DETECTION OF NARROW-BAND SONAR SIGNAL ON A RIEMANNIAN MANIFOLD

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ABSTRACT
We consider the problem of narrow-band signal detection in a passive sonar environment. The classical method employs a fast Fourier Transform (FFT) delay-sum beamformer in which the feature used in detection is the output of the FFT spectrum analyser in each frequency bin. This is compared to a locally estimated mean noise power to establish a likelihood ratio test (LRT). In this paper, we suggest to use the power spectral density (PSD) matrix of the spectrum analyser output as the feature for detection due to the additional cross-correlation information contained in such matrices. However, PSD matrices have structural constraints and describe a manifold in the signal space. Thus, instead of the widely used Euclidean distance (ED), we must use the Riemannian distance (RD) on the manifold for measuring the similarity between such features. Here, we develop methods for measuring the Fréchet mean of noise PSD matrices and optimum weighting matrices for measuring similarity of noise and signal PSD matrices. These are then used to develop a decision rule for the detection of narrow-band sonar signals using PSD matrices. The results yielded by the new detection method are very encouraging.

Index Terms— signal features, signal detection, Riemannian geometry

1. INTRODUCTION
A passive sonar system [1] consists of an array of hydrophones collecting information by "listening" to the acoustic signals emanated from the underwater targets. Using the collected signals, the sonar system performs the functions of detection, localization and classification of the targets. A common arrangement of sensors in a passive sonar system is in the form of a towed line of uniformly spaced hydrophones. Acoustic signals from different directions to the normal of the array are received at each sensor. The received signal is usually passed onto a Fourier analyzer which determines the frequency contents, followed by a frequency-domain beamformer for the directional features. The beam output may undergo further processing for signal detection to be performed [2].

The signals of a target vessel originate from the rotation of the propulsion system, vibration of the propeller, as well as from the rotation and vibration of auxiliary machinery in the vessel. These rotations and vibrations generate different sets of harmonics, propagated under multi-path environment. Thus, the received signals are usually considered as random narrow-band Gaussian. Acoustic noise is generated by wind, waves, currents, ocean creatures, and even distant shipping. In our consideration, the noise will be modeled as an ergodic zero mean Gaussian process with a power spectrum that is flat over the total system bandwidth.

In this paper, we examine the problem of detecting such sonar signals in noise, i.e., we examine the process of determining the presence of a signal in a particular frequency bin of a beam of the beamformer output. After reviewing the classical detection method, we introduce the idea of using the PSD matrices of the beamformer output as the feature for detection. However, PSD matrices are structurally constrained and are therefore not free elements in the signal space. Rather, they describe a manifold. Therefore, distance measurements between these matrices should be carried out along the surface of the manifold, i.e., we use the Riemannian distance (RD). Also, the mean of these matrices should be evaluated in terms of RD. The revision of these concepts leads us to derive the Fréchet mean of PSD matrices, from which, following similar strategy as the Neyman-Pearson test, a decision rule is established. Simulations results show that this novel detection method is very attractive.

2. SIGNAL MODEL AND CLASSICAL SIGNAL DETECTION
Consider a towed array of $P$ hydrophones. At the $p$th hydrophone sensor, the output discrete-time signal at $nT$ is denoted by $x_p(nT)$ where $T$ is the sampling period. The beam $b(nT, \theta)$ is the summation of the delayed time-domain signals coming from the same direction $\theta$ at each sensor, and can be written as [3]:

$$b(nT, \theta) = \sum_{p=1}^{P} x_p(nT - pd \sin \theta/c)$$

(1)

where $\theta$ is the angle between the normal to the arriving wavefront and the normal to the array axis, $d$ is the separation between sensors, and $c$ is the velocity of the sound in water. For
more stable power estimation, the output signal at each sensor is divided into $M$ segments: $X_{pm}(nT), m = 1, \ldots, M$. The corresponding beam is denoted as $b_m(nT, \theta)$. Taking the DFT of $b_m(nT, \theta)$, we have:

$$B_m(k\Omega, \theta) = \sum_{p=1}^{P} X_{pm}(k\Omega) e^{-j p (k\Omega/c) d \sin \theta}$$

Staggering the quantities $B_m(k\Omega, \theta)$ together, we form an $M \times 1$ vector

$$\beta(k\Omega, \theta) = [B_1(k\Omega, \theta) \cdots B_M(k\Omega, \theta)]^T \quad (2)$$

The power spectrum $Z_x$ of signals from direction $\theta$ at the frequency bin $k\Omega$ can be written as:

$$Z_x(k\Omega, \theta) = \frac{1}{M} |\beta^H \beta| = \frac{1}{M} \sum_{m=1}^{M} |B_m(k\Omega, \theta)|^2 \quad (3)$$

If $Z_x$ in a frequency bin is noise, we normalize its power to unity. The probability density function (PDF) is of a $\chi^2$-distribution and can be shown to be [3] [4]

$$f_{Z_x}(z) = \frac{M^M}{(M-1)!} z^{M-1} e^{-Mz} \quad \text{for } z \geq 0 \quad (4)$$

On the other hand, if $Z_x$ in a frequency bin is a signal having SNR $\rho$, the corresponding PDF can be shown to be

$$f_{Z_x}(z) = \frac{M/(1+\rho)^M}{(M-1)!} z^{M-1} e^{-Mz/(1+\rho)} \quad \text{for } z \geq 0 \quad (5)$$

Classical detection is a binary decision problem made between two possible hypotheses:

$H_0$: only noise is present, $H_1$: signal plus noise is present

Denoting by $f(z|H_0)$ and $f(z|H_1)$ respectively as the PDF of only noise, and of signal plus noise, then, the probability of false alarm and the probability of detection are given by

$$P_{FA} = \int_{Z_T}^{\infty} f(z|H_0) dz \quad P_D = \int_{Z_T}^{\infty} f(z|H_1) dz$$

The threshold, $Z_T$, is usually set larger than the mean noise power $\bar{Z}_\nu$, such that $Z_T = (1+r)\bar{Z}_\nu$, where $r$ is a positive constant power ratio. When a priori probability is not known, we use the Neyman-Pearson criterion so that the probability of false alarm is fixed at an acceptable value. For a maximum tolerable false-alarm constraint, we set the appropriate threshold $Z_T$ as shown in Fig. 1. If the true mean of the noise power $\bar{Z}_\nu$ is known, the detection of a signal is made by comparing the power in the frequency bin of interest, $Z_x(k\Omega)$, to the threshold $(1+r_0)\bar{Z}_\nu$. The decision rule is thus [3]

$$\frac{Z_x(k\Omega)}{\bar{Z}_\nu} \geq H_t \quad (1+r_0)$$

To obtain the nominal constant power ratio $r_0$, we use the expression of $f_{Z_x}(z)$ in Eq. (4) to evaluate the probability of allowable false alarm so that

$$P_{FA} = \int_{(1+r_0)\bar{Z}_\nu}^{\infty} \frac{M^M}{(M-1)!} z^{M-1} e^{-Mz} dz$$

In practice, an estimate $\hat{Z}_\nu$ of the mean noise power has to be made. $\hat{Z}_\nu$ is a random variable which depends on the data and the estimator applied. The expected values of $P_{FA}$ can be expressed in terms of $\hat{Z}_\nu$:

$$E[P_{FA}] = \int_{0}^{\infty} \int_{(1+r)\hat{Z}_\nu}^{\infty} f(z|H_0) f(\hat{Z}_\nu) d\hat{Z}_\nu dz$$

where $f(\hat{Z}_\nu)$ is the PDF of $\hat{Z}_\nu$. For a given value of $E[P_{FA}]$ together with a knowledge of $f(\hat{Z}_\nu)$, the value of $r$ can be obtained from Eq. (7). The decision rule then becomes:

$$\frac{Z_x(k\Omega)}{\hat{Z}_\nu} \geq H_t (1+r) \quad (8)$$

$\hat{Z}_\nu$ is a random variable which depends on the data and the estimator applied. There are several methods of estimating noise power [3] and each of these estimators is applied along the frequency axis of a beam of the data $Z_x$, and the corresponding noise power estimate is obtained for testing each frequency bin. Split window moving average (SWMA) filter is most commonly employed due to its simplicity and effectiveness. This method assumes that the frequency bins in the neighborhood of the bin of interest contain noise only, taking the average of $2L$ random samples, $L$-samples on either side of $k\Omega$. The coefficients of window size $2L + 1$ are given by

$$a_L = \cdots = a_{-1} = a_1 = \cdots = a_{L} = 1/2L$$

$$a_0 = 0$$

The SWMA improves the estimation by not taking the signal in the frequency bin $k\Omega$ into account for averaging. However, there may be signals falling within the neighboring $2L$ frequency bins, hence the SWMA estimate of the mean noise power will be biased due to the presence of these signals. Suppose within the $2L$ samples, there are $n$ signals, while the
remaining $2L - n$ are noise only samples, the characteristic function becomes:

$$
\Phi_{\hat{Z}_v}(\zeta, n) = \left[1 - \frac{j\zeta}{2LM}\right]^{-M(2L-n)} \prod_{i=1}^{n} \left[1 - \frac{j\zeta}{2LM}(1 + \rho_i)\right]^{-M}
$$

The PDF $f_{\hat{Z}_v}$ of the estimated power $\hat{Z}_v$ can be obtained from $\Phi_{\hat{Z}_v}(\zeta)$ by using the transform relationship [4]:

$$
f_{\hat{Z}_v}(\hat{Z}_v, n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{\hat{Z}_v}(\zeta, n) e^{-j\zeta\hat{Z}_v} d\zeta
$$

Let $Q$ and $P_n$ denote respectively the probabilities of finding a signal in a sample of the beam and of finding $n$ signals in the $2L$ samples other than that at $k\Omega$ of the window. The overall probability of false alarm $\hat{P}_{FA}$ given that the $k\Omega$ frequency sample contains noise only can be expressed as:

$$
\hat{P}_{FA} = \sum_{n=0}^{2L} P_n \cdot E[\hat{P}_{FA}(n)]
$$

where $P_n = \sum_{n=0}^{2L} \binom{2L}{n} Q^n (1 - Q)^{2L-n}$ and

$$
E[\hat{P}_{FA}(n)] = \int_{0}^{\infty} \int_{0}^{\infty} f(z|H_0) dz f_{\hat{Z}_v}(\hat{Z}_v, n) d\hat{Z}_v.
$$

When a signal exists in a particular frequency bin, what is concerned is whether this signal can be detected. For the SWMA estimator, since one of the $n$ signals falling within the filter window occurs at $k\Omega$ and has no effect on the outcome of the averaging ($a_0 = 0$), there are $n - 1$ signals in the rest of the $2L$ samples of the filter window that will be taken into the average. The average probability of detection is

$$
\hat{P}_D = \sum_{n=1}^{2L+1} P_{n-1} \cdot E[\hat{P}_D(n)]
$$

where $E[\hat{P}_D(n)] = \int_{0}^{\infty} \int_{0}^{\infty} f(z|H_1) dz f_{\hat{Z}_v}(\hat{Z}_v, n - 1) d\hat{Z}_v$.

Eqs. (11) and (12) respectively represent the overall average probabilities of false alarm and detection for the signal detection system using a SWMA filter for local noise estimation. Choosing different values of $r$ in these equations represents different chosen thresholds in the decision rule. A plot of the $\hat{P}_{FA}$ against $\hat{P}_D$ for different values of $r$ yields the receiver operating characteristic (ROC) [5] specifying the detection performance.

3. THE RIEMANNIAN MANIFOLD OF PSD MATRICES

We see from the last section that in classical signal detection, for the chosen signal feature of power spectrum $Z_x$, the mean of the normalized noise power spectrum has to be evaluated, a threshold of a prescribed distance from the noise mean has to be determined, and then, a decision rule can be established so that any observed signal feature within the distance of the threshold is classified as noise, otherwise it is a signal.

In this section, we propose the use of the PSD matrix as a feature for narrow-band sonar signal detection. This is because in addition to power information, PSD matrices provide extra correlation information between measured signals from different segments. For such a feature, the above concepts of distance, mean of noise feature, and threshold distance, have all to be revised and redefined.

3.1. PSD Matrices and the Riemannian Distance

We employ the PSD matrix $P$ of each frequency bin in a beam as the feature for detection. Such a feature can be obtained by forming the outer product of the beamformer output $\beta$ shown in Eq. (2) such that

$$
P(k\Omega, \theta) = \beta(k\Omega, \theta) \cdot \beta^H(k\Omega, \theta)
$$

Signal detection involves classification of an observed feature $P$ into signal or noise (binary hypothesis test). Thus, we need to measure the similarity between this feature of the observed signal and that of the noise by evaluating the distance between the two PSD matrices, the shorter is the distance, the greater is the similarity. Now, we note that $P(k\Omega, \theta)$ is Hermitian and positive semi-definite. Due to these structural constraints, the PSD matrices are not free elements in the signal space, rather they are constrained and form a manifold, $M$, in the signal space. Therefore, when we consider the distance between two PSD matrices, we should measure along the surface of the manifold. This concept is akin to finding the distance between two cities: The usual Euclidean distance (ED) (measured in a straight line) between two cities is neither informative nor accurate. The curve on the manifold linking two PSD matrices $P_m$ and $P_n$, having the minimum length is called a geodesic, and the length of the geodesic is called the Riemannian distance (RD) between the two points [6], i.e.,

$$
d_{R}(P_m, P_n) \triangleq \min_{\theta} \left\{ \ell(P(\theta)) \right\}
$$

In general, the evaluation of the RD on a manifold is difficult, however, by considering the decomposition of a PSD matrix $P$ such that $P = PP^H$ where $P$ is an $M \times M$ complex matrix in a Euclidean space $\mathcal{H}$, together with a certain Riemannian metric [6], three different closed-form expressions of RD for the PSD matrix manifold have been developed [7]. In particular, by choosing $P = P^{1/2}$, together with the Riemannian metric $\gamma_P(A, B) = \langle A, K \rangle$, such that $PK + KP + 2PKP = B$, the RD between two PSD matrices $P_m$ and $P_n$ is [7]:

$$
d_{R}^2(P_m, P_n) = trP_m + trP_n - 2tr[P^{1/2}_mP^{1/2}_n]
$$

Furthermore, it was found that such RD can be weighted so that prior information can be used to enhance or to deempha-
size certain parts of data, leading to a weighted RD given by
\[
\begin{align*}
d_{\text{WR2}}^2(P_m, P_n) &= \text{tr}(\Omega^H P_m \Omega) + \text{tr}(\Omega^H P_n \Omega) - 2\text{tr}[(\Omega^H P_m \Omega)^{1/2} (\Omega^H P_n \Omega)^{1/2}] \\
&= 2\text{tr}(\Omega^H P_m \Omega) - 2\text{tr}(\Omega^H P_n \Omega) + 2\text{tr}(\Omega^H P_m \Omega)^{1/2} (\Omega^H P_n \Omega)^{1/2}
\end{align*}
\] (15)
with \( W = \Omega^H \) being an \( M \times M \) positive definite weighting matrix. A weighted PSD matrix is given by \( P_W = \Omega^H P \Omega \).

The RD and weighted RD will be used as a distance measure between PSD matrices in our signal detection problem.

### 3.2. The Fréchet Mean of PSD Matrices

It is well-known [8] that the mean of a set of points has the minimum total squared distance to all the points in the set. Generalizing this concept to finding the mean of a set of PSD matrices \( \{P_n\}, n = 1, \cdots, N \), we arrive at:
\[
C_F = \arg \min_{C} \sum_{n=1}^{N} d^2 (P_n, C)
\] (16)
where \( C_F \) is known as the Fréchet mean of the set of PSD matrices and \( d \) denotes a generalized distance measure between two matrices. In this paper, we will apply the RD \( d_{\text{WR2}} \) and the weighted RD \( d_{\text{WR2}}^W \) given by Eqs. (14) and (15) to locate the Fréchet mean of the noise PSD matrices. The result is given by the following theorem:

**Theorem 1** For a group of PSD matrices \( \{P_n\}, n = 1, \cdots, N \), the Fréchet mean according to \( d_{\text{WR2}} \) is given by
\[
C_F = \bar{C}_F \cdot \hat{C}_F
\] (17)
where \( \bar{C}_F = \frac{1}{N} \sum_{n=1}^{N} P_n^{1/2} \).

**Theorem 2** For a group of PSD matrices \( \{P_n\}, n = 1, \cdots, N \), the Fréchet mean according to \( d_{\text{WR2}}^W \) is given by
\[
C_{F_w} = \bar{C}_{F_w} \cdot \hat{C}_{F_w} 
\] (18)
where \( \bar{C}_{F_w} = \Omega^{-H} \left[ \frac{1}{N} \sum_{n=1}^{N} (\Omega^H P_n \Omega)^{1/2} \right] \).

Proofs of Theorem 1 and 2 are shown in Appendix A. As described in Section 2, finding the noise mean is an essential step in sonar detection. We will apply both the Fréchet mean and the weighted Fréchet mean of noise PSD matrices in our new detection process.

### 3.3. Optimum Weighting Matrix for Detection

The purpose of a weighted distance in Eq. (15) is to highlight certain parts, and deemphasize other parts, of the features so as to increase the efficiency of detection. Optimum weighting for detection employs prior information to enhance similarities and dissimilarities between signal features. For the purpose of signal detection, we may define similarity as the amount of correlation between the two features. We can find a weighting matrix which maximizes the correlation between matrices of similar classes and minimizes the correlation between dissimilar classes.

We may define similarity by the correlation between two PSD matrices \( P_m \) and \( P_n \) such that
\[
\sigma(P_m, P_n) = \text{tr} (P_m^H P_n)
\] (19)
To facilitate detection, we find a weighting matrix which maximizes the correlation between noise PSD matrices, and minimizes the correlation between the PSD matrices of signal and noise. We can first divide the library of PSD matrices into noise group \( \mathcal{N} \) and signal group \( \mathcal{S} \). To determine whether a PSD matrix under observation is noise or signal, our detection algorithm makes the decision by judging the weighted RD measured between the mean noise matrix and the PSD matrix under observation, i.e., an unweighted RD between weighted mean noise and weighted PSD. Therefore we should find a weighting matrix which maximizes the average similarity between weighted noise and the weighted noise mean while minimizing the average similarity between weighted signal and the weighted noise mean. Specifically, since \( d_{WR2} \) is developed in the Euclidean space of \( \bar{P}_W \) such that \( \bar{P}_W = P_W^W \), we will find a weighting matrix optimizing the correlation between elements in this space, i.e.,
\[
\underbrace{\text{argmax}}_\Omega \underbrace{\frac{1}{N} \sum_{m \in N} \text{tr}(\tilde{P}_m^H \bar{C}_W \Omega)}_{N_p} - \underbrace{\frac{1}{N} \sum_{n \in S} \text{tr}(\tilde{P}_n^H \bar{C}_W \Omega)}_{N_s} \quad (20)
\]
where \( N_p \) and \( N_s \) are respectively the number of PSD matrices in the noise and signal groups in the library, and \( \bar{C}_W \) is the noise weighted Fréchet Mean in the isometric Euclidean space. A solution for the optimization problem of Eq. (20) is given in [9] such that
\[
\Omega_{op} = \sqrt{Q_{op} \Pi_{op}^{-1/2}}, \quad (21a)
\]
such that \( W_{op} = (Q_{op} \Pi_{op} Q_{op}^H)^{1/2} \) (21b)
where \( \Pi_{op} = \frac{1}{N_p} \sum_{m \in N_p} P_m \) and \( \Pi_s = \frac{1}{N_s} \sum_{n \in S} P_n \), and \( Q_{op} = [u_1, \cdots, u_M] \), with \( [u_1, \cdots, u_M] \) being the eigenvector matrix of \( \Pi_{op}^{-1/2} \).

### 4. SIGNAL DETECTION ON THE PSD MANIFOLD

Having established the concepts of distance and Fréchet mean on the PSD manifold, we are ready to set up the procedure of detection of narrow-band sonar signals using the PSD matrix as the detection feature:

1. From the \( M \) segments of the output beam of the delay-sum beamformer containing collected samples of normalized noise only from the direction \( \theta \), we form the noise only vectors \( \beta_0(k \Omega, \theta) \) for each frequency bin \( k \Omega \) as discussed in Eq. (2). Noise PSD matrices \( \{P_m\}_k \) for all frequency bins \( k \Omega \) are then formed using Eq. (13).
2. For a chosen SNR \( \rho \), we also generate \( M \) segments of the output beam of the delay-sum beamformer containing collected samples of signal+noise from the direction \( \theta \) forming the signal+noise vectors \( \beta (k\Omega, \theta) \) for each frequency bin \( k\Omega \). Also form signal+noise PSD matrices \( \{ P_{1k} \} \) for all frequency bins \( k\Omega \) using Eq. (13).

3. The Riemannian distance \( d_{R_2}(P_m, P_n) \), between any two of these PSD matrices is given by Eq. (14).

4. Using \( P_m \) and \( P_n \), the corresponding matrices \( \Pi_\nu \) and \( \Pi_s \) are evaluated. The optimum weighting matrix \( W_{op} \) can now be evaluated as in Eq. (21) and the optimally weighted Riemannian distance, \( d_{WR_2}(P_m, P_n) \), between any two of the PSD matrices can be obtained by applying \( W_{op} \) to Eq. (15).

5. With the establishment of \( d_{R_2} \) and \( d_{WR_2} \), the Fréchet mean \( C_F \) according to \( d_{R_2} \) and the Fréchet mean \( C_{F_w} \) according to \( d_{WR_2} \), of the normalized noise PSD matrices \( \{ P_m \} \) can be obtained using Theorem 1 and Theorem 2 respectively.

6. For signal detection, we can apply the Neyman-Pearson strategy by setting up a threshold RD (or weighted RD) \( d_{\text{test}} = s \) such that \( \alpha \% \) of the noise PSD matrix members in the histogram are outside this distance from the Fréchet mean \( C_F \). Thus, for a PSD matrix \( P_x \) under test, the decision rule is

\[
 d_{R_2}(C_F, P_x) \gtrless_{H_0}^H s
\]

with \( \alpha \% \) of false alarm. Similar detection procedure is repeated with \( d_{WR_2}(C_{F_w}, P_x) \).

7. By varying the values of \( s \), the whole range of false alarm rate with the corresponding detection rate can be examined and we obtain the ROC which allows us to evaluate the performance of the detection method.

8. In practice, we do not know the Fréchet mean, \( C_F \), of the noise PSD. Therefore, an estimate, \( \hat{C}_F \), has to be obtained. We employ the principle of SWMA estimate as described in Section 2 by examining the \( 2L \) PSD matrices in the neighbourhood of the PSD matrix under test. We obtain the estimate, \( \hat{C}_F \), by evaluating the Fréchet mean of the \( 2L \) neighbouring PSD matrices, assuming all of them are from noise only samples. Then, replacing \( C_F \) by \( \hat{C}_F \) in the decision rule of Eq. (22), we can carry out signal detection. A similar SWMA estimate is carried out for \( C_{F_w} \).

5. SIMULATION RESULTS

We now test the performance of our proposed detection scheme using simulated data. We generate a data sequence representing the received signal/noise sequence from the direction \( \theta = 30^\circ \). The data sequence is divided into \( M = 4 \) segments of equal length. For received data to be noise only, the generated data sequence is zero-mean normalized white Gaussian noise. For signal+noise, narrow-band Gaussian signals of certain frequencies are generated. Following Steps 1 and 2 in Section 4, we generate a library of noise and signal PSD matrices, \( \{ P_m \} \) and \( \{ P_n \} \), respectively. The collection of \( \{ P_m \} \) are divided into 5 categories: those of noise only, of SNR 3dB, 5dB, 10dB, and 15dB. The Fréchet mean \( C_F \) of the noise PSD matrices \( \{ P_m \} \) is then evaluated. The histogram of noise PSD distribution measured from \( C_F \) is plotted using the RD \( d_{R_2} \) and is shown in Fig. 2. From the noise PSD histogram, we can mark out a Riemannian distance \( s \) from \( C_F \) such that \( \alpha \% \) of the noise PSD matrices are outside. This distance is used in the decision rule of Eq. (22) for signal detection.

Fig. 3 shows the histograms of the signal PSD distributions (at 3dB, 5dB, 10dB, 15dB) measured from \( C_F \). This diagram illustrates how the signal PSD matrices are distributed further and further away from the noise Fréchet mean as the SNR increases. Following Step 4 in Section 4, we evaluate the optimum weighting matrix \( W_{op} \). Here, the two dissimilar groups of PSD matrices are respectively \( \{ P_m \} \) and \( \{ P_n \} \) where \( \{ P_m \} \) contains all normalized noise PSD matrices and \( \{ P_n \} \) contains the signal+noise PSD matrices of SNR 3dB and 5dB. We do not include the signal+noise PSD matrices of 10dB and 15dB in the set \( \{ P_n \} \) because the effect of optimum weighting of RD at high SNR is minimum. This optimum weighting matrix \( W_{op} \) is used in the evaluation of weighted RD between two PSD matrices. Again, from the noise PSD histogram based on the optimally weighted RD, we can mark out a weighted RD \( s \) away from \( C_{F_w} \) such that \( \alpha \% \) of the noise
It can be seen that while the ROC using $d_R^2$ also show the ROC for the classical signal detection scheme. The ROC of our detection method using $d_{R2}$ and $d_{WR2}$ are plotted in Fig. 4. On the same graph, we also show the ROC for the classical signal detection scheme. It can be seen that while the ROC using $d_{R2}$ shows significant improvement over that of classical detection method, the ROC using $d_{WR2}$ offers substantially greater improvements.

6. CONCLUSION

In this paper, we examine the problem of detection of narrowband sonar signals in Gaussian ambient noise. We suggest the use of the PSD matrix as the feature for detection. We reason that the usual Euclidean distance may not be appropriate for measurement of similarity between PSD matrices of signal and noise; rather, this should be measured using the Riemannian distance and/or its weighted form on the feature manifold. We further developed a closed form for the Fréchet mean of PSD matrices to facilitate the setting of threshold distance. The superior performance of such method in the detection of narrow-band sonar signals shows that this approach is a development in the right direction.

7. APPENDIX

Proof of Theorem 1

Solution to Problem (16) using $d_{R2}$ can be obtained by writ-