Model and algorithm of an inventory problem with the consideration of transportation cost

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Abstract

In this paper, we address the problem of deciding the optimal ordering quantity and frequency for a supplier–retailer logistic system in which the transportation cost as well as the multiple uses of the vehicles are considered. Based on the traditional economic order quantity (EOQ) formula, a modified EOQ model is set up and an algorithm for the model is presented. Computational results verify the proposed model as well as the efficiency of the algorithm.

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1. Introduction

Contemporary research in logistics management relies on an increased recognition that an integrated plan requires coordinating different functional specialties within a system. In keeping with this trend, we focus on the integration of production, inventory and transportation arising in a supplier–retailer logistic system.

Inventories are critical in the manufacturing as well as in the service industries. Either widespread shortage or slow-moving inventories (and often both) are clear signals of a company in trouble. Short-sighted attempts to correct these extremes, through frequent production changes or emergency orders, often just make matters worse.

From the point of providing high-level service for the customers and reducing the operation costs to the lowest, the coordination of inventories with other issues, such as production, order, shortage, etc, is
also of very important. In the general inventory models, costs of such issues are usually accounted according to the following assumptions: The production cost is proportional to the quantity of products produced. The ordering cost, which refers to the charge for preparing of production, is independent of the quantity ordered. The inventory cost (shortage cost) is proportional to the quantity of products stored (out of order) as well as the duration for which these items are stored (stock out).

When products are delivered from the supplier to the consumer, transportation costs are incurred. In the traditional economic order quantity (EOQ) model, the transportation cost is calculated together with the production cost, or with the ordering cost. However, in a practical logistic system, the transportation cost of a vehicle includes both of the fixed cost and the variable cost. The fixed cost, which is considered to be a constant sum in each period, refers to some necessary expenses such as parking fare and rewards to the driver. As to the variable cost, it depends mainly on the oil consumed, which is related directly to the distance traveled. In short, considering the real condition, it is unreasonable to assume that the transportation cost is proportional to the quantity delivered or is a constant sum. For example, the optimal ordering quantity $y^*$ gained according to the general EOQ formula may be partly loaded by the vehicles and the cost of the logistic system may not be the lowest.

In this study, we address the problem of minimizing the production, inventory and transportation costs for a supplier–retailer logistic system. Unlike most of the prior inventory models, both of the fixed cost and the variable cost of the vehicles are accounted in the model. In addition, since the multiple use of the vehicle can share the fixed cost and may reduce the total cost arising in the logistic system, the permitted working duration of the vehicle as well as the travel time of such vehicle along the trip is also considered.

It is worth noting that inventory lot-sizing models where transportation costs are considered explicitly are rich. Burns, Hall, Blumenfeld and Daganzo (1985) developed analytic methods for minimizing total inventory and distribution costs under know demand. Abdelwahab and Sargious (1990) presented a selected dispatching policy each time a demand arrives so as to specify a shipment release schedule. Russell and Krajewski (1991), Higginson and Bookbinder (1994) identified practical operating routines for temporal consolidation such that the service requirements are met and scale economics can be realized. An extensive review of the routing and inventory models for freight distribution problem was given by Baita, Ukovich, Pesenti, and Favaretto (1998). Cetinkaya and Lee (2000) presented an analytical model for coordinating inventory and transportation decisions in a vendor-managed inventory system. However, these earlier papers do not consider the effects of multiple uses of the vehicles in which both the fixed cost and the variable costs are considered. Thus, from a practical point of view, there is still a need for analytical models that take into account the transportation cost in the inventory model.

The study can also be viewed as an example of channel coordination problem (CCP), which has been studied by both marketing and supply chain researchers. The supply chain literature, recently surveyed by Tsay, Steven, and Naren (1998), develops an economic rationale for why retailers and vendors may choose different levels of inventory investment.

There is a large set of studies on channel coordination. Thomas and Griffin (1996) provide a comprehensive review of the topic. A few notable studies related to buyer–vendor coordination include Banerjee (1986), Kohli and Park (1994), Weng (1995) and Anupindi and Bassok (1998) which analyze the separate or joint optimal ordering policies with discount schedules. Through channel coordination, inventory and other production costs will likely be reduced while capacity utilization is increased, as demonstrated by Xu, Dong and Evers (2001), and Waller, Johnson and Davis (1999). However, no studies have examined the optimal ordering scheme under CCP. And the impact of CCP on both supplier and retailer aspects is seldom evaluated either.
In this paper we present a modified economic ordering quantity model for a single supplier–retailer system in which the transportation costs is calculated based on the practical operation. And an algorithm for such a model is designed so as to find the optimal solution within limited steps. The remainder of this paper is organized as follows. The modified model and some lemmas are presented in Section 2. An algorithm used to find the optimal solution for the model is given in Section 3. A few examples are followed in Section 4, and conclusions is given in Section 5.

2. The model

In the supplier–retailer problem considered, we assume demand is static during the whole planning horizon and the products can be delivered after they have been ordered for \( L \) time, where \( L \) is called as the lead time and it is a predetermined parameter. In addition, the replenishment should be completed without product shortage occurring. There exists a set of homogeneous vehicles with limited capacity for delivery. In this study, we assume the vehicles are hired from the third logistic party whenever the delivery needs to be finished. The objective of the study is to minimize the whole average costs of the logistic system on the long planning horizon.

Denote the demand quantity per unit time (referring to a day in this study) by \( b \); and \( y \) as the ordering quantity of products. Then the highest inventory occurs when \( y \) is received, and after \( y/b \) time periods the inventory quantity will be reduced to zero. Denote the capacity of the vehicle as \( p \); the fixed cost of such a vehicle as \( f \); in this study, \( f \) represents the lowest cost of hiring such a vehicle in a working day, no matter how long the vehicle will be traveled. And the variable transportation cost per trip is \( c \). \( U \) is the permitted working duration per day, \( t \) is the traveling time along each trip. \( K \) is the cost of preparing an order, \( h \) is the unit inventory cost per unit time, and \( s \) is the unit production cost. Then the problem can be formulated as the following model \((P_1)\):

Minimize \( TCU_0 = \frac{K}{y/B} + \frac{sy}{y/B} + \frac{nc}{y/B} + \frac{mf}{y/B} + \frac{hy}{2} \) \hspace{1cm} (1)

subject to  
\( (n - 1)p < y \leq np \) \hspace{1cm} (2)
\( d \leq U/t \) \hspace{1cm} (3)
\( md \geq n \) \hspace{1cm} (4)
\( m, n, d \) are integers \hspace{1cm} (5)

where \( TCU_0(y) \) is the total cost per unit time associated with the logistic system, \( y \) is the ordering size, \( m \) is the number of vehicles used for delivering \( y \), \( n \) is the total trips of these vehicles.

Constraint (2) specifies the number of trips finished by the vehicles for delivering quantity \( y \). Since \( d \) in constraints (3) and (4) represents the maximum trips each vehicle is able to complete in a working day, we can regard it as a predetermined parameter in the following paragraphs and sections. Let \( m = g(n) \), model \( P_1 \) can further be expressed as the following \( P_2 \) model:

Minimize \( TCU(y) = \frac{K}{y/B} + \frac{sy}{y/B} + \frac{nc + fg(n)}{y/B} + \frac{hy}{2} \) \hspace{1cm} (6)
subject to Eqs. (2) and (5) and the following constraint (7):

\[ g(n) = m_d \left[ \frac{n}{d} \right] \]  

(7)

In the following, we give an example to illustrate each item in function \( TCU(y) \).

In this example, assume \( d = 2 \), and the vehicle’s number used for delivering the quantity \( [y \in (n_i p, (n_i + 2)p)] \) is same, where \( n_i \) is a positive integer. We can deduce from the assumption that

\[ fg(n_i + 1) = fg(n_i + 2) \]

and

\[ fg(n_i + 3) = fg(n_i + 4) = fg(n_i + 2) + f \]

Based on the above assumption, the relationship of each item in \( TCU(y) \) with \( y \) is given in Fig. 1.

It can be seen from Fig. 1 that \( TCU(y) \) is not a continuous function, it cannot be differentiated during the whole interval. However, observe that when \( n \) is fixed, the value of \( g(n) \) is also a constant. Denote \( TCU(y) \) with a given \( n \) as \( TCU_n(y) \), that is

\[ TCU_n(y) = \frac{K}{y/\beta} + \frac{sy}{y/\beta} + \frac{cn + fg(n)}{y/\beta} + \frac{hy}{2} \]

(8)

It is obvious that when \( y > 0 \), function \( TCU_n(y) \) is continuous. To derive formulation (8), and let

\[ \frac{dTCU_n(y)}{dy} = 0 \]

then

\[ y_n^* = \sqrt{\frac{2\beta[K + cn + fg(n)]}{h}} \]

(9)
Since variables $n$ and $g(n)$ are taken as the constant in function $TCU_n(y)$, model $P_2$ can be expressed as the following formulation:

$$\min_{n \in N} \left\{ \min TCU_n(y) \right\} \quad \text{st} \ (2)$$

Based on the above analysis, the following conclusions can be derived:

**Conclusion 1.** The function $TCU_n(y)$ is convex and there exists a unique lowest solution at the point $y_n = y_n^*$, where $y_n^*$ can be given by formulation (9).

**Conclusion 2.** The optimal solution of model $P_2$ can be obtained by following the steps below:

1. For different positive integer $n$, find the lowest solution $TCU_n(y)$ which satisfies constraint (2), denote this value as $f(n)$
2. The optimal solution of model $P_2$ can be obtained by comparing all of $f(n), n \in N$

It is time consuming and impractical to compare $f(n)$ for each $n \in N$, however, based on the following theorems, we can limit the number of $n$ need to considered and the optimal solution of model $P_2$ can be found within limited steps.

**Theorem 1.** For any Function $TCU_n(y)$, if $y_n^*$ gained by formulation (9) also satisfies constraint (2), then $f(n) = TCU_n(y_n^*)$, where the meaning of $f(n)$ is the same as that defined in conclusion 2; otherwise, $f(n) = \min\{TCU_n(y_1), TCU_n(y_2)\}$, where $y_1 = (n - 1)p + 1, y_2 = np$.

**Proof.** Based on conclusion 1, we know that $TCU_n(y)$ is convex and there exists a unique optimal solution at $y_n = y_n^*$. If $y_n^*$ is within the interval given by constraint (2), clearly $f(n) = TCU_n(y_n^*)$. On the other hand, if $y_n^*$ is not within the interval given by constraint (2), since the function $TCU_n(y)$ is either increased or decreased within the given interval, we can find the lowest value of $TCU_n(y)$ by comparing the value of the two side nodes of the interval. □

**Theorem 2.** If $f(n_i)$ satisfies either of the following two conditions, then for all $n_i > n_j$, the lowest value of $TCU_{n_i}(y), TCU_{n_j}(y)$, that is, $f(n_i)$, cannot be lower than $f(n_j)$:

1. $f(n_i) = TCU_{n_i}(y_{n_i}^*)$;
2. $f(n_i) = TCU_{n_i}(y_{n_i}^*), where n_k = n_j + 1$.

**Proof.**

1. As $g(n)$ is a non-decreased function of $n$, it can be seen from formulations (9) and (10) that, with the increment of $n, y_n^*$ and $TCU_n(y_n^*)$ become larger. So, for all $n_i > n_j, TCU_{n_i}(y_n^*) > TCU_{n_j}(y_n^*)$, then
2. It can be deduced from the given condition that \( f(n_j) \leq TCU(n_j(y^*_j)) \leq f(n_k) \). Since \( TCU(n_j(y^*_j)) \) is a non-decreasing function of \( n_k \), we can further conclude that for all \( n_i > n_k, f(n_j) \leq TCU(n_i(y^*_i)) < TCU(n_i(y^*_i)) \leq f(n_i) \). \( \square \)

**Corollary 2.1.** If \( TCU_{\min}(y^*_i) \leq TCU(n_i(y^*_i)) \), where \( TCU_{\min}(y^*_i) = \min\{f(n_j)\} \) for all \( n_r \leq n_j \), then for all \( n_i > n_k, TCU_{\min}(y^*_i) \leq TCU(n_i(y^*_i)) \leq TCU(n_i(y^*_i)) \leq f(n_i) \).

**Proof.** It can be deduced from above theorems.

3. The algorithm

Based on the above analysis, if denote \( TCU_{\min} \) as the optimal objective value of model \( P_2 \), and \( y^* \) is the ordering quantities associated with \( TCU_{\min} \) we can follow the steps listed below to find \( TCU_{\min} \) as well as \( y^* \):

**Step 1.** For any positive integer \( n_i \) (the initial value of \( n_i = \max\{1, \lceil \beta/p \rceil \} \)), calculate \( y^*_i \) according to formulation (9). If \( y^*_i \) satisfies constraint (2), record \( f(n_i) = TCU(n_i(y^*_i)) \), where \( TCU(n_i(y^*_i)) \) is calculated according to formulation (10), let \( TCU_{\min} = \min\{TCU_{\min}, f(n_i)\} \) (the initial value of \( TCU_{\min} \) is set to infinite), then stop; else if \( y^*_i \) does not satisfy constraint (2), go to step 2.

**Step 2.** According to formulation (8), calculate \( TCU(n_i(y_1)) \) and \( TCU(n_i(y_2)) \), respectively, where \( y_1 = [(n_i - 1)p + 1], y_2 = np \); let \( f(n_i) = \min\{TCU(n_i(y_1)), TCU(n_i(y_2))\}, TCU_{\min} = \min\{TCU_{\min}, f(n_i)\} \). If \( TCU_{\min} \leq TCU(n_i(y^*_i)) \), where \( n_k = n_j + 1 \), stop; else let \( n_i = n_i + 1 \), and go to Step 1.

Since the value of \( TCU(n_i(y^*_i)) \) increases with the increment of \( n \), the algorithm can be finished within limited iterations process.

4. The examples

Three examples are given in this section in order to verify the given model as well as the algorithm In the first example, it is assumed that the transportation cost is proportional to the quantities delivered and no traveling duration constraint is considered. In the second example, the transportation cost is calculated based on travel distance of the vehicles and no fixed cost is considered. Whereas in the third example, the transportation costs include not only the fixed cost which is a fixed sum whenever a vehicle is employed, but also the variable cost which is calculated based on the travel distance of the vehicle. In addition, in the last two examples, the permitted working duration as well as the travel time of any vehicle along the trip is taken into account.

The meanings of the parameters in the examples are the same as those in the prior sections. In addition, for the purpose of simplicity, the units of the parameters in the examples are omitted. It is reasonable since the computational results as well as the conclusions cannot be affected by such simplification.
4.1. Example 1

It is assumed that in this problem, $\beta = 100, K = 100, h = 0.02, s = 0.3, L = 2$ In addition, we assume the transportation cost is proportional to the quantity delivered and the unit transportation cost, defining by $a$, equal to 0.1. The objective is to decide optimal value of $y^*, x^*$ and $r^*$ with the respect of minimizing the total average cost of the logistic system, where $y^*$ is the economic ordering quantity, $x^*$ is the optimal ordering points and $r^*$ is the ordering cycle.

According to the traditional EOQ formula, we find

$$y^* = \sqrt{\frac{2K\beta}{h}} = \sqrt{\frac{2 \times 100 \times 100}{0.02}} = 1000$$

$$r^* = y^*/\beta = \frac{1000}{100} = 10$$

$$x^* - L\beta = 2 \times 100 = 200$$

$$\text{TCU}(y^*) = \sqrt{2K\beta h} + (s + a)\beta = \sqrt{2 \times 100 \times 100 \times 0.02} + (0.3 + 0.2) \times 100 = 70$$

The results show that in the optimal solution, when the storage quantities reduce to 200, an order for 1000 unit products should be sent. The optimal ordering cycle is 10 days and the total transportation cost in each ordering cycle is 200. Base on the results, the next example is given.

4.2. Example 2

It is assumed that in this problem, $\beta = 100, K = 100, h = 0.02, s = 0.3, p = 200, c = 40, f = 0, U = 8, t = 4$, to decide the optimal solutions of $y^*, x^*$ and $r^*$

The algorithm described in Section 3 is coded by Visual C++, and the computational results are listed in Table 1.

We can see from the results that, based on the stopping criterion in step 2, the algorithm stops when $n = 8$. The optimal solution occurs at $n = 5$, and $\text{TCU}_{\min} = 70.00, y^* = 1000, r^* = 10, x^* = 200$. The number of vehicle used for delivery is 3.

The reason that the results of the above two examples are equal is that we design example 2 based on the results from example 1, that is, the parameters given in example 2 guarantee that vehicle used for delivery is 3.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$y_n$</th>
<th>$\text{TCU}_n(y_n^*)$</th>
<th>$\text{TCU}_n(y_1)$</th>
<th>$\text{TCU}_n(y_2)$</th>
<th>$f(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>200</td>
<td>53.66</td>
<td>14030.00</td>
<td>102.00</td>
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</tr>
<tr>
<td>2</td>
<td>400</td>
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<td>121.56</td>
<td>79.00</td>
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</tr>
<tr>
<td>3</td>
<td>600</td>
<td>59.66</td>
<td>88.87</td>
<td>72.67</td>
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</tr>
<tr>
<td>5</td>
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<td>75.46</td>
<td>70.00</td>
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</tr>
<tr>
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<td>73.98</td>
<td>70.33</td>
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<tr>
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<td>73.65</td>
<td>71.14</td>
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</tr>
<tr>
<td>8</td>
<td>1600</td>
<td>70.99</td>
<td>73.99</td>
<td>72.25</td>
<td>72.25</td>
</tr>
</tbody>
</table>
The delivery of $y^*$ in example 1 is fully loaded. However, in example 2, when $n = 5$, the value of $y^*_n$ calculated according to formulation (9) is 1732, whereas based on the algorithm, $y^*$ equals to 1000. In other words, the solution method using the traditional economic ordering quantity formula is not suitable for the given problem in this example.

We can also see from the results that for all $n \not\equiv 8; f(n) < \text{TCU}_n(y^*_n)$, and the value of $\text{TCU}_n(y^*_n)$ increase along with the increment of $n$. Following computation indicates that when $n = 13, f(n) = \text{TCU}_n(y^*_n) = 79.83 > \text{TCU}_{\min}$. Such results further verify the algorithm.

### 4.3. Example 3

It is assumed that in this problem, $\beta = 100, K = 100, h = 0.02, s = 0.3, p = 200, c = 40, U = 8, t = 4, f = 30$, to decide the optimal solutions of $y^*, x^* \text{ and } r^*$.

The computational results are listed in Table 2.

We can see from the results that, based on the criterion in step 2, the algorithm stops when $n = 9$. The optimal solution occurs at $n = 6$, and $\text{TCU}_{\min} = 77.83, y^* = 1200, r^* = 12, x^* = 200$. The number of vehicle used for delivery is 3.

Following computation indicates that when $n = 16, f(n) = \text{TCU}_n(y^*_n) = 92.6 > \text{TCU}_{\min}$. When the fixed cost of the vehicle is considered, the value of $y^*$ and $r^*$ are different from those gained in example 2. It shows that under $n = 5$, one of the vehicles used will finish only one trip, thus the marginal transportation cost of such vehicle for delivering unit product will be higher contrasting to that of other vehicles which can complete two trips. On the other hand, if the vehicles are utilized to the greatest efficiency, the inventory quantities may increase. So the optimal solution of the problem is the results of the trade-off of the transportation cost and the inventory cost.

### 5. Conclusion

In this study, we address an inventory problem arising from supplier–retailer logistic system on the integration of production, inventory and transportation. In our study, unlike most of the prior inventory models, both of the fixed transportation cost and the variable transportation cost are accounted.
In addition, multiple use of the vehicle is also considered since such arrangement can share the fixed transportation cost. A model for such problem is set up for the purpose of trading-off all of the costs related to the logistic system and an algorithm for such model is presented. Computational results verify the proposed model as well as the efficiency of the algorithm.

Future research will consider the situation in which there exists more than one retailers. In addition, the relationship of each item in the model will be further invested so that a more efficient and general algorithm will be proposed.

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