Online Chasing Problems for Regular Polygons

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Abstract

We consider a server location problem with only one server to move. In this paper we assume that a request is given as a region and that the service can be done anywhere inside the region. Namely, for each request an online algorithm chooses an arbitrary point in the region and moves the server there. Note that if every request is a single point and the server must exactly go there in the given order as conventional server problems, there is no choice for the online player and the problem is trivial. Our main result shows that if the region is a regular \( n \)-gon, the competitive ratio of the greedy algorithm is \( \frac{1}{\sin \frac{\pi}{2n}} \) for odd \( n \), and \( \frac{1}{\sin \frac{\pi}{n}} \) for even \( n \). In particular for a square region, the greedy algorithm turns out to be optimal.

Keywords. On-line Algorithms, Analysis of Algorithms, Competitive Analysis, Server Location Problem.

1 Introduction

In the \( k \)-server problem the player manages \( k \) servers by changing their location so that at least one of them serves the request at each time step [MMS90]. In this paper we study a 1-server problem on the \( xy \)-plane. One may think that if the player owns only one server, there is no choice in server location and therefore the competitive ratio is always one. This is obviously true if the player has to move the server to the exact position of the request. However, if it is enough to move it somewhere near the request, then the player does have a choice. Namely, in the online chasing problem, each request has a certain region such that the request can be served if the server moves or stays inside the region. The single server can choose an arbitrary point in the region in order to reduce the total travel distance. Natural applications include hierarchical transportation system with mobile destinations and relay service for low-power high-fidelity mobile terminals, such as logistics for mobile forces and routing of a broadcast van to follow up wireless camera crews.

Friedman and Linial [FL93] first studied this problem and proved that there exists a competitive online algorithm for the online chasing problem for any convex region. They began with a line chasing and extended the analysis to a half plane and general convex bodies. However, they did not consider any specific shape of the region or specific values of the competitive ratio (only such a result is an upper bound of 28.53 for the competitive ratio of the line chasing). The problem has not appeared in the literature since then.

Our Contribution. In this paper we focus our attention on regular \( n \)-gons (without rotation) as the shape of the region and investigate how we can take advantage of the information of the specific shape. It is shown that: (i) The greedy algorithm (GRD) that moves the server to the nearest position in the request region from the current position works quite well for a small \( n \). For instance, it achieves competitive ratios of 2, \( \sqrt{2} \), and 3.24 for a regular triangle, a square, and a regular pentagon, respectively. In particular for a square region, the ratio equals the lower bound for an arbitrary convex region [FL93], that is, GRD is optimal. Our result for a general \( n \) is \( \frac{1}{\sin \frac{\pi}{2n}} \) for odd \( n \) and \( \frac{1}{\sin \frac{\pi}{n}} \) for even \( n \). Interestingly, there is a remarkable difference between odd \( n \)'s and even \( n \)'s. (The reason is given at the end of Section 2.) (ii) Our analysis is tight, namely, there are request sequences for which the competitive ratio of GRD
asymptotically coincides with the above values. (iii) We give a definition of the work function algorithm and some preliminary observation.

Related Work. Several online versions of the Traveling Salesman Problem are considered in [AFL+01, ABL05]. Although our problem seems quite similar to them, there is a significant difference that in the online chasing also an offline algorithm has to serve requests in the given order. In a broader sense, the CNN problem [KT04, SS06] can be regarded as a special case of the online chasing problem: One can just set the region \((x, y) \mid x = r_x \text{ or } y = r_y\) for a request (scene) on \((r_x, r_y)\). Although the upper bound remains still large, [IY04] showed that with a nontrivial restriction the competitive ratio decreases to 9. Chrobak and Sgall provided a 5-competitive work function algorithm for the weighted 2-server problem which corresponds to a special case of the CNN problem [CS04]. Ausiello et al illustrated the application of the work function algorithm to some problems belonging to metrical service systems [ABL04]. It is also famous that the work function algorithm is \((2N - 1)\)-competitive and optimal for general metrical task systems where the size of the space is \(N\) [BLS92]. See [BCL02] for the recent progress in the \(k\)-server problem.

2 Greedy Algorithm

In the online chasing problem a request region \(D_i\) is given somewhere on the \(xy\)-plane at each time step \(i = 1, 2, \ldots, m\). Then an online algorithm \(\text{ALG}\) sets the sole server on a point \(A_i\) in region \(D_i\). For an input sequence \(\sigma = (D_1, D_2, \ldots, D_m)\), the cost of \(\text{ALG}\) is defined as

\[
\text{ALG}(\sigma) = \overline{SA_1} + \sum_{i=2}^{m} \overline{A_{i-1}A_i},
\]

where \(S\) is the initial location of the server and \(\overline{A_{i-1}A_i}\) denotes the Euclidean distance between \(A_{i-1}\) and \(A_i\). The offline problem, i.e., minimization of \(\overline{SA_1} + \sum_{i=2}^{m} \overline{A_{i-1}A_i}\) subject to \(A_i \in D_i\) given in advance for all \(i\), is solved in polynomial time if every region \(D_i\) is convex [NN94]. In this paper we let region \(D_i\) be the union of a regular polygon and its interior, and be congruent for all \(i\). The polygon does not rotate and therefore we can assume without loss of generality that its bottom side is always parallel to the \(x\)-axis. We set the length of sides in the polygon as one. It should be noticed that the size of the polygon does not matter to the online competitiveness. We use the definition of the competitive ratio as in [ST85, BE98]: The competitive ratio of an online algorithm \(\text{ALG}\) is \(c\) if there exists a constant \(b\) such that, for all input sequences \(\sigma\),

\[
\text{ALG}(\sigma) - c \cdot \text{OPT}(\sigma) \leq b,
\]

where \(\text{OPT}\) is an optimal offline algorithm.

The greedy algorithm for the online chasing problem is defined as below.

Algorithm \(\text{GRD}\): For each request \(i\), (i) if the server’s current position \(A_{i-1}\) is not in \(D_i\), then move the server to \(X \in D_i\) that minimizes \(\overline{A_{i-1}X}\). (ii) Otherwise, do not move the server.

For a regular polygon one can easily see that there are two types on the server’s behavior: The server arrives on one of the sides after vertical movement against that side, or on one of the vertices.

Theorem 1. For a regular \(n\)-gon \((n \geq 3)\) the competitive ratio of \(\text{GRD}\) is at most \(1/\sin \frac{\pi}{n}\) for odd \(n\), and \(1/\sin \frac{\pi}{n}\) for even \(n\).

Proof. We investigate the behavior of \(\text{GRD}\) and an arbitrary offline solution \(\text{OFF}\) for a given input sequence \(\sigma\). Let \(A_i\) and \(P_i\) denote the positions of \(\text{GRD}\)’s and \(\text{OFF}\)’s servers, respectively, for serving the \(i\)-th request. Also we use the notation of \(\Delta \text{GRD}_i = A_{i-1}A_i\) and \(\Delta \text{OFF}_i = P_{i-1}P_i\).
It is impossible to bound simply $\Delta GRD_i$ by $\Delta OFF_i$ since $\Delta GRD_i > 0$ and $\Delta OFF_i = 0$ may occur for some request. Therefore, we carry out an amortized analysis as in [ST85, BE98]. To prove $c$-competitiveness of GRD, it suffices to show that there exists a potential function $\{\Phi_i\}$ such that

$$f := \Delta GRD_i + \Phi_i - \Phi_{i-1} - c \cdot \Delta OFF_i \leq 0,$$

$\Phi_0 = 0$, and $\Phi_i \geq 0$ for $1 \leq i \leq m$. The reason is confirmed by just summing up the inequality; $\Phi_1, \Phi_2, \ldots, \Phi_{m-1}$ are cancelled and finally we have $GRD(\sigma) \leq GRD(\sigma) + \Phi_m \leq c \cdot OFF(\sigma)$.

When $\Delta OFF_i = 0$, the server of GRD always approaches the server of OFF since $\angle A_{i-1}A_i P_i$ is obtuse. Indeed, if $\angle A_{i-1}A_i P_i$ is acute, there exists a point $Y$ on the line segment $A_i P_i \in D_i$ such that $\overline{A_{i-1}Y} < \Delta GRD_i$, which contradicts the definition of GRD. So some function of the distance between the two servers appears to help, but it turns out that simple ones are insufficient. We adopt different potential functions depending on $n$. In what follows we shall complete first the proof for odd $n$. As a preliminary, let us define

$$\phi(x, y) := \sum_{k=0}^{n-1} \left| x \sin \frac{2k\pi}{n} + y \cos \frac{2k\pi}{n} \right|$$

and denote by $\phi[P, A]$ the value of $\phi$ for the displacement $(x, y)$ from point $P$ to point $A$. For odd $n$ we employ $\phi[P_i, A_i]$ as the potential function $\Phi_i$. In the later analysis we need the following four facts. Let $P$ be a fixed point, $AA'$ be any line segments such that $P$ lies on the line $AA'$ but not on the line segment excluding the end points and also $\overline{PA} > \overline{PA'}$ holds, and $BB'$ be any general line segments. (a) If $AA'$ is orthogonal to one of the sides of $D_i$, then

$$\phi[P, A'] - \phi[P, A] = \overline{AA'}/\sin \frac{\pi}{2n}.$$  

(b) If $AA'$ is parallel to one of the sides of $D_i$, then

$$\phi[P, A'] - \phi[P, A] = \overline{AA'}/\tan \frac{\pi}{2n}.$$ 

(c) If $BB'$ is parallel to one of the sides of $D_i$, then

$$|\phi[P, B'] - \phi[P, B]| \leq \overline{BB'}/\tan \frac{\pi}{2n}.$$ 

(d) Generally,

$$|\phi[P, B'] - \phi[P, B]| \leq \overline{BB'}/\sin \frac{\pi}{2n}.$$ 

These facts are confirmed by the property that for a fixed point $P$ and a positive constant $a$, the set of $A$ which satisfies $\phi[P, A] = a$ is a regular $2n$-gon with the centroid $P$, the Euclidean distance from $P$ to each of the vertices is $a \tan \frac{\pi}{2n}$, and one pair of the sides are parallel to the $x$-axis. Note that every side of the $2n$-gon is parallel to one of the edges of $D_i$. Furthermore, this property is implied by the two equations: For all $x$ and $y$,

$$\phi(x \cos \frac{\pi}{n} - y \sin \frac{\pi}{n}, x \sin \frac{\pi}{n} + y \cos \frac{\pi}{n}) = \sum_{k=0}^{n-1} \left| x \sin \frac{2(k+1)\pi}{n} + y \cos \frac{2(k+1)\pi}{n} \right| = \phi(x, y).$$
that is, the value of \( \phi \) is invariant with respect to rotation by \( \frac{2\pi}{n} \). Also, for all points represented as \((r \cos \theta, r \sin \theta)\) with a positive \( r \) and \( 0 \leq \theta < \frac{2\pi}{n} \),

\[
\phi(r \cos \theta, r \sin \theta) = r \left( \sum_{k=0}^{n-1} \sin \left( \frac{2k\pi}{n} + \theta \right) - \sum_{k=0}^{n-1} \sin \left( \frac{2k\pi}{n} + \theta \right) \right)
\]

\[
= \frac{1}{\sin \frac{\pi}{2n}} \left( r \cos \theta \cos \frac{\pi}{2n} + r \sin \theta \sin \frac{\pi}{2n} \right).
\]

Namely, the value of \( \phi \) equals \( 1/\sin \frac{\pi}{2n} \) times the distance between the point \((r \cos \theta, r \sin \theta)\) and the line \( x \cos \frac{\pi}{2n} + y \sin \frac{\pi}{2n} = 0 \) and therefore the points with the same value of \( \phi \) form a line segment parallel to this line.

Firstly, we claim that it is sufficient to see only the movement of the servers with \( A_{i-1} = P_{i-1} \), i.e., GRD’s server and OFF’s server start from the same location. Consider a point \( X \) on the line segment \( A_{i-1}P_{i-1} \) and the destination point \( Y \) by GRD when the server departs from \( X \). We show that

\[
f(X) := -XY + \phi[P_i, Y] - \phi[P_{i-1}, X] - c \cdot \Delta OFF_i
\]

does not decrease as \( X \) moves from \( A_{i-1} \) to \( P_{i-1} \). Let us investigate the following three possible cases of the position of \( Y \). (i) \( Y \) is on a side of \( D_i \). Let \( X' \) and \( Y' \) be the point after walking toward \( P_{i-1} \) from \( X \) and GRD’s destination starting from \( X' \), respectively. Assume that \( Y' \) is still on the same side as \( X' \). Let \((\theta + \gamma)\) denote the angle between \( A_{i-1}P_{i-1} \) and \( YY' \) by choosing \( \theta \) from \((2j - 1)\pi/2n \) \((j \in \mathbb{Z}, 1 \leq j \leq (n + 1)/2)\) and \( \gamma \) from \([-\pi/2n, \pi/2n]\). This choice guarantees that there exists a line \( l \) through \( P_{i-1} \) that is orthogonal to one of the sides of \( D_i \) and forms an angle of \( |\gamma| \) with \( XX' \) (see Figure 1). Considering the projection of \( XX' \) on \( l \), we obtain \( \phi[P_{i-1}, X'] - \phi[P_{i-1}, X] = -XX' \cos \gamma / \sin \frac{\pi}{2n} \) by using the fact (a). We have also \( XX' = YY' \cos(\theta + \gamma) \) and \( YY' = XX' \cos(\theta + \gamma) \). The fact (c) yields \( |\phi[P_i, Y'] - \phi[P_i, Y]| \leq YY' / \tan \frac{\pi}{2n} = XX' \cos(\theta + \gamma) / \tan \frac{\pi}{2n} \). Thus

\[
f(X') - f(X) \geq -[XX' - YY'] - [\phi[P_i, Y'] - \phi[P_i, Y]] - (\phi[P_{i-1}, X'] - \phi[P_{i-1}, X])
\]

\[
= \frac{XX'}{\sin \frac{\pi}{2n}} \cdot \left( -\cos(\theta + \gamma) \cos \frac{\pi}{2n} + \cos \gamma \right)
\]

\[
\geq 0.
\]

(ii) \( Y \) is on a vertex of \( D_i \). Let \( X' \) be the point after walking toward \( P_{i-1} \) and assume that its destination is also \( Y \). Since \( XX' + YY > XX' \) and \( |\phi[P_{i-1}, X'] - \phi[P_{i-1}, X]| \leq XX' / \sin \frac{\pi}{2n} \) from the fact (d), \( f(X') - f(X) \geq -XX' (1 + 1 / \sin \frac{\pi}{2n}) > 0 \). (iii) \( Y = X \in D_i \). Suppose that \( X \) moves to \( X' \in D_i \). By a similar argument as (i), we have \( \phi[P_{i-1}, X] - \phi[P_{i-1}, X'] \geq [\phi[P_i, X'] - \phi[P_i, X]]. \) Therefore \( f(X') - f(X) \geq 0 \).

The remainder is to seek \( c \) that satisfies \( f \leq 0 \) for all possible movements with \( A_{i-1} = P_{i-1} \). The following two cases should be considered depending on the position of \( A_i \). (I) \( A_i \) is on a side of \( D_i \). We can focus on the case in which \( P_i \) is also on the same side. The reason is as follows: Fix \( \Delta OFF_i \). The candidate of \( P_i \) is then on the intersection of the circle with the center \( P_{i-1} \) and \( D_i \). When \( P_i \) and \( A_i \) are on the same side, \( P_iA_i \) is maximized and so is \( \phi[P_i, A_i] \). By applying the fact (b) for \( P_iA_i \) we have

\[
f = \Delta GRD_i + \frac{1}{\tan \frac{\pi}{2n}} \sqrt{(\Delta OFF_i)^2 - (\Delta GRD_i)^2 - c \cdot \Delta OFF_i}.
\]
Figure 1: X moves from $A_{i-1}$ to $P_{i-1}$ ($n$ odd.)
which is maximized to
\[
\left(1 - \sqrt{c^2 + 1 - \frac{1}{\sin^2 \frac{\pi}{2n}}} \right) \cdot \Delta GRD_i
\] (1)
when \(\Delta OFF_i = \left(\frac{c}{\sqrt{c^2 - 1/\tan^2 \frac{\pi}{2n}}} \right) \cdot \Delta GRD_i\). (1) is non-positive if and only if \(c \geq 1/\sin \frac{\pi}{2n}\).

Under the setting of \(c = 1/\sin \frac{\pi}{2n}\), \(f\) is maximized to zero when \(\angle A_{i-1}P_iA_i = \frac{\pi}{2n}\) as shown in Figure 2. (II) \(A_i\) is on a vertex of \(D_i\). For \(c = 1/\sin \frac{\pi}{2n}\), \(f \leq 0\) is shown as followings. By a similar argument as (I) it can be assumed that \(P_i\) is also on the adjacent side to \(A_i\). We can write \(\angle A_{i-1}A_iP_i = \frac{\pi}{2} + \beta\) \((0 \leq \beta \leq \frac{\pi}{n})\). By the fact (b) we have

\[
f = \Delta GRD_i + \frac{P_iA_i}{\tan \frac{\pi}{2n}} \cdot \frac{1}{\sin \frac{\pi}{2n}} \cdot \sqrt{(\Delta GRD_i)^2 + P_iA_i^2 + 2P_iA_i \cdot \Delta GRD_i \cdot \sin \beta},
\]

which is maximized to
\[
g := \left(1 - \cos \beta + \sin \beta/ \tan \frac{\pi}{2n}\right) \cdot \Delta GRD_i
\]
when \(P_iA_i = (\cos \beta/ \tan \frac{\pi}{2n} - \sin \beta) \cdot \Delta GRD_i\). Since \(\frac{2n \beta}{\Delta GRD} = (1/ \tan \beta - 1/ \tan \frac{\pi}{2n}) \cdot \cos \beta \cdot \Delta GRD_i < 0\), \(g\) is maximized when \(\beta = 0\). By applying the analysis in (I) again we have \(f \leq 0\).

For even \(n\) we employ
\[
\phi_e(x, y) := \sum_{k=0}^{n/2-1} \left| x \sin \frac{2k\pi}{n} + y \cos \frac{2k\pi}{n} \right|
\]
as a potential function instead of \(\phi\). By a similar analysis \(f \leq 0\) can be proved for \(c = 1/\sin \frac{\pi}{n}\).

In particular for \(n = 4\), the lemma below implies that GRD is an optimal online algorithm. 

\[
\text{Figure 2: Case that } f \text{ is maximized (}\text{n odd}).
\]
Lemma 1 ([FL93]). There exists no online algorithm whose competitive ratio is smaller than $\sqrt{2}$ for any convex region on $\mathbb{R}^2$.

Although GRD may not be optimal for other values of $n$, we can show that the analysis in Theorem 1 is tight.

**Lemma 2.** For any $\varepsilon > 0$, there exists an input sequence $\sigma_1$ such that $\frac{GRD(\sigma_1)}{OPT(\sigma_1)} > 1/\sin \frac{\pi}{2n} - \varepsilon$ for odd $n$ and an input sequence $\sigma_2$ such that $\frac{GRD(\sigma_2)}{OPT(\sigma_2)} > 1/\sin \frac{\pi}{n} - \varepsilon$ for even $n \geq 6$.

**Proof.** Firstly, we prove the lemma for odd $n$. Let us label all the sides of the $n$-gon, beginning from the bottom side parallel to the $x$-axis as side 1, the next side in counterclockwise as side 2, and so on until side $n$. Then we consider an input sequence for which the server of GRD moves drawing a zigzag between side 1 and side $\frac{n+3}{2}$ of another $n$-gon. These sides form an angle of $\frac{\pi}{n}$. Although a zigzag travel can be forced by a simple alternation of these two requests, it is possible to obtain even a larger cost ratio by the following trick: On each request, we slide the lower $n$-gon slightly to the right so that the travel distance back from side 1 to side $\frac{n+3}{2}$ is equal to that from side $\frac{2n+3}{2}$ to side 1 (see Figure 3.) Suppose that the first cost from side $\frac{n+3}{2}$ to side 1 is $a$. For any $\delta > 0$, we can choose a positive integer $m$ such that for this input sequence $\sigma_1$ with length $2m$,

$$GRD(\sigma_1) = 2a \left( 1 + \cos \frac{\pi}{n} + \cos^2 \frac{\pi}{n} + \cdots + \cos^m \frac{\pi}{n} \right) > \frac{2a}{1 - \cos \frac{\pi}{n}} - 2a\delta.$$

On the other hand, an offline solution is just to move to P shown in Figure 3 by traveling a distance of $a/\sin \frac{\pi}{n}$, since all requests can be served there. Therefore $OPT(\sigma_1) \leq a/\sin \frac{\pi}{n}$.

We derive

$$\frac{GRD(\sigma_1)}{OPT(\sigma_1)} > \frac{2\sin \frac{\pi}{n}}{1 - \cos \frac{\pi}{n}} - 2\delta \sin \frac{\pi}{2n} = \frac{1}{\sin \frac{\pi}{2n}} - 2\delta \sin \frac{\pi}{2n}.$$

By choosing $\delta = \varepsilon/(2 \sin \frac{\pi}{2n})$ the lemma is proved for odd $n$. As for an even $n$-gon, we can composite a similar sequence $\sigma_2$ using side 1 and side $\frac{n+1}{2}$ with an angle of $\frac{2\pi}{n}$, which is acute for $n \geq 6$. Similarly, we can have $\frac{GRD(\sigma_2)}{OPT(\sigma_2)} > 1/\sin \frac{\pi}{n} - \varepsilon$. □

**Theorem 2.** For the online chasing problem the tight competitive ratio of the greedy algorithm is

$$\begin{cases} 
\frac{1}{\sin \frac{\pi}{n}}, & n \text{ is odd;} \\
\frac{1}{\sin \frac{\pi}{2n}}, & n \text{ is even,}
\end{cases}$$

if the request region is a regular $n$-gon ($n \geq 3$). In particular for the case of a square, the greedy algorithm is an optimal online algorithm.

Figure 4 illustrates the competitive ratio of GRD for a regular $n$-gon. It follows that the ratio for a regular $(2n + 1)$-gon is at least twice that for a regular $2n$-gon, which means that regular odd polygons are more difficult to chase. This difference is explained as follows: If one gives an input sequence that forces a zigzag between two sides forming as small angle as possible, the angle is $\frac{\pi}{n}$ for odd $n$-gons and $\frac{2\pi}{n}$ for even $n$-gons. Consequently, the server approaches more slowly to a position that the server does not need to move any more.

The competitiveness of GRD is independent of the size of the regular polygon. Indeed, the proof of Theorem 1 does not depend on the length of sides. No matter how small the polygon is, it can still oblige a zigzag travel as the proof of Lemma 2. Furthermore, GRD achieves the same competitive ratio as a regular $n$-gon for any irregular $n$-gon which is obtained by modifying
Figure 3: GRD for $\sigma_1$ ($n$ odd.)

Figure 4: Competitive ratio of GRD for a regular $n$-gon.
only the length of sides from a regular $n$-gon. For example, the competitive ratio is $\sqrt{2}$ for any rectangle.

If the rotation of the polygon is allowed, GRD is no longer competitive. Suppose that the polygon continues to rotate gradually by $\eta > 0$ radian for each step around a fixed point $P$. GRD’s server approaches $P$ turning around $P$, whereas the optimal solution is just to go to $P$. GRD’s travel distance can become arbitrarily long by choosing a small $\eta$.

As $n$ grows, one can see that the competitive ratio of GRD for an $n$-gon without rotation becomes large. We can show that for a circle GRD is not competitive with a simple input sequence.

3 Concluding Remarks

According to [ABL04], for a positive parameter of $\alpha$, the work function algorithm is defined as follows: Let $w_i(X)$ be

$$w_i(X) = \min_{B_j \in D_i, 1 \leq j \leq i-1} \left( SB_1 + \sum_{j=2}^{i-1} B_{j-1}B_j + B_{i-1}X \right).$$

For each request $i$, (i) if the server’s current position $A_{i-1}$ is not in $D_i$, then move the server to $X \in D_i$ that minimizes $A_{i-1}X + \alpha \cdot w_i(X)$. (ii) Otherwise, do not move the server. We can show that this algorithm performs better than GRD against an input sequence that forces a zigzag trip such as $\sigma_1$ in the proof of Lemma 2, and also that $\alpha$ should be smaller than $\cos \frac{2\pi}{n} / \sin \frac{\pi}{n}$ to achieve better performance for another example. Unfortunately, the analysis for general input sequences seems much more difficult.

As for the general lower bound, we cannot find a good way of exploiting a specific shape to improve $\sqrt{2}$ of [FL93]. However, if we drop the convex condition from a simple polygon, then we can get a lower bound of $2 - \varepsilon$ by a standard ski-rental argument. Note that if one uses an unbounded and non-convex region, a large lower bound is obtained as $6 + \sqrt{17}$ for the CNN problem [KT04].

Apparently many problems remain to be attacked, including (I) more formal analysis of the work function algorithm and (II) investigation of other shapes than regular polygons, especially a circle and some simple shape which is not convex.

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