PID-Type Controller Tuning for Unstable First Order Plus Dead Time Processes Based on Gain and Phase Margin Specifications

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Abstract—The control of unstable first-order plus dead-time (UFOPDT) processes using proportional-integral (PI) and proportional-integral-differential (PID) type controllers is investigated in this brief. New tuning rules based on the exact satisfaction of gain and phase margin specifications are proposed. The tuning rules are given in the form of iterative algorithms, as well as in the form of accurate, analytical approximations. Moreover, several specific functions, related to the crossover frequencies of the Nyquist plot and to the feasible design specifications for a given process, are derived. These functions, which are particularly useful for the general design of PI- and PID-type controllers for UFOPDT processes are accurately approximated, in order to simplify the tuning procedure. With the proposed approximations, the tuning rules reported in this brief require relatively small computational effort and are particularly useful for online applications.

Index Terms—Controller tuning, dead-time processes, gain and phase margins, proportional-integral-differential (PID) controllers, process control, unstable processes.

I. INTRODUCTION

Research on tuning methods of two- or three-term controllers for unstable first-order-plus-dead-time (UFOPDT) processes, has been very active in the last 15 years [1]–[12]. The most widely used feedback configurations for the control of such processes are the proportional-integral-differential (PID), the pseudo-derivative feedback (PDF) [12], and the proportional-proportional–integral–derivative (P-PID) controller [4], [7]. These control schemes are identical in practice, provided that the parameters of the controllers and of the prefilters needed in some cases are selected appropriately.

Two- or three-term controllers for UFOPDT processes have been tuned according to several methods, the most popular of them being several variations of the direct synthesis tuning method, the ultimate cycle method, the method based on the minimization of various integral criteria, etc. [1]–[13]. Moreover, due to the wide practical acceptance of the gain and phase margins (GPM) in characterizing system robustness, some tuning methods for UFOPDT models, based on the satisfaction of GPM specifications, have also been reported [6], [11], [14]. Unstable processes exhibit two gain margins, designated as the increasing (or upward) and the decreasing gain margin (or gain reduction margin). Nevertheless, in [6], a proportional-integral (PI) tuning method, based only on the phase and the increasing gain margin specifications is proposed. This method uses some approximations of the \( \tan^{-1} \) function to simplify the PI controller design, but due to the less accurate approximations used, it is not applicable for large values of the time delay and for small gain and phase margin specifications. In [11], an inner feedback loop with a \( P \) controller is used to stabilize the UFOPDT process and, subsequently, an outer PID controller is designed using known tuning methods for stable processes, in order to achieve one particular design specification \( \left( PM = 60^\circ, GM = 3 \right) \). In [14], inverse function mapping was done by fuzzy multilayer feed-forward neural networks, in order to tune the PID controller parameters according to the given phase and increasing gain margin specifications. The problem of designing PI controllers for UFOPDT processes, based on phase and increasing gain margin specifications could also be treated by applying the method reported in [15]. However, this method is a graphical one and it cannot be easily applied online. Finally, it is worth mentioning, that performance and robustness measures other than GPM have also been proposed as design specifications for PID controllers. Such a type of robust design methods, which rely on the maximum sensitivity function, the disturbance rejection measure and the control activity measure, are those reported in [16] and [17]. Preliminary investigations show that, some of the tuning methods reported in [16] and [17], although originally intended for stable open-loop systems, can be used in the case of UFOPDT models, but for a restricted region of values of the dead time \( d < 0.16 \) or \( d < 0.3 \), depending on the method) and with rather unsatisfactory closed-loop performance.

This brief is focused on the tuning of PI/PID-type controllers based on the exact satisfaction of gain and phase margin specifications. Here, the phase margin, the increasing, as well as the decreasing gain margins, have simultaneously been taken into account for the design of the PID controller for UFOPDT processes. The development of the proposed tuning methods starts from the design of a PI type controller that satisfies the desired specifications. These tuning methods are subsequently extended to PID controllers, when the derivative term of the controller is appropriately selected. Since the transfer function of a UFOPDT system includes a dead time term, the tuning formulas of a PID type controller cannot be directly expressed in an explicit analytic form. For this reason, to calculate the controller parameters, iterative algorithms are derived in this brief. One of the main innovations of this brief is that, in addition to the solution of the controller-tuning problem provided by iterative algorithms, an analytic expression that approximates the exact solution quite accurately and which is particularly useful for online

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applications is also provided. The reported approximate solutions have been obtained using appropriate curve-fitting optimization techniques. Moreover, the admissible values of the stability robustness specifications for a particular process are also given in analytic forms. The results obtained by performing a variety of simulation studies verify the accuracy and efficiency of the proposed methods for the design of PI- and PID-type controllers for a wide range of UFOPDT models.

II. FREQUENCY DOMAIN ANALYSIS FOR CLOSED-LOOP UFOPDT PROCESSES

Consider the control configuration of Fig. 1, where \( G_F(s) \) is the UFOPDT transfer function model, \( G_F(s) \) is the set-point prefilter transfer function associated with a series form PID controller \( G_C(s) \). In Fig. 1, \( K_r, d \) and \( T \) are the process gain, the time delay, and the time constant, respectively, while \( K'_r, \tau'_r, \tau'_p \), and \( \tau'_d \) are the gain, the integral reset time, and the derivative time of the controller, respectively. Note that, in practice, the control structure of Fig. 1 is identical to the PDF and P-PID controller configurations [4], [7], [12].

It is well known that the stability properties of the closed-loop system are not affected by the prefilter, which is used here, only to filter the set point and to prevent excessive overshoot in closed-loop responses to set-point changes, which are common in the case of UFOPDT systems [4]. Note also that the prefilter does not affect the regulatory control performance. Hence, in the analysis that follows, the presence of the prefilter is actually ignored and the obtained results hold for all PI- and PID-type control schemes.

In the sequel, to simplify the analysis and in order to facilitate comparisons, all system and controller parameters are normalized with respect to \( T' \) and \( K' \). The normalized parameters are \( T = 1, K = 1 \), and \( d = d'/T, \tau_I = \tau_I'/T, \tau_D = \tau_D'/T, \omega = \omega'/T, K_C = K'_r K'_C \).

The loop transfer functions of a UFOPDT system controlled by a PI- or a PID-controller, are given by

\[
G_{L,\text{PI}}(s) = \frac{K_C(\tau_I s + 1)}{\tau_I s (s - 1)} \exp(-ds) \tag{1a}
\]

\[
G_{L,\text{PID}}(s) = \frac{K_C(\tau_I s + 1)(\tau_D s + 1)}{\tau_I s (s - 1)} \exp(-ds). \tag{1b}
\]

These transfer functions are the starting point for the derivation of tuning methods proposed in this brief.

A. Frequency Domain Analysis for PI-Type Controllers

The argument and the magnitude of the loop transfer function \( G_{L,\text{PI}}(s) \) are given by

\[
\phi_L(\omega) = -\frac{3\pi}{2} - d\omega + \tan^{-1}(\omega) + \tan^{-1}(\tau_p\omega) \tag{2}
\]

\[
A_{L,\text{PI}}(\omega) = |G_{L,\text{PI}}(j\omega)| = K_C \sqrt{1 + \frac{T^2 \omega^2}{\omega^2}} \frac{1}{\tau_p \omega}. \tag{3}
\]

In Fig. 2, the Nyquist plots of \( G_{L,\text{PI}}(s) \) are depicted for several values of the parameters \( \tau_I \). From this figure it becomes clear that, for a specific \( d \), and for \( \tau_I \) greater than a critical value, say \( \tau_I_{\text{min}} \) (which can be determined as suggested in Section III-A), there exists two crossover points which determine the critical frequencies \( \omega_{\text{min}} \) and \( \omega_{\text{max}} \) and the critical gains \( K_{\text{min}} = 1/A_L(\omega_{\text{min}}) \) and \( K_{\text{max}} = 1/A_L(\omega_{\text{max}}) \). Moreover, for given \( d \) and \( \tau_I \), there exists one point of the Nyquist plot corresponding to the maximum argument \( \varphi_{L,\text{max}}(d, \tau_I) \). If \( \varphi_{L,\text{max}} < -\pi \), then the system can be stabilized with an appropriate choice of \( K_C \). From the change of the Nyquist plot with respect to \( \tau_I \), one can observe that the stability region is reduced when \( \tau_I \) is decreased, starting from the maximum region of stability when \( \tau_I \rightarrow \infty \) (i.e., a P-controller) and degenerate to a single point when \( \tau_I = \tau_I_{\text{min}} \). It is also well known that the stability region is reduced, when \( d \) is increased. In fact, when \( d > 1 \), a PI controller is not sufficient to stabilize the system [1].

The two critical frequencies \( \omega_{\text{min}} \) and \( \omega_{\text{max}} \) are the solutions of the equation \( \varphi_L(\omega_C) = -\pi \), or, equivalently, using (2), of the equation

\[
-\frac{3\pi}{2} - d\omega + \tan^{-1}(\omega) + \tan^{-1}(\tau_p\omega) = 0 \tag{4}
\]

when the values of the \( \tan^{-1} \) function are assigned in the range \((-\pi/2, \pi/2)\).

Having computed \( \omega_{\text{min}} \) and \( \omega_{\text{max}} \), one can determine the acceptable values for the controller gain \( K_C \), for which the

Fig. 1. PID controller configuration with a prefilter.
closed-loop system is stable. In particular, \( K_{\text{min}} < K_C < K_{\text{max}} \), where \( K_{\text{min}} \) and \( K_{\text{max}} \) are given by

\[
K_{\text{min}} = \tau_1 \omega_{\text{min}} \sqrt{(1 + \omega_{\text{min}}^2)/(1 + \tau_1^2 \omega_{\text{min}}^2)}, \quad (5a)
\]

\[
K_{\text{max}} = \tau_1 \omega_{\text{max}} \sqrt{(1 + \omega_{\text{max}}^2)/(1 + \tau_1^2 \omega_{\text{max}}^2)}. \quad (5b)
\]

The frequency \( \omega_p \) at which \( \varphi_L(\omega) \) is maximized is given by the maximum real root of equation \( d\varphi_L/d\omega = 0 \), having the value

\[
\omega_p = \sqrt{(\tau_1^2(1 - d) + \tau_1 - d) + \Delta} / 2\tau_1^2 \quad (6)
\]

where

\[
\Delta = (\tau_1^2(1 - d) + \tau_1 - d)^2 + 4\tau_1^2(1 + \tau_1 - d).
\]

Substituting \( \omega_p \) in (2), the respective maximum argument \( \varphi_L(\omega) \) can be computed.

Taking into account that the phase margin is defined as \( PM = \varphi_L(\omega_C) + \pi \), where \( \omega_C \) is the frequency at which \( G_{L,T}(\omega_C) = 1 \), one can easily conclude that the maximum phase margin for given \( d \) and \( \tau_1 \) can be obtained if the controller gain \( K_C \) is selected such that \( \omega_C = \omega_p \), that is

\[
K_C = \tau_1 \omega_p \sqrt{(1 + \omega_p^2)/(1 + \tau_1^2 \omega_p^2)} \quad (7)
\]

where \( \omega_p \) is given by (6). With this choice for \( K_C \), the maximum phase margin is given by

\[
PM(d, \tau_1) = -\pi/2 - d\omega_p + \tan^{-1}(\omega_p) + \tan^{-1}(\tau_1 \omega_p). \quad (8)
\]

The largest phase margin \( PM_{\text{max}} \) is obtained by a P-controller. Using (6) and (8) for \( \tau_1 \to \infty \), we obtain

\[
PM_{\text{max}}(d) = -d \sqrt{d - 1} + \tan^{-1}(\sqrt{d - 1}). \quad (9)
\]

Next we define the increasing gain margin \( GM_{\text{inc}} \) and the decreasing gain margin \( GM_{\text{dec}} \) as follows:

\[
GM_{\text{inc}} \equiv K_{\text{max}}/K_C \quad \text{and} \quad GM_{\text{dec}} \equiv K_C/K_{\text{min}}. \quad (10)
\]

These margins essentially describe the increasing and decreasing uncertainty of the process gain \( K \) for which the closed-loop system remains stable. Obviously, if \( 1/GM_{\text{dec}} < K < GM_{\text{inc}} \) and if there is no uncertainty in \( T \) and \( d \) (no phase uncertainty), then the system is stable. Note that, if \( K_C = (K_{\text{max}}/K_{\text{min}})^{1/2} \), then from (10), it follows that \( GM_{\text{dec}} = GM_{\text{inc}} = (K_{\text{max}}/K_{\text{min}})^{1/2} \).

Given the increasing and decreasing gain margins \( GM_{\text{inc}} \) and \( GM_{\text{dec}} \) of the closed-loop system, we next define the gain margin product

\[
GM_{\text{prod}} \equiv GM_{\text{inc}} GM_{\text{dec}} = K_{\text{max}}/K_{\text{min}}. \quad (11)
\]

A very useful property of \( GM_{\text{prod}} \) is that it does not depend upon \( K_C \). The largest value \( GM_{\text{prod,\max}} \) of the gain margin product is obtained with a P-controller. Since \( K_{\text{min}}(\tau_1 \to \infty) = 1 \) (in which case \( \omega_{\text{min}} = 0 \)), it follows that \( GM_{\text{prod,\max}} = K_{\text{max}} \). From (4) and (5) and for \( \tau_1 \to \infty \), this results in the following system of equations:

\[
GM_{\text{prod,\max}} = \sqrt{1 + \omega_{\text{max}}^2}, \quad d\omega_{\text{max}} + \tan^{-1}(\omega_{\text{max}}) = 0. \quad (12)
\]

**B. Frequency Domain Analysis for PID-Type Controllers**

In Fig. 3, the Nyquist plots of \( G_{L,T}(s) \) are presented for several values of the controller parameter \( \tau_D \). From these plots, it is clear that, for small values of \( \tau_D \), all stability margins increase with \( \tau_D \), whereas for larger values of \( \tau_D \), \( GM_{\text{inc}} \) starts to decrease. An extensive search (for all \( d \) and \( \tau_1 \) and \( \tau_D \)) has shown that for

\[
\tau_D < \tau_{D\text{\max}} \approx d(0.5 + 0.2779d + 0.2171d^2) \quad (13)
\]

all stability margins of a PID controller are larger than those obtained by a PI controller with the same \( \tau_1 \). Note that the limit provided by (13) can be readily used in order to evaluate an additional derivative term, which when added to a PI controller, leads to a closed-loop system with at least the same stability margins as those produced by the PI-only controller. Another very useful property of the Nyquist plot for both PI and PID controllers is that all stability margins increase when \( \tau_1 \) is increased.
The argument and the magnitude of the loop transfer function of a UFOPDT system controlled by a PID-type controller are given by

\[ \phi_{L,PID}(\omega) = -3\pi/2 - d\omega + \tan^{-1}(\omega) + \tan^{-1}(\tau_I\omega) \]
\[ + \tan^{-1}(\tau_D\omega) \]
\[ A_{L,PID}(\omega) = |G_{L,PID}(j\omega)| \]
\[ = K_C \frac{\sqrt{(1 + \tau^2_I\omega^2)(1 + \tau^2_D\omega^2)}}{1 + \omega^2}. \]  

The values of \(\omega_{\text{min}}, \omega_{\text{max}}, K_{\text{min}},\) and \(K_{\text{max}}\) can be calculated from the following relations:

\[ -\pi/2 - d\omega_M + \tan^{-1}(\omega_M) + \tan^{-1}(\tau_I\omega_M) + \tan^{-1}(\tau_D\omega_M) = 0 \]
\[ K_M = \tau_I\omega_M \frac{(1 + \omega^2_M)\left[\left(1 + \tau^2_I\omega^2_M\right)(1 + \tau^2_D\omega^2_M)\right]}{\left(1 + \tau^2_I\omega^2_M\right)\left(1 + \tau^2_D\omega^2_M\right)} \]  

where the subscript “\(M\)” is used for either “\(\text{min}\)” or “\(\text{max}\).” The frequency \(\omega_G\) at which \(A_{L,PID}(\omega_G) = 1\) is now given by the maximum real root of the biquadratic equation

\[ \omega^4_c \left(\tau^2_I - K^2_c\tau^2_D\right) + \omega^2_c \left[\tau^2_I - K^2_c\left(\tau^2_I + \tau^2_D\right)\right] - K^2_c = 0 \]  

obtained from (15). Furthermore, the phase margin of the system is given by

\[ PM(d, K_C, \tau_I, \tau_D) = -\pi/2 - d\omega_G + \tan^{-1}(\omega_G) \]
\[ + \tan^{-1}(\tau_I\omega_G) + \tan^{-1}(\tau_D\omega_G). \]  

Finally, the frequency \(\omega_p\), at which the maximum phase is obtained, for given \(d, \tau_I,\) and \(\tau_D,\) is calculated from the solution of the equation \(d\phi_{L,PID}/d\omega = 0,\) or equivalently from the solution of the equation

\[ -d + 1/(1 + \omega^2) + \tau_I/(1 + \tau^2_I\omega^2) + \tau_D/(1 + \tau^2_D\omega^2) = 0. \]

This equation results in a third-order linear equation with respect to \(\omega^2\) with only one acceptable (positive real) root. The maximum phase and gain margin in the case of a PID controller for a given \(\tau_D,\) are obtained when \(\tau_I \to \infty\) (PD controller). In particular, the maximum phase margin is given by (19) for \(\tau_I \to \infty(\omega_{\text{min}} = 0)\) and the maximum value of the product \(GM_{\text{prod, max}}\) is given by

\[ GM_{\text{prod, max}}(d, \tau_D) = \sqrt{(1 + \omega^2_{\text{max}})(1 + \tau^2_D\omega^2_{\text{max}})} \]  

where \(\omega_{\text{min}}\) and \(\omega_{\text{max}}\) are the crossover frequencies of the PD controller.

III. ALGORITHMS FOR THE TUNING PROBLEM AND THEIR APPROXIMATE ANALYTIC SOLUTIONS

A. Algorithms for Phase Margin Specifications and PI-Type Controllers

From the system of (6) and (8), one can specify the value of parameter \(\tau_I(d, PM^{\text{des}})\) which provides a maximum phase margin equal to the desired phase margin \(PM^{\text{des}}.\) To solve this system of equations, the following simple fixed-point algorithm is proposed.

**PM Algorithm:**

Step 1) Check if \(PM^{\text{des}}\) has an acceptable value, that is
\[ 0 < PM^{\text{des}} < PM_{\text{max}}, \]
where \(PM_{\text{max}}\) is given by (9).

Step 2) Start with an initial guess for \(\tau_I,\) e.g., \(\tau_I = \tau_{I,\text{min}}(d),\) where \(\tau_{I,\text{min}}(d)\) is defined in (22).

Step 3) Calculate the frequency \(\omega_p\) for this value of \(\tau_I\) using (6).

Step 4) Select the new value of \(\tau_I\) from the solution of (8) with respect to \(\tau_I,\) which is

\[ \tau_I = \tan[PM_{\text{des}} + \pi/2 + d\omega_p - \tan^{-1}(\omega_p)]/\omega_p. \]

Step 5) Repeat Steps 3) and 4) until convergence.

This algorithm always converges to the value of \(\tau_I\) which gives the desired maximum phase margin, if \(PM^{\text{des}}\) has an acceptable value (i.e., when \(0 < PM^{\text{des}} < PM_{\text{max}}\)).

Using the above algorithm for \(PM^{\text{des}} = 0,\) one can obtain \(\tau_{I,\text{min}}(d),\) which is the smallest value of \(\tau_I\) that renders the closed-loop system stable. An approximation provided by an optimization algorithm designed to minimize the normalized error

\[ \tilde{\tau}_{I,\text{min}}(d) \approx (0.0029 - 0.0682\sqrt{d} + 1.4941d)/(1.003 - d)^2, \]  

Note that the minimum normalized error (MNE) of this approximation is always less than 1.4% when \(d < 0.9.\)

For online tuning, in order to avoid the iteration involved in the PM algorithm, an analytic approximation of \(\tau_I(d, PM^{\text{des}})\) is proposed here. Taking into account that \(\tau_I(0) = \tau_{I,\text{min}}\) and \(\tau_I(d, PM_{\text{max}}) = \infty,\) an approximation was derived with the use of an optimization algorithm designed to minimize the normalized error

\[ PM_{\text{des}} = \left[PM^{\text{des}} - PM(d, \tilde{\tau}_I)\right]/PM^{\text{des}}. \]

This approximation is as follows:

\[ \tilde{\tau}_I(d, PM^{\text{des}}) = \tilde{\tau}_{I,\text{min}}(d) \left[1 + f_{PM}(d) \frac{PM^{\text{des}}/PM_{\text{MAX}}(d)}{1 - [PM^{\text{des}}/PM_{\text{MAX}}(d)]}\right] \]  

where \(f_{PM}(d) = (1 - 0.0153 + 0.436\sqrt{d} + 0.632d)/d.\) The normalized error \(PM^{\text{des}}\) obtained using the approximation (23) is less than 5% when \(PM^{\text{des}} > 0.2PM_{\text{max}}\) and \(d < 0.9.\)
B. Estimation of the Two Crossover Frequencies for PI-Type Controllers

The frequencies $\omega_{\text{min}}$ and $\omega_{\text{max}}$, in the case of PI-type controllers, can be obtained from the solution of equations (4), which, unfortunately, has no analytic solution. To circumvent this difficulty the following two iterative algorithms are proposed. The main steps of these algorithms are given below.

$\omega_{\text{min}}$ Algorithm:
Step 1) Start with an initial estimate for $\omega_{\text{min}} = (\tau_I - d(1 + \tau_I))^{-1/2}$.
Step 2) Calculate the error of this approximation using
\[ e_r = -\frac{\pi}{2} - d\omega_{\text{min}} + \tan^{-1}(\omega_{\text{min}}) + \tan^{-1}(\tau_I\omega_{\text{min}}). \]
Step 3) Take the new value of $\omega_{\text{min}}$ as $\omega_{\text{min}}(1 + e_r)$.
Step 4) Repeat Steps 2)–4 until convergence.

$\omega_{\text{max}}$ Algorithm:
Step 1) Start from a very large initial estimate of $\omega_{\text{max}}$, say $\omega_{\text{max}} = 10^4$.
Step 2) Using (4), assume the new value of $\omega_{\text{max}}$ is $\omega_{\text{max}} = [-\frac{\pi}{2} + \tan^{-1}(\omega_{\text{max}}) + \tan^{-1}(\tau_I\omega_{\text{max}})]/d$.
Step 3) Repeat Steps 2) and 3) until convergence.

These two algorithms always converge to the correct values of $\omega_{\text{min}}$ and $\omega_{\text{max}}$, respectively, when such values exist, or equivalently when $\tau_I > \tau_{I,\text{min}}$. In [12], the approximation $\omega_{\text{min}} = [\tau_I - d(1 + \tau_I)]^{-1/2}$ is proposed for the estimation of $\omega_{\text{min}}$. This approximation is satisfactory for values of $\tau_I > 2\tau_{I,\text{min}}$. An improved approximation of $\omega_{\text{min}}$ has been derived here using an additional correction factor

\[ \hat{\omega}_{\text{min}}(d, \tau_I) = f_{\omega_{\text{min}}}(d, \tau_I)/\sqrt{\tau_I - d(1 + \tau_I)} \tag{24} \]

where
\[ f_{\omega_{\text{min}}}(d, \tau_I) = 1 + \frac{[0.006 + 0.03d/(1.14 - d)\hat{\tau}_{I,\text{min}}]}{[0.973 + 0.05/(1 - d)\hat{\tau}_{I,\text{min}}]} \]

For the estimation of the second crossover frequency $\omega_{\text{max}}$, using the $\tan^{-1}$ function properties
\[ \tan^{-1}(x) = \tan^{-1}(1/x) + \pi/2 \]
\[ \tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}[(x - y)/(1 + xy)] \]
relation (4), can be rewritten in the form $\tan(d\omega_{\text{max}}) = (\tau_I/\omega_{\text{max}} - 1)/[\omega_{\text{max}}(1 + \tau_I)]$. Using the assumption that $\tau_I/\omega_{\text{max}} \gg 1$, and the approximation
\[ \tan(x) \approx x[0.9463 + 0.3854x/(0.5x - x)] \]
a first approximation for $\hat{\omega}_{\text{max}}$ is obtained as

\[ \hat{\omega}_{\text{max}} = \frac{(0.5\pi/d)(\tau_I/[(\tau_I + 1)d] - 0.9463)}{(\tau_I/[(\tau_I + 1)d] - 0.5660)^{-1}}. \]

To further improve the accuracy of this approximation, an additional correction function $f_{\omega_{\text{max}}}(d, \tau_I)$ is used to obtain

\[ \hat{\omega}_{\text{max}} = f_{\omega_{\text{max}}}(d, \tau_I) \frac{(0.5\pi/d)(\tau_I/[(\tau_I + 1)d] - 0.9463)}{(\tau_I/[(\tau_I + 1)d] - 0.5660)^{-1}} \tag{25} \]

where
\[ f_{\omega_{\text{max}}}(d, \tau_I) = (1 + 0.22d) \frac{[1 + (0.1 - 0.3\sqrt{d})(\tau_I/\tau_I\text{, min})^2]}{\tau_I}. \]

The MNE (defined by $\hat{\omega}_M = (\omega_M - \hat{\omega}_M)/\omega_M$, where $M$ is either $\text{min}$ or $\text{max}$) of the approximations presented above are smaller than 1.8%, when $d < 0.9$ and $\tau_I > 2\tau_{I,\text{min}}$. Moreover, when (24) and (25) are used for the estimation of the critical gains $K_{\text{min}}$ and $K_{\text{max}}$, then the MNE of the critical gains are less than 2.2%.

C. Algorithms for Gain Margin Specifications and PI-Type Controllers

Using relations (4), (5), and (11), one can specify the value of the parameter $\tau_I(d, G_{\text{prod}})$ that satisfies (11), for desired gain margin specifications $G_{\text{prod}}^{\text{des}}$ and $G_{\text{dec}}^{\text{des}}$. To evaluate $\tau_I(d, G_{\text{prod}})$ the following algorithm is proposed based on the dissection method.

$GM$ Algorithm:
Step 1) Check if the $G_{\text{prod}}^{\text{des}}$ has an acceptable value, that is if
\[ 1 < G_{\text{prod}}^{\text{des}} < G_{\text{prod, max}}. \tag{26} \]
Step 2) Start with $\tau_{I,1} = \tau_{I,\text{min}}$ and give at $\tau_{I,2}$ a very large value, say $\tau_{I,2} = 1000\tau_{I,\text{min}}$.
Step 3) Take the new value of $\tau_I$ as the average of $\tau_{I,1}$ and $\tau_{I,2}$, that is $\tau_I = 0.5(\tau_{I,1} + \tau_{I,2})$.
Step 4) Calculate the values of $\omega_{\text{min}}$ and $\omega_{\text{max}}$ using the $\omega_{\text{min}}$ algorithm and the $\omega_{\text{max}}$ algorithm for the given $\tau_I$, and obtain $K_{\text{min}}$ and $K_{\text{max}}$ from (5).
Step 5) Calculate the value of $G_{\text{prod}}$ from (11).
Step 6) If $G_{\text{prod}} < G_{\text{prod}}^{\text{des}}$, then $\tau_{I,1} = \tau_I$, or else $\tau_{I,2} = \tau_I$.
Step 7) Repeat Steps 2)–7 until convergence.

The GM algorithm always converges to the value of $\tau_I$ that attains the desired $G_{\text{prod}}^{\text{des}}$. If (26) is satisfied. The maximum gain margin product $G_{\text{prod, max}}$ can be calculated using the GM algorithm with $\tau_I \to \infty$. A very good estimate of the function $G_{\text{prod, max}}(d)$ is given by

\[ G_{\text{prod, max}}(d) = (0.5\pi/d)[1 + 0.4085d/(1 - 0.2864d)]^{-1} \tag{27} \]

where the MNE of this estimate is smaller than 0.03% when $d < 0.9$.

To obtain an accurate analytic approximate solution of $\tau_I(d, G_{\text{prod}}^{\text{des}})$, one can observe that $\tau_I(d, 1) = \tau_{I,\text{min}}$
and \( \tau_I(d, GM_{\text{prod}}^{\text{max}}) = \infty \). Based on these properties and with the use of an optimization algorithm designed to minimize the normalized error defined by \( GM_{\text{prod}} = \frac{[GM_{\text{res}} - GM_{\text{prod}}(d, \tau_I)]}{GM_{\text{res}}} \) the following approximation was derived:

\[
\hat{\tau}_I(d, GM_{\text{prod}}^{\text{res}}) = \hat{\tau}_{I, \text{min}}(d) \left( 1 + 0.65 \frac{A^{d+1}}{1 - A} \right) + g(d) \tag{28}
\]

where

\[
A = \frac{\sqrt{GM_{\text{prod}}^{\text{res}} - 1}}{\sqrt{GM_{\text{prod}}^{\text{max}}(d) - 1}}
\]

and where

\[
g(d) = 10^{-2}[-0.18 + 5\sqrt{d} - 32d + 75d^2 - 51d^3 + (-2.3d^2 + 3d^4)/(1 - d^2)]
\]

is a correction term needed when \( GM_{\text{prod}}^{\text{res}} < 1 + 0.6(GM_{\text{prod}}^{\text{max}} - 1) \). The MNE of \( GM_{\text{prod}}^{\text{res}} \) is less than 3% for \( d < 0.9 \) and \( GM_{\text{prod}}^{\text{res}} > 1 + 0.2(GM_{\text{prod}}^{\text{max}} - 1) \).

D. Approximations for PID-Type Controllers

For the calculation of the critical frequencies \( \omega_{\text{min}}(d, \tau_I, \tau_D) \) and \( \omega_{\text{max}}(d, \tau_I, \tau_D) \), in the case of a PID controller, the \( \omega_{\text{min}} \) algorithm and the \( \omega_{\text{max}} \) algorithm can be used with the obvious modifications (namely, in Step 2) of these algorithms, instead of using (4), the corresponding relations for the PID-type controllers, derived from (16), should be used. From \( \omega_{\text{min}} \) and \( \omega_{\text{max}} \), one can then calculate the values of \( K_{\text{min}} \) and \( K_{\text{max}} \) using relation (17). Hence, the GM algorithm can be used also for PID controllers if \( \tau_I \) is previously selected. Similarly, the PM algorithm can be used in the case of PID controllers, if in Step 3) of the algorithm the frequency \( \omega_{\text{wp}} \) is calculated from (20), and in Step 4) the new value of \( \tau_I \) is selected from the solution of (19), with respect to \( \tau_I \).

Although an exact design of the PID controller, which achieves a desired gain or phase margin, is possible using the aforementioned corresponding algorithms, our aim here is to derive an approximate solution, which satisfies the required specifications without resorting to iterative algorithms. To this end, the approximate solutions that have been previously derived for the PI controller will be appropriately modified, in order to be applicable in the case of PID controller design. For this purpose, the exponential term of \( G_L, \text{PID}(s) \) in (1) can be written in the form \( \exp(-\alpha ds) = \exp[-(1 - \alpha)ds]/\exp(\alpha ds) \).

When \( ds \ll 1 \), then the term \( \exp(\alpha ds) \) can be approximated by \( \exp(\alpha ds) \approx ds + 1 \) and \( G_L, \text{PID}(s) \) is approximated by

\[
G_L, \text{PID}(s) \approx \frac{K_C(\tau_IS+1)(\tau_DS+1)}{\tau_IS(s-1)} \times \left[ \frac{\exp(-(1-\alpha)ds)}{ds+1} \right]. \tag{29}
\]

If \( \tau_D = \alpha \), then the loop transfer function of a PID controller can be approximated (when \( \alpha \ll 1 \)) by that of a PI controller and a UFOPDT process with smaller time delay, namely \( d_P = d(1-a) = d - \tau_D \). Using this property, it is only when \( a < 0.15 \) that all approximations presented above are good enough and the MNE is always smaller than 5%. For larger values of the parameter \( \tau_D \), the following more accurate approximations have been derived and are valid in the range:

\[
0.02 < d < 0.9 \text{ and } 0 < a < 0.55 + d[0.31 - 0.00096/(1-d)]. \tag{30}
\]

These constraints are mainly imposed by the range in which the following approximations are valid.

1) The value of the parameter \( \tau_I \) which results in a marginally stable closed-loop system in the case of PID-type controllers, can be approximated by

\[
\hat{\tau}_{I, \text{min}}(d, \tau_D) = \hat{\tau}_{I, \text{min}}(d - \tau_D)[1 + a^3(0.367 + 1.78d)/(1 - a^2)] \tag{31}
\]

where \( a = \tau_D/d \) and \( \hat{\tau}_{I, \text{min}}(d - \tau_D) \) is given by (22). The MNE of this approximation is smaller than 2%.

2) The frequency \( \omega_{\text{min}} \) can be approximated by

\[
\hat{\omega}_{\text{min}}(d, \tau_I, \tau_D) = \hat{\omega}_{\text{min}}(d - \tau_D, \tau_I) \times \left[ 1 + \frac{(0.14 + 1.5d)a^3}{1 + 2\alpha^2} \hat{\tau}_{I, \text{min}}(\tau_I)^2 \right]. \tag{32}
\]

3) The frequency \( \omega_{\text{max}} \) can be approximated by

\[
\hat{\omega}_{\text{max}}(d, \tau_I, \tau_D) = \hat{\omega}_{\text{max}}(d - \tau_D, \tau_I) \times \left[ \frac{(5.3 - 0.41d)a^4}{1 - (0.1 + 0.5\alpha)d^2}[1.27 - 0.4(\hat{\tau}_{I, \text{min}}(\tau_I))^2] \right]^{-1} \tag{33}
\]

where \( \hat{\omega}_{\text{min}}(d - \tau_D, \tau_I) \) and \( \hat{\omega}_{\text{max}}(d - \tau_D, \tau_I) \) are given by (24) and (25), respectively. The MNE of these approximations is smaller than 3% when \( \tau_I > 1.2\hat{\tau}_{I, \text{min}} \). When approximations (32) and (33) are used in combination with (17) for the estimation of the critical gains \( K_{\text{min}} \) and \( K_{\text{max}} \), then the resulting MNE is also smaller than 3%.

4) An approximation of the function \( \tau_I(d, \tau_D, PM^{\text{res}}) \) is given by

\[
\hat{\tau}_I(d, \tau_D, PM^{\text{res}}) = \hat{\tau}_{I, \text{min}}(d, \tau_D) \times \left[ 1 + f_{PM}(d, \tau_D) \cdot \frac{PM^{\text{res}}/PM^{\text{max}}(d, \tau_D)}{1 - [PM^{\text{res}}/PM^{\text{max}}(d, \tau_D) \cdot 0.1]} \right] \tag{34}
\]

where

\[
f_{PM}(d, \tau_D) = \frac{[(1 + 0.4a - 0.22\alpha)(-0.0153 + 0.436\sqrt{d} + 0.63d)]}{d}
\]

and \( PM^{\text{max}}(d, \tau_D) \) is given by (19) for \( \tau_I \rightarrow \infty \). When \( \hat{\tau}_I \) is used, then the MNE of the phase margin obtained is always smaller than 5%. 

5) An approximation of the function \( \tau_I(d, \tau_D, GM_{\text{prod}}^{\text{des}}) \) is given by

\[
\hat{\tau}_I(d, \tau_D, GM_{\text{prod}}^{\text{des}}) = f_{GM}(d, \tau_D) \\
\times \left[ \frac{\tau_{\text{min}}(d, \tau_D)}{1 + 0.65\frac{Bd^{1.1}}{1 - B}} + g(d - \tau_D) \right]
\]

where

\[
B = \sqrt{GM_{\text{prod}}^{\text{des}} - 1} \\
\sqrt{GM_{\text{prod}}^{\text{max}}(d, \tau_D) - 1}
\]

and where the correction term \( f_{GM}(d, \tau_D) \) is given by

\[
f_{GM}(d, \tau_D) = 1 + a^2(2d^3 - 3.32d^2 + 1.2d - 0.27) \\
- \left( \frac{GM_{\text{prod}}^{\text{des}} - 1}{GM_{\text{prod}}^{\text{max}}(d, \tau_D) - 1} \right)^3
\]

while an approximation (with MNE 3%) of \( GM_{\text{prod}}^{\text{max}}(d, \tau_D) \) is given by

\[
GM_{\text{prod}}^{\text{max}}(d, \tau_D) = GM_{\text{prod}}^{\text{max}}(d = \tau_D) \\
\times \left[ 1 + \left( \frac{0.47 \pm 0.49d}{1.41 - 3.52d} \right)^{2} \right]^{-1}
\]

The MNE of \( GM_{\text{prod}} \) is smaller than 4%, when \( \tau_I(d, \tau_D, GM_{\text{prod}}^{\text{des}}) \) is used instead of \( \tau_I(d, \tau_D, GM_{\text{prod}}^{\text{des}}) \).

IV. PROPOSED TUNING METHODS

The tuning methods presented in this section are based on the analysis and observations made in Section II, as well as on the algorithms and approximations derived in Section III. In particular, for a certain normalized time delay \( d \) and given desired gain margin specifications \( GM_{\text{inc}}^{\text{des}} \) and \( GM_{\text{dec}}^{\text{des}} \) and phase margin specification \( PM^{\text{des}} \), the methods presented below provide the controller that satisfies these specifications.

A. Tuning Method for PI-Type Controllers Based on a Phase Margin Specification

When the only specification for the closed-loop system is the desired phase margin \( PM^{\text{des}} \), then according to the analysis presented in Section II, it is recommended to tune the PI controller in such a way that this single specification is achieved at the maximum phase margin corresponding to the frequency \( \omega_p \), namely, when \( \omega_C = \omega_p \). This way, the integral time constant \( \tau_I \) is the smallest possible that satisfies the specification and, hence, the controller derived this way gives the fastest possible response for both set-point tracking and regulatory control. The main steps of this tuning method are listed as follows.

**PM Tuning Method:**

- **Step 1** Check if the phase margin specification is achievable by a PI controller (i.e., \( 0 < PM^{\text{des}} < PM^{\text{max}}(d) \)).
- **Step 2** Given the time delay \( d \) and the phase margin specification \( PM^{\text{des}} \), calculate the integral time constant \( \tau_I(d, PM^{\text{des}}) \) of the PI controller using either (23), if \( PM^{\text{des}} > 0.2PM_{\text{max}} \) or, for better accuracy, the GM algorithm, if \( PM^{\text{des}} < 0.2PM_{\text{max}} \).
- **Step 3** With \( \tau_I \) known, calculate the corresponding frequency \( \omega_p \) from (6) and the controller gain \( K_C \) from relations (7). This completes the method.

The advantage of this method is that the obtained \( GM_{\text{inc}} \) and \( GM_{\text{dec}} \) are quite symmetrical, especially in the case where the stability region is small (large \( d \) or small \( PM^{\text{des}} \)). It should be noted here, that it is not recommended to choose \( PM^{\text{des}} < 0.2PM_{\text{max}} \) since, in this case, the system is not sufficiently robust.

B. Tuning Method for PI-Type Controllers Based on Gain Margin Specifications

This tuning method is applicable in the case where the specifications are described in the form of increasing and decreasing gain margins (\( GM_{\text{inc}} \) and \( GM_{\text{dec}} \)). From the results presented in Section III-C, it becomes clear that when both specifications are exactly achieved, the integral time constant \( \tau_I \) is the smallest that satisfies these specifications. The main steps of the so-called GM tuning method are as follows.

**GM Tuning Method:**

- **Step 1** Calculate the desired gain margin product \( GM_{\text{prod}}^{\text{des}} \) and check if this product is achievable by a PI type controller (i.e., check if \( 1 < GM_{\text{prod}}^{\text{des}} < GM_{\text{prod}}^{\text{max}} \)).
- **Step 2** Given the time delay \( d \) and \( GM_{\text{prod}}^{\text{des}} \), calculate the integral time constant \( \tau_I(d, GM_{\text{prod}}^{\text{des}}) \) of the PI controller, using either (28), if \( GM_{\text{prod}}^{\text{des}} > 1 + 0.2(GM_{\text{prod}}^{\text{max}} - 1) \), or, for better accuracy, the GM algorithm, if \( GM_{\text{prod}}^{\text{des}} < 1 + 0.2(GM_{\text{prod}}^{\text{max}} - 1) \).
- **Step 3** If \( \tau_I > 1.2\tau_{\text{min}} \), evaluate the crossover frequencies \( \omega_{\text{min}} \) and \( \omega_{\text{max}} \) using (24) and (25), and the critical gains \( K_{\text{min}} \) and \( K_{\text{max}} \) using (5). If \( \tau_I < 1.2\tau_{\text{min}} \), then the \( \omega_{\text{min}} \) algorithm and the \( \omega_{\text{max}} \) algorithm should be used for the estimation of \( \omega_{\text{min}} \) and \( \omega_{\text{max}} \), respectively. Note that when the GM algorithm is used to obtain \( \tau_I \), the algorithm produces the values of \( \omega_{\text{min}}, \omega_{\text{max}}, K_{\text{min}} \), and \( K_{\text{max}} \).
- **Step 4** The controller gain can now be evaluated from one of the following relations:

\[
K_C = K_{\text{max}}/GM_{\text{inc}} \text{ or } K_C = GM_{\text{dec}}K_{\text{min}}.
\]

This completes the method. □

When applying the GM tuning method, it is preferable to select symmetrical gain margin specifications (\( GM_{\text{inc}} = GM_{\text{dec}} \)). In this case, the phase margin of the closed-loop system is close to the maximum phase margin obtained when \( \omega_C = \omega_p \) especially in the case of small phase margins (large \( d \) and small \( GM_{\text{prod}} \)).
TABLE I

<table>
<thead>
<tr>
<th>PM Method</th>
<th>GM Method</th>
<th>PGM Method</th>
<th>PI-Nyquist Plots</th>
<th>PID-Nyquist Plots</th>
</tr>
</thead>
<tbody>
<tr>
<td>PM=0.3, GMm=4, GMs=2</td>
<td>( K_c = 5.2293 )</td>
<td>( K_c = 3.0225 )</td>
<td>( K_c = 3.1333 )</td>
<td>( K_c = 3.2544 )</td>
</tr>
<tr>
<td>PIS=0.1</td>
<td>( \tau_i = 0.3010 )</td>
<td>( \tau_i = 0.3184 )</td>
<td>( \tau_i = 0.3598 )</td>
<td>( \tau_i = 0.4509 )</td>
</tr>
<tr>
<td></td>
<td>( \tau_i = 0.2980 )</td>
<td>( \tau_i = 0.3216 )</td>
<td>( \tau_i = 0.3597 )</td>
<td>( \tau_i = 0.4509 )</td>
</tr>
<tr>
<td>PM=0.3, GMm=4, GMs=2, ( \tau_i = 0.0579 )</td>
<td>( K_c = 6.8658 )</td>
<td>( K_c = 3.0341 )</td>
<td>( K_c = 3.2544 )</td>
<td>( K_c = 3.2627 )</td>
</tr>
<tr>
<td>PIS=0.1</td>
<td>( \tau_i = 0.1196 )</td>
<td>( \tau_i = 0.1247 )</td>
<td>( \tau_i = 0.1709 )</td>
<td>( \tau_i = 0.1711 )</td>
</tr>
<tr>
<td></td>
<td>( \tau_i = 0.1183 )</td>
<td>( \tau_i = 0.1249 )</td>
<td>( \tau_i = 0.1711 )</td>
<td>( \tau_i = 0.1711 )</td>
</tr>
<tr>
<td>PM=0.15, GMm=1.3, GMs=1.5</td>
<td>( K_c = 1.5614 )</td>
<td>( K_c = 1.7581 )</td>
<td>( K_c = 1.6933 )</td>
<td>( K_c = 1.6933 )</td>
</tr>
<tr>
<td>PIS=0.5</td>
<td>( \tau_i = 6.5667 )</td>
<td>( \tau_i = 5.2836 )</td>
<td>( \tau_i = 6.9007 )</td>
<td>( \tau_i = 6.9007 )</td>
</tr>
<tr>
<td></td>
<td>( \tau_i = 6.5160 )</td>
<td>( \tau_i = 5.7115 )</td>
<td>( \tau_i = 6.9008 )</td>
<td>( \tau_i = 6.9008 )</td>
</tr>
<tr>
<td>PM=0.6, GMm=1.3, GMs=1.5, ( \tau_i = 0.3474 )</td>
<td>( K_c = 1.8108 )</td>
<td>( K_c = 1.7677 )</td>
<td>( K_c = 1.8108 )</td>
<td>( K_c = 1.8108 )</td>
</tr>
<tr>
<td>PIS=0.5</td>
<td>( \tau_i = 4.3471 )</td>
<td>( \tau_i = 0.8473 )</td>
<td>( \tau_i = 4.3471 )</td>
<td>( \tau_i = 4.3471 )</td>
</tr>
<tr>
<td></td>
<td>( \tau_i = 4.8747 )</td>
<td>( \tau_i = 0.8543 )</td>
<td>( \tau_i = 4.8747 )</td>
<td>( \tau_i = 4.8747 )</td>
</tr>
<tr>
<td>PM=0.018, GMm=1.07, GMs=1.07</td>
<td>( K_c = 1.0602 )</td>
<td>( K_c = 1.0681 )</td>
<td>( K_c = 1.0756 )</td>
<td>( K_c = 1.0756 )</td>
</tr>
<tr>
<td>PIS=0.9</td>
<td>( \tau_i = 77.17 )</td>
<td>( \tau_i = 51.12 )</td>
<td>( \tau_i = 97.46 )</td>
<td>( \tau_i = 97.46 )</td>
</tr>
<tr>
<td></td>
<td>( \tau_i = 74.56 )</td>
<td>( \tau_i = 59.00 )</td>
<td>( \tau_i = 93.70 )</td>
<td>( \tau_i = 93.70 )</td>
</tr>
<tr>
<td>PM=0.3, GMm=1.15, GMs=1.15, ( \tau_i = 0.6671 )</td>
<td>( K_c = 1.2162 )</td>
<td>( K_c = 1.4662 )</td>
<td>( K_c = 1.1515 )</td>
<td>( K_c = 1.1596 )</td>
</tr>
<tr>
<td>PIS=0.9</td>
<td>( \tau_i = 4.7672 )</td>
<td>( \tau_i = 2.9524 )</td>
<td>( \tau_i = 5.8637 )</td>
<td>( \tau_i = 5.7544 )</td>
</tr>
</tbody>
</table>

C. Tuning Method for PI-Type Controllers Based on Simultaneous Gain and Phase Margin Specifications

Clearly, it is not always possible to design a PI controller that gives a closed-loop system with stability margins \( GM_{inc}, GM_{dec}, \) and \( PM \) equal to some prespecified desired values \( GM_{inc,d}, GM_{dec,d}, \) and \( PM_{d}. \) This is due to the fact that it is not always possible to exactly assign three independent specifications with only two independent controller parameters, namely \( K_C \) and \( \tau_I. \) In this respect, our intention here is to present a PI controller tuning method (and in the following Section IV-D, a PID controller tuning method) that does not necessarily provide stability margins equal to the desired ones, but, in the general case, it attains actual stability margins larger than or at least equal to their prespecified desired values. Therefore, from this point on, simultaneous stability margins satisfaction is considered in the above sense, unless otherwise stated.

The proposed PI controller tuning method is based on the tuning methods presented in Sections IV-A and IV-B. The basic steps of the method are the following.

**PGM Tuning Method:**

Step 1) First, verify that the desired specifications are admissible, namely, if

\[
0 < PM_{d} < PM_{max} \\
1 < GM_{d} < GM_{max}
\]

and if there exists a value of \( K_C \) that can satisfy all three specifications, when \( \tau_I \to \infty. \)

Step 2) Calculate the two controllers obtained by the PM and the GM methods. If the controller with the largest value of \( \tau_I \) satisfies all three specifications, then this is the requested controller. In the opposite case continue with Step 3).

Step 3) Assume that \( K_{CPM} \) and \( \tau_{I,PM} \) are the controller parameters obtained from the application of the PM-tuning method and \( K_{CGM} \) and \( \tau_{I,GM} \) are the controller parameters obtained from the GM-tuning method. Then, if none of these two controllers satisfy all specifications, check which controller gives the largest gain \( K_C. \) In the case where:

1) \( K_{CPM} > K_{CGM}, \) then in order to satisfy all specifications with the smallest value of \( \tau_I, \) gradually increase \( \tau_I \) (starting from the \( \max(\tau_{I,GM,\tau_{I,PM}}) \)), while maintaining the same \( GM_{inc} \) (by choosing \( K_C = K_{max}/GM_{inc} \)), until the phase margin specification is also satisfied;

2) \( K_{CPM} < K_{CGM}, \) then gradually increase \( \tau_I \) (starting from the \( \max(\tau_{I,GM,\tau_{I,PM}}) \)), while maintaining the same \( GM_{dec} \) (by choosing \( K_C = K_{min}/GM_{dec} \)), until the phase margin specification is also satisfied (see examples in Table I).

Step 4) This completes the method.
Although there are several ways to select the controller parameters in order to satisfy (of course not exactly) all three specifications, the method presented here is preferred because it requires the smallest computational effort, since for a given \( \tau_f, PM(d, \tau_f) \) can be calculated exactly without the use of iterative algorithms.

It is noted here that when the tuning methods are applied using the approximations proposed in the brief, it is recommended to select the specifications slightly larger (i.e., by 5%) to compensate for the errors of the approximations. Moreover, in all PI tuning methods mentioned above, if the response obtained is too oscillatory (due to the small value of \( \tau_f \)), then, by increasing the value of \( \tau_f \), the damping of the closed-loop system increases. From the analysis presented in Section II, it is clear that, when \( \tau_f \) is increased, the resulting closed-loop system is more robust, and hence, all the stability robustness specifications are still satisfied.

D. Extension to PID-Type Controllers and USOPDT Systems

The simplest way to tune the PID controller is to first tune the parameters \( \tau_f \) and \( K_C \) of a PI controller, in order to satisfy the desired specifications, on the basis of the tuning methods presented above, and, subsequently, to add the derivative action. If \( \tau_D < \tau_{\text{max}} \), then the PID controller has a larger stability region than that of the initially tuned PI controller and it satisfies the desired gain and phase margin specifications.

To further improve the design of the PID controller, \( \tau_D \) should \textit{a priori} be selected on the basis of the designer’s knowledge relative to the process. If there are no restrictions imposed by the process, then it is recommended to select \( \tau_D \) as large as possible in the range defined by (30). This way, the resulting closed-loop system has the fastest possible response, from both the set-point tracking and the load attenuation point of view, and the minimum possible error in the case of regulatory control. In the case where the response of the system is too oscillatory, as it was previously suggested, it is preferable to increase the parameter \( \tau_f \). In this case, the response of the system becomes smoother and the resulting closed-loop system is more robust. When \( \tau_D \) is selected as suggested above, the remaining parameters are obtained using the methods presented in Sections IV-A–IV-C for PI-type controllers with the obvious modifications.

In practical applications, derivative action of the PID controller is often combined with a first-order filter. The PID-tuning methods presented here can easily be applied in combination with such a filter if the time delay used in the design is selected larger than the real process dead time by a factor of \( \tau_f \), where \( \tau_f \) is the time constant of the filter (e.g., \( \tau_f = 0.1 \tau_D \)).

Moreover, the tuning methods presented above can easily be applied for the control of unstable second-order plus dead-time (USOPDT) systems where an additional time lag \( \tau_S \) is included in the model of the system (i.e., \( G_S(s) = K e^{-\tau_S/d}/[(s - 1)(s\tau_S + 1)] \)). In the case of USOPDT systems, the loop transfer function is given by

\[
G_{L,S}(s) = \frac{K_C (\tau_f s + 1)(\tau_D s + 1)}{\tau_f s (\tau_f s + 1)(s - 1)} \exp(-d s). \tag{37}
\]

From (37) it is clear that a PID controller can be tuned based on the PI-tuning methods presented in this brief if the derivative time constant \( \tau_D \) of the controller is selected as \( \tau_D = \tau_S \). When \( \tau_S > 0.3 \), the obtained controller has sufficiently large derivative action to give a fast response. When \( \tau_S < 0.3 \), the proposed methods can be used for tuning a PID controller, by setting the time delay equal to \( d + \tau_S \). The resulting controller provides slightly larger stability margins than the desired ones.

V. Numerical Examples

Using the proposed methods it is possible, by selecting appropriate specifications, to produce all the PI and PID controllers proposed by other tuning methods, as for example, those reported in [1]–[12]. To this end one can use the GM-tuning method with the same specifications \( GM_{\text{inc}} \) and \( GM_{\text{dec}} \) as those attained from the other methods, and in the case of a PID controller by selecting the same \( \tau_D \). Moreover, other PID tuning methods which are based on gain and phase margin specifications [6], [11], do not rely on all three specifications (namely PM, \( GM_{\text{inc}} \), \( GM_{\text{dec}} \)). The tuning methods proposed in [6] and [11] are developed using simple approximations of the exponential and the \( \tan^{-1} \) functions and are accurate for small values of \( d \) (i.e., \( d < 0.3 \)) and large values for the specifications (e.g., \( PM_{\text{des}} > 0.6 PM_{\text{max}} \)). For these reasons, a comparison with existing tuning methods would not provide any information regarding the efficiency of the proposed methods. However, to check the validity and the accuracy of the proposed tuning methods, a series of numerical examples are presented in this section.

In particular, the PM-, GM-, and PGM-tuning methods are applied to three UFOPDT processes with normalized dead time \( 0.1, 0.5, \) and \( 0.9 \), for both PI- and PID-type controllers. The controller parameters obtained from the application of these tuning methods are presented in the left part of Table I for both the exact methods and the approximate methods. In the right part of Table I, the Nyquist plots of the resulting closed-loop systems are presented separately for the PI- and PID-type controllers. Solid and dashed lines are used for the exact and the approximate controller, respectively. The gain margin specifications are indicated by the symbol “\( \bigcirc \)” and the point on the unit circle which determines the phase margin specification is indicated by the symbol “\( \bigotimes \).” From all these polar plots, it becomes obvious that the approximate solution is very accurate and cannot be easily distinguished from the exact solution. Moreover, the use of the approximate solution makes the proposed tuning methods significantly faster (e.g., using Matlab, the GM method for PID controllers is 97 times faster when the approximations are used instead of the iterative algorithms with the same requirements for accuracy).

In the first example, with \( d = 0.1 \), a \( PM_{\text{inc}} = 0.3 \) rad, \( GM_{\text{inc}} = 4 \) (12.04 dB) and \( GM_{\text{dec}} = 2 \) (6.02 dB) are selected for both the PI and PID controllers, while \( \tau_D \) for the PID controller, is selected as the maximum, suggested by (30). Fig. 4(a) shows the responses for both the servo and the regulatory control problem, in the case of a PI and of a PID controller, when the
PGM method is applied. It is clear that for small \( d \), the servo responses are similar for the two controllers, whereas for the regulatory control problem, the derivative term is important since it significantly reduces the maximum error.

For a process with \( d = 0 \), we select \( GM_{\text{inc}} = 1.3 \) (2.28 dB) and \( GM_{\text{dec}} = 1.5 \) (3.52 dB) for both the PI and the PID controller, while \( PM_{\text{des}} \) is selected four times larger for the PID controller than the respective margin for the PI controller. The responses for the PI and PID controllers obtained from the application of the PGM method are shown in Fig. 4(b). From this figure, it is clear that by using a large derivative term, the response obtained is smoother. Moreover, significantly faster load attenuation is achieved, while satisfying substantially larger stability margins. The benefit of a large derivative term can also be verified from the last example, where a process with \( d = 0.9 \) is considered. In this example, the gain margin specifications for the PID controller are selected more than two times larger than those of the PI controller and the phase margin almost 17 times larger (see Table I). From the unit input load response presented in Fig. 4(c), it is clear that the PI controller is not practically useful, while the PID controller provides a rather acceptable response.

VI. CONCLUSION

In this brief, new tuning rules for PI and PID controllers are developed for UFOPDT processes, based on the exact satisfaction of phase and gain margin specifications. These methods take into account all three desired specifications (i.e., \( PM \), \( GM_{\text{dec}} \), and \( GM_{\text{inc}} \) and they are valid and accurate for a wider range of process and controller parameters than other existing methods based on stability margin specifications. The proposed tuning methods have the advantage that the derivative term is not imposed by the design methods and its choice is left to the process designer. Although iterative algorithms are required to accurately solve the tuning problem, analytic equations are also provided to avoid iterations in the case of online tuning. The simulation results obtained from the application of the proposed tuning methods to a variety of UFOPDT process models show the accuracy of the proposed methods and their efficiency in meeting the desired design specifications.

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REFERENCES


