx} + u_{4x}, \\
\Psi(v)_x = -6v_xv_{xx} + v_{4x}.
\]  
(56)

Assuming \( \alpha = -1, \beta = 1 \) and substituting (56) into (41), we obtain

\[
U_{,t} = 3(u_x^2 - v_x^2) - (u - v)_{xxx},
\]  
(57)

which is equivalent to the second Bäcklund transformation (28).
Polynomial (52) is of the following form:
\[ P'' U_{,x}^3 + 3(P'' - 1)U_{,x} U_{,xx} = 0. \]  \hspace{1cm} (58)

The resulting system of equations for \( P(U) \) is very simple:
\[ P'' = 0, \] \hspace{1cm} (59)
\[ P'' - 1 = 0. \] \hspace{1cm} (60)

The solution is a square polynomial:
\[ P(U) = \frac{1}{2} U^2 + bU + c, \] \hspace{1cm} (61)

where \( b \) and \( c \) are arbitrary real numbers. Substituting (61) into (40) we derive the first Bäcklund relation (27). Eqs. (27) and (57) establish the Bäcklund transformation.

4.2. The fifth order equations of the Korteweg–de Vries hierarchies

There are two nonequivalent KdV hierarchies, both of which contain the classical KdV equation (15) as one of their members (Caudrey, 1978). However, the higher-order equations are not equivalent. In this subsection we apply the plus–minus algorithm to the fifth order equations of both hierarchies.

4.2.1. The fifth order equation of the lax hierarchy

Using the inverse scattering method Lax obtained a hierarchy of soluble KdV-type equations (Lax, 1968). The third equation of this hierarchy is of the following form:
\[ \bar{u},t + 30\bar{u}^2\bar{u}_{,x} - 20\bar{u}_{,x}\bar{u}_{,xx} - 10\bar{u}\bar{u}_{,xxx} + \bar{u}_{,5x} = 0. \] \hspace{1cm} (62)

After applying the transformation \( \bar{u} \rightarrow u_{,x} \), \( \bar{v} \rightarrow v_{,x} \) we combine two independent equations:
\[ u_{,xx} + \left[ 10u_{,x}^3 - 10u_{,xx}^2 - 10u_{,x}u_{,3x} + 5u_{,xx}^2 + u_{,5x} \right]_{,x} = 0, \] \hspace{1cm} (63)
\[ v_{,xx} + \left[ 10v_{,x}^3 - 10v_{,xx}^2 - 10v_{,x}v_{,3x} + 5v_{,xx}^2 + v_{,5x} \right]_{,x} = 0. \] \hspace{1cm} (64)

Comparing (63) and (33), (34) we get
\[ \Phi(u) = 10u_{,x}^3 - 10u_{,xx}^2 - 10u_{,x}u_{,3x} + 5u_{,xx}^2 + u_{,5x}, \] \hspace{1cm} (65)
\[ \Psi(v) = 10v_{,x}^3 - 10v_{,xx}^2 - 10v_{,x}v_{,3x} + 5v_{,xx}^2 + v_{,5x}. \] \hspace{1cm} (66)

According to the assumptions of the plus–minus algorithm, the Bäcklund transformation takes the following form:
\[ u_{,t} - v_{,t} = H(u, v, u_{,x}, v_{,x}, \ldots, u_{,5x}, v_{,5x}), \] \hspace{1cm} (67)
\[ u_{,x} + v_{,x} = P(u - v), \] \hspace{1cm} (68)

where the free parameters of (35)–(38) have been assumed to be \( \alpha = -1, \beta = 1. \) Using (64) and following the algorithm we obtain the first Bäcklund relation:
\[ u_{,t} - v_{,t} = 10v_{,x}^3 - u_{,x}^3 - 5(v_{,xx}^2 - u_{,xx}^2) - 10(v_{,x}v_{,xxx} - u_{,x}u_{,xxx}) + (v - u)_{,5x}. \] \hspace{1cm} (69)

Introducing \( U \) and \( V \):
\[ U = u - v, \] \hspace{1cm} (70)
\[ V = u + v, \] \hspace{1cm} (71)
\[ u = \frac{1}{2}(U + V), \] \hspace{1cm} (72)
\[ v = \frac{1}{2}(V - U). \] \hspace{1cm} (73)
we transform the right side of (67) to the following form
\[
H = -10 U_x \left( \frac{1}{4} U_{xx}^2 + \frac{3}{4} V_x^2 \right) + 5 U_{xxx} V_{xx} + 5 (U_x V_{xxx} + V_x U_{xxxx}) - U_{5x}.
\]

Introducing (70), (69) and (46)–(51) to (45) we derive the following form for (52):
\[
P^{(5)} U_x^3 + 10 P^{(4)} U_x^2 U_{xxx} + 10 P^{(3)} U_x^2 U_{xxx} + 15 P^{(2)} U_x U_{xxxx} + 10 (P^{(2)} - 1) U_x U_{xxx} + 5 (P^{(2)} - 1) U_x U_{xxx} - 5 \left[ P^{(2)} (P^{(2)} - 1) + P P^{(3)} \right] U_x^3 = 0,
\]

which becomes a tautology when \( P \) satisfies the following simultaneous set of equations:
\[
P^{(5)} = 0, \quad P^{(4)} = 0, \quad P^{(3)} = 0, \quad P^{(2)} - 1 = 0, \quad P (P^{(2)} - 1) = 0, \quad P^{(2)} (P^{(2)} - 1) + P P^{(3)} = 0.
\]

The solution of (72)–(77) is a square polynomial of \( U \):
\[
P(U) = \frac{1}{2} (U)^2 + C_1 U + C_2.
\]

where \( C_1 \) and \( C_2 \) are arbitrary constants. And finally, substituting (78) into (66) we derive the second relation of the Bäcklund transformation:
\[
u_{,x} = \frac{P(u - v)}{2} (u - v)^2 + C_1 (u - v) + E.
\]

### 4.2.2. The fifth order equation of the Caudrey hierarchy

The third equation of the Caudrey hierarchy is of the following form:
\[
\tilde{u}_{,t} + 45 \tilde{u}_{,x}^2 \tilde{u}_{,x} - 15 \tilde{u}_{,xxx} + 15 \tilde{u}_{,xxx} = 0.
\]

After using the transformation \( \tilde{u} \rightarrow u_{,x} \), (80) takes the conservation form:
\[
u_{,xt} + (15 u_{,xxx}^3 - 15 u_{,xxx} + u_{,5x}), = 0.
\]

Combining (81) and its twin equation for \( v \)
\[
v_{,xt} + (15 v_{,xxx}^3 - 15 v_{,xxx} + v_{,5x}), = 0,
\]

we derive the following form of (52):
\[
\frac{15}{2} (P^{(2)} - 1) U_x^3 + P^{(5)} U_x^3 + 5 \left( \frac{2 P^{(2)} - 3}{2} \right) U_{xxx} U_{xxx} + 5 \left( P^{(2)} - \frac{3}{2} \right) U_x U_{xxx} + 10 P^{(4)} U_x^3 U_{xxx} + 10 P^{(4)} U_x^3 U_{xxx} + 15 P^{(2)} U_x U_{xxx} - \frac{45}{2} P (P^{(2)} - 1) U_x U_{xxx} = 0.
\]

It is enough to analyze only two terms of (83) in order to show that the Bäcklund transformation of (81) does not exist in the frame of the plus–minus algorithm. Indeed, let us take into account coefficients of \( U_x U_{xxx} \) and \( U_x U_{4x} \):
\[
P^{(2)} - 1 = 0, \quad P^{(2)} - \frac{3}{2} = 0.
\]

It is impossible to satisfy (84) and (85) simultaneously.
5. The Gardner and Burgers equations

Let us put the following question: which PDEs that have the conservation form possess the Bäcklund transformation allowed by the plus–minus algorithm? The answer to this question depends on the form of the flux $\Phi(u)$. We show that in the frame of $\Phi(u)$’s form constrained to (92) the following PDEs: the Gardner, Burgers and two generalized KdV equations (Fordy, 1990) possess the Bäcklund transformation allowed by the plus–minus algorithm. The Gardner and Burgers equations are

\begin{align}
\bar{u}_t & - 6\bar{u}\bar{u}_x + \bar{u}_{xxx} - 12\delta\bar{u}^2\bar{u}_x = 0, \\
\bar{u}_t & + \bar{u}\bar{u}_x - \nu\bar{u}_{xx} = 0,
\end{align}

(86) (87)

where $\delta$ is an arbitrary real number and $\nu$ is positive.

A character of solutions of (86) depends on $\text{sign}(\delta)$. For $\delta = 0$, (86) becomes the Korteweg–de Vries equation (Drazin and Johnson, 1989). The substitution $\bar{u} \rightarrow u_x$ transforms (86) and (87) into the conservation forms:

\begin{align}
u_{xx} + (-3u_x^2 + u_{xxx} - 4\delta u_x^3)_{xx} = 0, \\
\frac{1}{2}u_x^2 - \nu u_{xx} = 0.
\end{align}

(88) (89)

5.1. Generalized equation possessing conservation form

Let us consider two uncoupled equations possessing the conservation forms (33) and (34):

\begin{align}
u_{xx} + \Phi(u)_x = 0, \\
\nu_{xx} + \Phi(u)_x = 0.
\end{align}

(90) (91)

Let the flux $\Phi(u)$ be of the sufficiently general form, namely,

\begin{align}
\Phi(u) & = s_1 u_x + s_2 u_{xx} + s_3 u_{xxx} + p_{1,1} u_{x}^2 + p_{1,2} u_x u_{xx} \\
& + p_{1,3} u_x u_{xxx} + p_{2,2} u_{xx}^2 + p_{2,3} u_x u_{xxx} + p_{3,3} u_{xxx}^2 \\
& + q_{1,1,1} u_x^3 + q_{1,1,2} u_x^2 u_{xx} + q_{1,1,3} u_x u_{xxx} + q_{1,2,2} u_x u_{xx}^2 \\
& + q_{1,2,3} u_x u_{xx} u_{xxx} + q_{1,3,3} u_x u_{xxx}^2 + q_{2,2,2} u_{xx}^3 \\
& + q_{2,2,3} u_{xx}^2 u_{xxx} + q_{2,3,3} u_{xx} u_{xxx}^2 + q_{3,3,3} u_{xxx}^2.
\end{align}

(92)

This is an algebraic sum of three homogeneous polynomials of the first, second and third degree, in variables $u_x, u_{xx}, u_{xxx}$. The coefficients $s_i, p_{i,j}$ and $q_{i,j,k}$ are arbitrary real numbers. The aim of this section is to determine all possible $\Phi(u)$ which allow the Bäcklund transformation in the form of (33) and (34).

5.2. Symbolic computations

A description of the plus–minus algorithm is presented in Section 3. Now we present its MAPLE implementation applied to generalized equation possessing conservation form (90) and (92). Expected results are: all possible partial differential equations possessing the Bäcklund transformation derived from the plus–minus algorithm and the corresponding Bäcklund transformations.

For convenience of presentation of the implemented code we change the notation for $P(U)$ and its derivatives: $P, P', P''$, $\ldots, P^{(n)} \rightarrow P[0], P[1], P[2], \ldots, P[n]$, respectively. Next, in the text and formulas we will return to the old notation. Below, we present the MAPLE code:

**Step 1**

Input: $r$ - the highest order of derivative, degrees of polynomial terms coded by the structures of the indexed variables $s[i], p[i,j], c[i,j,k]$.

Output: The flux (92).

\[
r := 3; \quad \Phi_0 := \sum (s[i]u[i], i = 1…r); \quad \Phi_1 := 0; \quad \Phi_2 := 0;
\]

for i from 1 by 1 to r do
    for j from i by 1 to r do
        Phi1 := Phi1 + p[i,j]u[i]u[j]
    for k from 1 by 1 to r do
        if i <= j and j <= k then
            Phi2 := Phi2 + q[i,j,k]u[i]u[j]u[k];
        od;
    od;
od:

\[
\Phi[0] := \Phi_0 + \Phi_1 + \Phi_2:
\]

• Step 2

Input: \( \Phi[0] \). Output: \( H \) function (41) - generator of the first Bäcklund transformation.

\[
ulist := [u[0]]; \quad uvlist := [u[0] = v[0]]; \quad VVflist := [V[0] = Vf[0]]; \quad r1 := r + 1;
\]

for i from 1 to r1 do
    ulist := [op(ulist), u[i]];
    uvlist := [op(uvlist), u[i] = v[i]];
    VVflist := [op(VVflist), V[i] = Vf[i]]:
    Rho[0] := subs(uvlist, Phi[0]);
    Phi[1] := diff(Phi[0], u[0])u[1];
end do:

\[
\Phi[1] := \text{collect}(\Phi[1], \text{ulist}, \text{distributed}, \text{factor});
\]

\[
\Phi[1] := \text{subs}(\text{ulist}, \Phi[1]); \quad \Phi_{\text{eq}[0]} := \Phi[0];
\]

for i from 1 to r do
    Phi[1] := subs(u[i] = (U[i] - alpha*V[i])/(1-alpha*beta), Phi[1]);
end do:

\[
H := -(\Phi[0] + alpha*\Phi[0]); \quad Hf := \text{subs}(VVflist, H);
\]

• Step 3

Input: \( H, \Phi[1] = \text{Diff}(\Phi[0], x), \ Rho[1] = \text{Diff}(\Rho[0], x) \) 

Output: result - generator of the left hand side of (45).

\[
V[1] := \text{P}(U(x)); \quad r2 := r + 3;
\]

for i from 2 to r2 do
    V[i] := (diff(V[i-1], x)):
end do;

for i from 2 to r2 do
    for j from r2 to 1 by -1 do
        V[i] := subs('@@'(D,j)(P)(U(x))=P[j], V[i])
    end do;
end do;

\[
V[1] := \text{P}[0]; \quad \text{result} := \text{P}[1]*H + beta*\Phi[1] + \Rho[1];
\]

• Step 4

Input: result. Output: result evaluated to the form of (52).

for i from 1 to 10 do:
    if diff(result, U[i]) <> 0 then Nu := i: fi
od:

\[
Ulist := [U[0]];
\]

for i from 1 to Nu do
    Ulist := [op(Ulist), U[i]]:
result := collect(result, Ulist, distributed, factor);
• Step 5
  **Input:** result, evaluated to the form of (52).
  **Output:** \( \{C[i]=0\} \) - system of simultaneous equations (53).

\[ C:=\text{coeffs}(\text{result}, \text{Ulist}, 'R'): \text{nn}: =\text{nops(\text{result})}: \text{sys}: =\{\}: \text{for} \ i \text{ from} \ 1 \text{ to} \ \text{nn} \text{ do} \]
\[ \text{sys}: =\text{sys union} \{C[i]=0\} \text{ od}; \]

• Step 6
  **Input:** sys=\( \{C[i]=0\} \). **Output:** \( S \) - set of solutions of \( \{C[i]=0\} \).

\[ S:=\text{solve(\text{sys})}: \text{n}: =\text{nops(\{\text{\textit{S}}\})}: \]

• Step 7
  **Input:** \( S \) - set of solutions, \( n \) - number of solutions, numroz \( <= n \) - label of the particular solution \( B \). **Output:** \( \text{DDD} \) the subset of the algebraic solutions for \( \text{alpha}, \text{beta}, \text{s}, \text{p}, \text{c} \) and \( \text{BBB} \) the subset of ordinary differential equations for \( \text{P}[0] \).

\[ \text{numroz}: =17: \text{B}: =\text{S}[\text{numroz}]: \text{convert(\text{B}, \text{list})}: \text{BBB}: =\{\}: \text{NN}: =\text{nops(\text{B})}: \text{convert(\text{BBB}, \text{list})}: \]
\[ \text{for} \ i \text{ from} \ 1 \text{ to} \ \text{NN} \text{ by} \ 1 \text{ do} \]
\[ \text{WL}: =\text{lhs(\text{B}[i])}: \text{WR}: =\text{rhs(\text{B}[i])}: \text{n}: =0: \]
\[ \text{for} \ j \text{ from} \ r2 \text{ to} \ 0 \text{ by} \ -1 \text{ do} \]
\[ \text{if} \ \text{diff(WL,} \text{P}[j]) \neq 0 \text{ or} \ \text{diff(WR,} \text{P}[j]) \neq 0 \text{ then} \ n:=1: \text{fi}: \text{od}: \]
\[ \text{if} \ n=1 \text{ then} \text{BBB}: =[\text{op(\text{BBB}),B[i]}]: \text{fi}: \text{od}: \]
\[ \text{BBB}: =\text{convert(\text{BBB}, \text{set})}: \text{B}: =\text{convert(\text{B}, \text{set})}: \text{DDD}: =\text{B minus BBB}: \]

• Step 8
  **Input:** \( \text{BBB} \) - the subset of the ordinary differential equations for \( \text{P}[0] \).
  **Output:** test for \( \text{BBB} \) to be a trivial set of equations \( \{\text{P}[0]=\text{P}[0],\text{P}[1]=\text{P}[1], \ldots,\text{P}[r]=\text{P}[r]\} \), representation of \( \text{P}[i]-\text{s by the derivative operators} \), \( \text{SP} \) - solutions of \( \text{BBB} \) for \( \text{P}[0] \).

\[ \text{Btest}: =[\{}: \]
\[ \text{for} \ i \text{ from} \ 0 \text{ to} \ r \text{ do} \]
\[ \text{Btest}: =\text{[op(\text{Btest}),P[i]=P[i]] \text{ od}:} \]
\[ \text{Btest}: =\text{convert(\text{Btest}, \text{set})}: \text{L}: =0: \]
\[ \text{if} \ (\text{Btest}==\text{BBB}) \text{ then} \]
\[ \text{SP}: =\text{P}[0]=\text{F}(\text{U}) \]
\[ \text{else} \]
\[ \text{for} \ i \text{ from} \ 1 \text{ to} \ r2 \text{ do} \]
\[ \text{BBB}: =\text{subs(P[i]='@@'(D,i)(F)(U),BBB)}: \text{od}: \]
\[ \text{BBB}: =\text{subs(P[0]=(F)(U),BBB)}: \text{SP}: =\text{dsolve(BBB,F)} \]
\[ \text{fi}: \text{fi}: \text{od}: \]
\[ \text{if} \text{L}=0 \text{ then} \text{ERROR('system is inconsistent, choose another solution')} \text{ fi}; \]

• Step 9
  **Input:** solution of \( \text{SP} \) for \( \text{P}[0] \), numsol - label of the solution for \( \text{P}[0] \), \( \text{DDD} \) - solutions for \( \text{c}, \text{p}, \text{s}, \text{alpha} \) and \( \text{beta} \), \( \text{H} \) - function expressed by its initial form \( \text{Hf}, \text{Phi}[0] \) - flux in its initial form \( \text{Phieg}[0] \).
  **Output:** \( \text{Ba}[0] \) - particular form of the considered Eq. (90) defined by the solution's set \( \text{B} \), \( \text{Ba}[1] \) and \( \text{Ba}[2] \) - the Bäcklund transformations.

\[ \text{numsol}: =1: \]
\[ \text{if} \ \text{whattype(\text{SP})=} \text{set} \text{ then} \text{P}[0]:=\text{rhs(\text{SP}[1]) \fi}: \]
\[ \text{if} \ \text{whattype(\text{SP})=} \text{list} \text{ then} \text{P}[0]:=\text{rhs(\text{SP}[1][\text{numpsol}]) \fi}: \]
\[ \text{DD}: =\text{convert(\text{DDD}, \text{list})}: \text{Hf}: =\text{subs(\text{DD}, \text{Hf})} : \text{for} \ i \text{ from} \ 1 \text{ to} \ r2 \text{ do} \]
\[ \text{Hf}: =\text{subs(P[i]=diff(P[0],'$$'(U,i)),\text{Hf}) \od}; \]
The number of the formal solutions $nops(S)$ is equal to 56. The subset of 28 solutions consists of inconsistence systems of ordinary differential equations for $P$. The next 22 solutions are trivial which means that the resulting $P$ is constant. And finally, the 6 solutions are nontrivial, however only 4 of them are independent. The full list of partial differential equations resulting from these solutions and the corresponding Bäcklund transformations is presented below in the following order: partial differential equation and the Bäcklund relationship.

\[
\frac{\partial}{\partial x}(s_1 u_x + s_3 u_{3x} + p_{1,1} u_{x}^2 + q_{1,1,1} u_{x}^3) = 0,
\]

(93)

\[
u_t - v_t = (u_x - v_x)(q_{1,1,1} u_{x}^2 + p_{1,1} u_{x} + q_{1,1,1} v_x u_x
\]

\[+ s_1 + v_x p_{1,1} + q_{1,1,1} v_{x}^2) - s_3(u_x - v_{3x}),\]

(94)

\[
u_x + v_x = C_2 \sin(\phi) + C_1 \cos(\phi) - 2/3 \frac{p_{1,1}}{q_{1,1,1}},
\]

(95)

where $\phi = \sqrt{\frac{q_{1,1,1}}{2s_3}}(u - v)$. (93)-(95) are valid both for positive and negative ratios $q_{1,1,1}/s_3$. The $q_{1,1,1} = 0$ is allowed only for $p_{1,1} = 0$, however, then the problem becomes linear. (93) represents the one parameter family\(^2\) of the nonlinear partial differential equations. The corresponding Bäcklund transformations are new result. For $s_1 = 0$, $s_2 = 1$, $p_{1,1} = -3$ and $q_{1,1,1} = -4 \delta$ the Gardner equation (88) is recovered.

The second result corresponds to the Burgers equation and constitutes no free parameter equation. For $s_1 = 0$, $s_2 = -v$, $p_{1,1} = \frac{1}{2}$ we recover (89). The found Bäcklund transformations are new:

\[
u_x + \frac{\partial}{\partial x}(s_1 u_x + s_2 u_{2x} + p_{1,1} u_{x}^2) = 0,
\]

(96)

\[
u_t - v_t = -s_1 u_x - s_2 u_{2x} + s_1 v_x + s_2 v_{2x} - p_{1,1} (u_x - v_x)(u_x + v_x),
\]

(97)

\[v_x = C_3 + C_4 e^{\frac{p_{1,1}(u_x)}{r_2}}.
\]

(98)

\(^2\) Two parameters can be scaled off by the scaling of the independent variables as well as the dependent ones.
Two last derived equations are linear:

\[
\begin{align*}
    u_{xx} + \frac{\partial}{\partial x}(s_1u_x + s_2u_{xx} + s_3u_{xx}) &= 0, \\
    u_t + \alpha v_t &= -(u_{3x} + \alpha v_{3x})s_3 - (u_{2x} + \alpha v_{2x})s_2 - (u_x + \alpha v_x)s_1, \\
    \beta u_x + v_x &= C_5(u + \alpha v) + C_6, \\
\end{align*}
\]  

and

\[
\begin{align*}
    u_{xx} + \frac{\partial}{\partial x}(s_1u_{xx}) &= 0, \\
    u_t + \alpha v_t &= -(u_x + \alpha v_x)s_1, \\
    \beta u_x + v_x &= F(u + \alpha v),
\end{align*}
\]  

where \(F(\cdot) \in C^1\) and \(C_1, C_2, \ldots, C_6\) in (95)–(101) are arbitrary constants. Despite the linearity of these systems they show something new. On the basis of (99) and (102) one can consider appearance of an arbitrary function \(F\) in the Bäcklund transformations. This property enables consideration of a boundary problem on the level of dual equations (103) and (104). Moreover, the occurrence of the meromorphic function parameters and \(F\) in the Bäcklund transformations brings the following question: Since \(F(\cdot)\) in (104) is an arbitrary function (nonlinear in general case) can one derive a nonlinear superposition formula for solutions of (102) basing on the Bianchi permeability theorem (see to (Rogers and Schief, 2002))?

### 6. Computational complexity

Time computational complexity (TCC) of the algorithm implemented in MAPLE (Section 5.2) consists of the two following contributions: the plus–minus algorithm TCC and the MAPLE functions TCC. There is not access to the MAPLE algorithms, therefore TCC can be estimated only experimentally. Assuming the polynomial degree of the flux (92) to be fixed we were varying the highest derivative order \(r\) and estimated TCC to be double exponential: \(T = O(\exp(\exp(r)))\). Whereas, the plus–minus algorithm itself exhibits exponential TCC: \(T = O(\exp(r))\). The double exponential TCC makes problem when one uses the functions \texttt{solve} in Step 6 for \(r > 3\). Therefore, in the commercial version of the code one must support the \texttt{solve} function with auxiliary procedures which reduce TCC to small enough value. The time computational complexity of the reduced problem should be of polynomial computational complexity.

#### 6.1. Reduction of \(\Phi\) by the term reduction method

In this Subsection we present a procedure satisfying the formulated above requirements. For this purpose we generate the flux \(\Phi(u)\) for \(r = 5\), using the Step 1 of the code. The derived flux consists of derivatives of \(u\) up to the fifth order and constitutes polynomial form of the third degree with respect to \(u_x, u_{xx}, u_{xxx}, u_{4x}, u_{5x}\), (we have not introduced the fourth degree terms represented by indices coefficients like \(a_{i,j,k,l}\)). Next, applying the Step 2–Step 5 we derive the system of 344 simultaneous algebraic nonlinear equations for \(\alpha, \beta, s_i, p_{ij}, q_{ij,k} \) and \(P, P', \ldots, P^{(5)}\). Temporarily \(P\) and its derivatives \(P^{(i)}\) are treated as independent unknown algebraic variables. At this stage of considerations, in the case of \(r = 3\) (see Section 5.2) we have applied the \texttt{solve} function (Step 6). However, since in the case of nonlinear algebraic equations the number 344 is large, first, we reduce the system's size to 13 irreducible equations and after that we use the \texttt{solve}. Applied here reduction is based on the reduction of monomials by the use of parameters occurring in monomial's coefficient.

**Definition.** A system of algebraic equations is reducible if it contains at least one monomial equation.
Table 1
An example of reduction.

<table>
<thead>
<tr>
<th>Loop’s number</th>
<th>Number of equations</th>
<th>Coefficients to be equal zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>344</td>
<td>$\alpha + 1$</td>
</tr>
<tr>
<td>2</td>
<td>306</td>
<td>$\beta - 1$</td>
</tr>
<tr>
<td>3</td>
<td>305</td>
<td>$q_{1,2.5}, q_{1,3.5}, q_{1,4.5}, q_{1,5.5}, q_{2,3.5}, q_{2,4.5}, q_{2,5.5}, q_{3,4.5}, q_{3,5.5}, q_{4,4.5}, q_{4,5.5}, q_{5,5.5}$</td>
</tr>
<tr>
<td>4</td>
<td>147</td>
<td>$q_{3,3.4}, q_{2,4.5}, q_{4,4.4}, q_{2,2.5}, q_{2,4.5}, q_{3,3.5}, q_{1,4.4}, q_{2,4.4}, q_{1,3.4}, q_{3,4.4}, P_{5.5}, P_{4.5}, P_{5.5}, P_{2.5}$</td>
</tr>
<tr>
<td>5</td>
<td>65</td>
<td>$q_{2,3.4}, q_{2,3.3}, q_{2,2.4}, P_{2.4}, P_{4.4}$</td>
</tr>
<tr>
<td>6</td>
<td>42</td>
<td>$q_{3,3.3}, q_{1,1.4}$</td>
</tr>
<tr>
<td>7</td>
<td>36</td>
<td>$q_{1,1.5}, q_{1,2.3}, P_{1.4}$</td>
</tr>
<tr>
<td>8</td>
<td>31</td>
<td>$q_{2,2.2}, q_{1,2.4}, P_{1.5}, P_{2.3}$</td>
</tr>
<tr>
<td>9</td>
<td>23</td>
<td>$q_{1,3.3}, q_{2,2.3}, P_{2.4}$</td>
</tr>
<tr>
<td>10</td>
<td>14</td>
<td>$P_{3.3}$</td>
</tr>
<tr>
<td>11</td>
<td>13</td>
<td>Irreducible</td>
</tr>
</tbody>
</table>

For instance in the considered example the left side of the equation number 334 is monomial of the $U^{(p)}$ variables:

\[
\beta \ (\alpha - 1) (\alpha + 1) \left( \frac{2 q_{3,5.5} P^{(4)} q_{4,4.5} P^{(2)} \beta}{-1 + \alpha \beta} \right) (U')^6 U^{(6)} = 0. \tag{105}
\]

(105) has to be satisfied for any values of $U'$ and $U^{(6)}$. Now, we can perform the first loop of reduction by choosing one of the four solutions obtained by equating to zero the coefficient of (105) and substituting that to the remaining 343 equations. Each choice initializes a reduction process. For instance for $\alpha = -1$ we reduce the system of 334 equations to the 306 ones. The considered system is still reducible, it contains the following monomial equation:

\[
3 q_{3,5.5} \beta \frac{(-1 + \beta)}{(1 + \beta)^2} (U^{(5)})^2 U^{(6)} = 0 \tag{106}
\]

and we perform the second loop of reduction. From the three possibilities we choose $\beta = 1$ and we get new a system consisting of 305 equations. After the 10 loops we have got the irreducible system of 13 equations.

**Definition.** A system of algebraic equations is irreducible if it does not contain any monomial equation.

A complete particular reduction process is presented in the Table 1. Code of the module performing the flux reduction by the reduction of monomials has to be substituted after the Step 1.

### 6.2. Consistency of the set of ODEs for P in a subspace of the equation coefficients

It has been assumed that all coefficients $S = \{\alpha, \beta, s, p, q\}$ present in the set of ODEs as well as $P, P', \ldots, P^{(r)}$ satisfy the formal algebraic equations (53). By the word *formal* we mean $P, P', \ldots, P^{(r)}$ treated as independent algebraic variables. Solutions sets for (53) are derived by the Step 6. Each set constitutes relations between $S, P, P', \ldots, P^{(r)}$. Now, the relations which contain $P^{(r)}$’s are separated and collected in the set of ODEs Step 7. In the Step 8 the ODEs system is solved by the function *dsolve*. All coefficients occurring in the considered equations belong to $S$ and they are treated as independent ones. All nontrivial integrable equations and their Bäcklund transformations resulting from considerations in Section 5.1 are presented in (93)–(104). The derived results are valid for any value of appearing coefficients $S$, excluding possible singular values corresponding to roots of denominators. The majority of the derived sets of ODEs are inconsistence. Therefore, we are able to derive only very few integrable equations. However, some of inconsistent equations can be made self-consistency by constraining the feasible space of $S$ to a subspace. By subspace of consistency (inconsistency) of the ODEs we call a collection of parameter values in which the set of ODEs is

---

3 For instance, $S$ for (109) reads $[s_2, s_4, p_{2.2}, q_{1,1.2}]$. 

---
consistent (inconsistent). Let \( R^N = R^{n_\alpha} \times R^{n_\beta} \times R^{n_\gamma} \times R^{n_\eta} \) be the full space of \( S \). It is rather obvious that in the generally case this is not a space of consistency. Therefore, in order to derive all integrable ODEs corresponding to the general form of flux, we need a tool for extracting the subspaces of consistency from \( R^N \). We are interested in a procedure which can be constructed on the basis of existing tools in the MAPLE. Progress in this direction can be achieved by supplementing the set of ODEs with the subset of equations governing the coefficients \( S \) occurring in the considered set:

\[
\{f_1 = 0 \ldots , f_M = 0\} \to \{f_1 = 0 \ldots , f_M = 0\} \cup \left\{ \frac{\partial s}{\partial U} = 0, \frac{\partial p}{\partial U} = 0, \frac{\partial q}{\partial U} = 0 \right\}, \tag{107}
\]

where, \( f_i = f_i(P, P' \ldots P^{(N)}) = 0 \) represents the \( i \)th ODE for \( P \), \( M \) is the number of ODEs and \( N \) is the highest order of derivatives occurring in (107). In order to illustrate application of (107), we consider the formal solution of (53) derived with the Step 6 and characterized by \( DDD \cup BBB \) where,

\[
DDD = \{\beta = -1, \alpha = 1, s_2 = 0, q_{1,2,2} = 0, p_{1,2} = 0, q_{1,1,3} = 0, p_{1,3} = 0, s_5 = 0, q_{1,5,5} = 0, q_{3,4,5} = 0, p_{5,5} = 0, q_{4,4,5} = 0, q_{3,5,5} = 0, q_{2,5,5} = 0, q_{4,4,4} = 0, q_{4,5,5} = 0, q_{5,5,5} = 0, q_{1,1,1} = 0, s_4 = s_4, p_{1,1,1} = 0\}, \tag{108}
\]

and

\[
BBB = \left\{ p^{(4)} = \frac{p^{n_2}}{P}, p^{(3)} = 1/3 P''(P'' P + 2 P^2), s_2 = s_4(-q^{1/2} + P'' P), q_{1,1,2} = -4 \frac{s_4 P''}{p}, p_{2,2} = -2 \frac{s_4 P''}{p^3}, p^{(5)} = q^{(5)}, P = P, P' = P', P'' = P'' \right\}. \tag{109}
\]

Therefore, we have to solve the following system:

\[
BBB \cup \left\{ \frac{d q_{1,1,2}}{d U} = 0, \frac{d p_{2,2}}{d U} = 0, \frac{d s_2}{d U} = 0, \frac{d s_4}{d U} = 0, \right\}, \tag{110}
\]

which possesses only one solution:

\[
\left\{ s_2 = 0, s_4 = -\frac{p_{2,2}^2}{q_{1,1,2}}, P = C_1 e^{1/2 q_{1,1,2}^2 U} \right\}. \tag{111}
\]

(111) leads to the following new integrable equation and its Bäcklund transformations:

\[
u_{xt} + \frac{\partial}{\partial x} \left( s_1 u_x - \frac{p_{2,2}^2 u_{4x}}{q_{1,1,2}} + p_{2,2} u_{2x}^2 + q_{1,1,2} u_x^2 u_{2x} \right) = 0, \tag{112}
\]

\[
u_t + v_t = -s_1 (v_x + u_x) - p_{2,2} (u_{2x}^2 + v_{2x}^2) - q_{1,1,2} (u_x^2 u_{2x} + v_x^2 v_{2x}) + \frac{(v_{4x} + u_{4x}) p_{2,2}^2}{q_{1,1,2}}, \tag{113}
\]

\[-u_t + v_x = C_1 e^{1/2 q_{1,1,2}^2 (t+x)} \frac{q_{1,1,2} (u_{2x})}{p_{2,2}^2}. \tag{114}\]

All equation parameters in (112)–(114) can be scaled off. Another new integrable equations may result from different reduction ways of the flux. For instance by replacing illustrated in Table 1 the following sublist \([q_{2,4,5}, p_{2,5}]\) with \([q_{1,1,3}, q_{1,2,2}, q_{1,1,2}, p_{1,2}, s_2, s_4]\) we derive unknown integrable equation and its Bäcklund transformations:

\[
u_{xt} = \frac{\partial}{\partial x} \left( s_3 u_{3x} + s_5 u_{5x} + p_{2,2} u_{2x}^2 + 2 p_{2,2} u_x u_{3x} + 3 \frac{p_{2,2}^2}{s_5} u_x^2 + 2 \frac{p_{2,2}^2}{5 s_5} u_{3x} \right), \tag{115}\]
\[ u_t - v_t = s_3 (v_{,3x} - u_{,3x}) + s_5 (v_{,5x} - u_{,5x}) + p_{2,2} (v_{xx}^2 - u_{xx}^2) + 2 p_{2,2} (v_{,x} v_{,3x} - u_{,x} u_{,3x}) + \frac{3 s_3 p_{2,2}}{s_5} (v_x^2 - u_x^2) + \frac{2 p_{2,2}^2}{5 s_5} (v_x^3 - u_x^3), \]  
\[ u_x + v_x = -\frac{1}{10} p_{2,2} (u + v)^2 + C_1 (u + v) + C_2. \]

Equation (115) represents the one parameter family. For the particular values of the parameters: \( s_5 = 1, \) \( s_3 = 0, \) \( p_{2,2} = -5 \) we reproduce the results of Section 4.2 for the fifth order equation of the Lax hierarchy.

Code of the module performing reduction of \( R^N \) to a subspace of consistency has to be substituted after Step 8.

7. Summary

The plus–minus algorithm was applied in two different ways. The first one consists in deriving the Bäcklund transformations for the known NPDE (Section 4). However, the purpose of the second way is double: (1) to derive NPDEs having assumed general structure determined by the highest degree of derivative and by the polynomial’s order of the flux \( \Phi \) with respect to derivatives as well as possessing the Bäcklund transformations, (2) to derive the Bäcklund transformations (Sections 5 and 6). As we have shown in the Sections 4 and 5 the plus–minus algorithm works quite well. However, an attempt to derive the Bäcklund transformations for the Sawada–Kotera equation failed. It means that there is no Bäcklund transformation in the frame of the plus–minus algorithm, which however, does not exclude its existence. Nonetheless, one must stress that the parameter values set of the Sawada–Kotera equation is an isolated point in the parameter space of (115).

The presented code of the MAPLE implementation is complete for \( r < 4 \) and for the consistent set of ODEs (assigned by BBB in the code ). Otherwise one must supply appropriate modules according to Section 6. Applying the plus–minus algorithm we have derived the auto-Bäcklund transformation only for the three new integrable equations. Nevertheless, systematic search for integrable equations in richer parameter spaces then presented here is open.

Efficiency of the algorithm enables us to widen its application to the class of generalized KdV equations (Fordy, 1990):

\[ u_t = \left( u_{xx} + a_0 (u) + a_1 (u) u_x + \frac{1}{2} a_2 (u) u_x^2 \right)_{,x}, \]

where \( a_i (u) \) are arbitrary functions. Unfortunately, in the frame of plus–minus algorithm and polynomial forms of \( a_0 (u), \) \( a_1 (u) \) and \( a_2 (u) \) the Bäcklund transformations do not exist.

At the end we would like to remind the three advantages resulting from the Bäcklund transformations: (1) The existence of the auto-Bäcklund transformations that is equivalent to the proof that the considered equation is integrable. (2) Having one solution of the considered equation (at least trivial one \( u = 0 \)), substituting that to the Bäcklund transformations one can easily generate a new solution. (3) Applying the Bianchi permutability theorem and using the auto-Bäcklund transformations that enables a derivation of the nonlinear superposition formula for the solutions.

References


