On consecutive edge magic total labeling of graphs

K.A. Sugeng a,b,*, M. Miller a

a School of Information Technology and Mathematical Sciences, University of Ballarat, VIC 3353, Australia
b Department of Mathematics, University of Indonesia, Depok 16424, Indonesia

Received 5 December 2005; received in revised form 18 May 2006; accepted 22 May 2006
Available online 27 October 2006

Abstract

Let $G = (V, E)$ be a finite (non-empty) graph, where $V$ and $E$ are the sets of vertices and edges of $G$. An edge magic total labeling is a bijection $\alpha$ from $V \cup E$ to the integers $1, 2, \ldots, n + e$, with the property that for every edge $xy \in E$, $\alpha(x) + \alpha(y) + \alpha(xy) = k$, for some constant $k$. Such a labeling is called an $a$-vertex consecutive edge magic total labeling if $\alpha(V) = \{a + 1, \ldots, a + n\}$ and a $b$-edge consecutive edge magic total if $\alpha(E) = \{b + 1, b + 2, \ldots, b + e\}$. In this paper we study the properties of $a$-vertex consecutive edge magic and $b$-edge consecutive edge magic graphs.

© 2006 Elsevier B.V. All rights reserved.

Keywords: Graph; Magic labeling; Consecutive edge magic total labeling

1. Introduction

All graphs considered are finite, simple and undirected. The graph $G$ has vertex set $V = V(G)$ and edge set $E = E(G)$ and we let $e = |E|$ and $n = |V| > 1$. A bijection

$$\alpha: V(G) \cup E(G) \to \{1, 2, \ldots, n + e\}$$

is called a total labeling of $G$ and the associated weight $w_\alpha(xy)$ of an edge $xy$ in $G$ is $w_\alpha(xy) = \alpha(x) + \alpha(y) + \alpha(xy)$. All labelings considered in this paper are total labelings, and so from now on by a labeling we shall always mean a total labeling. The labeling $\alpha$ of $G$ is edge magic if every edge has the same weight, and $G$ is called an edge magic graph if an edge magic total labeling of $G$ exists. If $\alpha(V) = \{1, \ldots, n\}$ then $\alpha$ is called a super edge magic labeling. Sedláček [2] introduced magic labeling of graphs in 1963, and since then there have been many results in magic labeling, especially in edge magic labeling. For new results in graph labeling see [1].

In this paper we introduce consecutive edge magic labeling and studied some properties of such labelings. Preliminary result on this labeling can be found in [3].

A bijection $\beta: V(G) \cup E(G) \to \{1, 2, \ldots, n + e\}$ is called an $a$-vertex consecutive edge magic labeling of $G = G(V, E)$ if $\beta$ is an edge magic labeling and $\beta(V) = \{a + 1, \ldots, a + n\}$, $0 \leq a \leq e$. On the other hand, $\gamma: V(G) \cup E(G) \to \{1, 2, \ldots, n + e\}$ is called a $b$-edge consecutive edge magic labeling of $G = G(V, E)$ if $\gamma$ is an edge magic labeling.

* Corresponding author.
E-mail addresses: k.sugeng@ballarat.edu.au, kikiariyanti@yahoo.com (K.A. Sugeng), m.miller@ballarat.edu.au (M. Miller).
labeling and \( \gamma(E) = \{b + 1, \ldots, b + e\}, 0 \leq b \leq n \). A graph \( G \) that has \( a \)-vertex consecutive (respectively, \( b \)-edge consecutive) edge magic labeling is called an \( a \)-vertex consecutive (respectively, \( b \)-edge consecutive) edge magic graph.

Define \( M = e + n \). Let \( \gamma : V \cup E \rightarrow 1, 2, \ldots, M \) be a super edge magic labeling for a graph \( G \). Define the labeling \( \gamma' : V \cup E \rightarrow 1, 2, \ldots, M \) as follows.

\[
\gamma'(x) = M + 1 - \gamma(x), \quad x \in V, \\
\gamma'(xy) = M + 1 - \gamma(xy), \quad xy \in E.
\]

Then \( \gamma' \) is called the dual of \( \gamma \). From [6] we know that the dual of an edge magic labeling for a graph \( G \) is also an edge magic labeling.

In the next section, we present similar results in dual labeling for \( a \)-vertex consecutive and \( b \)-edge consecutive edge magic labelings.

Let \( V(G) = \{x_1, x_2, \ldots, x_n\} \) be the set of vertices in \( G \) with labels in \( \{1, 2, \ldots, n + e\} \). A symmetric matrix \( A = (a_{ij}), i, j = 1, \ldots, n \), is called an adjacency matrix of \( G \) if

\[
a_{ij} = \begin{cases} 
1 & \text{if there is an edge between } x_i \text{ and } x_j, \\
0 & \text{if there is no edge between } x_i \text{ and } x_j. 
\end{cases}
\]

A bijection \( \alpha : V(G) \rightarrow \{1, 2, \ldots, n\} \) is called an \((a, d)\)-edge-antimagic vertex (EAV) labeling of \( G = G(V, E) \) if the set of the edge-weights of all edges in \( G \) is \( \{a, a + d, \ldots, a + (e - 1)d\} \), where \( a > 0 \) and \( d \geq 0 \) are two fixed integers.

Note that a graph that has EAV labeling can be represented by a special adjacency matrix.

If \( G \) is an EAV graph then the rows and columns of \( A \) can be labeled by \( 1, 2, \ldots, n \). \( A \) is symmetric and every skew-diagonal (diagonal of \( A \) which is traversed in the “northeast” direction) line of matrix \( A \) has at most two “1” elements. The set \( \{\alpha(x) + \alpha(y) : x, y \in V(G)\} \) generates a sequence of integers of difference \( d \). Each entry “1” in a skew-diagonal line has a one-to-one correspondence to an element of the edge-weight set \( \{\alpha(x) + \alpha(y) : x, y \in V(G)\} \).

If \( d = 1 \) then the non-zero off diagonal lines form a band of consecutive integers. In this paper, EAV labeling always refers to an \((a, 1)\)-EAV labeling.

2. On \( a \)-vertex consecutive edge magic graphs

Let \( G \) be an \( a \)-vertex consecutive edge magic graph and \( \beta \) be an \( a \)-vertex consecutive edge magic labeling of \( G \). Then \( \beta(x) \in \{a + 1, a + 2, \ldots, a + n\} \), for every \( x \in V(G) \), \( 0 \leq a \leq e \). If \( a = 0 \) then the labeling is called a super edge magic labeling. In this paper we consider \( a \)-vertex consecutive edge magic labeling for \( 1 \leq a \leq n - 1 \). For further results, see [1, 6].

By using the dual property, we obtain the following theorem.

**Theorem 2.1.** The dual of an \( a \)-vertex consecutive edge magic labeling for a graph \( G \) is an \((e - a)\)-vertex consecutive edge magic labeling.

The following theorem characterises the graphs that can have \( a \)-vertex consecutive edge magic labeling.

**Theorem 2.2.** If \( G \) has an \( a \)-vertex consecutive edge magic labeling, \( a \neq 0 \) and \( a \neq e \), then \( G \) is a disconnected graph.

**Proof.** Let \( A \) be an adjacency matrix of \( a \)-vertex consecutive edge magic graph \( G \). Since all labels of vertices in \( G \) are consecutive integers, then \( A \) consists of all elements \( a + 1, a + 2, \ldots, a + n \) in its rows and columns. The maximum number of edges in this graphs will be \( 2n - 3 \) [4, 5]. If \( a \neq 0 \) and \( a \neq e \), and we know that \( \beta(E) = \{1, \ldots, a\} \cup \{a + n + 1, \ldots, n + e\} \), then there will be a gap in the set of edge-weights \( \{\beta(x) + \beta(y) : x, y \in V(G)\} \).

Thus, the labels are divided into two blocks. The width of the gap must be \( n \), the same as the gap in edge labels. Thus we can conclude that the maximum number of edges in \( G \) is \((2n - 3) - n = n - 3\). As a consequence, \( G \) cannot be connected. Thus, the theorem holds. \( \Box \)

A natural guess is that \( G \) might be the union of 3 trees. However, the following results show that this is not the case.
Corollary 1. If $a 
eq 0$ and $a 
eq e$ then there is no $a$-vertex consecutive edge magic labeling for $3K_2$.

Furthermore, we can prove that $G$ cannot be a union of any three trees.

Theorem 2.3. If $a 
eq 0$ and $a 
eq e$, and $G$ has an $a$-vertex consecutive edge magic labeling $\beta$ then $G$ cannot be the union of three trees $T_1, T_2$ and $T_3$, where $|V(T_i)| \geq 2$, $i = 1, 2, 3$.

Proof. Let $G$ be an $a$-vertex consecutive edge magic graph, $a \neq 0$ and $a \neq e$. Suppose that $G = T_1 \cup T_2 \cup T_3$, where $T_1, T_2, T_3$ are three arbitrary trees. Let $n_1 \geq 2$ (respectively, $n_2 \geq 2$, $n_3 \geq 2$) and $e_1$ (respectively, $e_2$, $e_3$) be the number of vertices and the number of edges in $T_1$ (respectively, $T_2$, $T_3$). Since $G$ is an $a$-vertex consecutive edge magic graph then the edge labels are divided into two blocks, each block consists of consecutive integers, then the set of vertices in $G$ also forms two disjoint subsets, say $S_1$ and $S_2$, following the edge blocks. Suppose that vertices in $T_1$ and $T_2$ are in the same block $S_1$ and the vertices of $T_3$ is $S_2$.

Since $n_3 \geq 2$ then $a + n$ and $a + n - 1$ are in $S_2$. Let $W_{S_1} = \{\beta(x) + \beta(y) \mid x, y \in S_1\}$ and $W_{S_2} = \{\beta(x) + \beta(y) \mid x, y \in S_2\}$ be the edge-weights sets of $\beta$. Since the number of edges must be the maximum ($= n - 3$), then the minimal and maximal values of edge-weight in the sets $W_{S_1} = \{\beta(x) + \beta(y) \mid x, y \in S_1\}$ and $W_{S_2} = \{\beta(x) + \beta(y) \mid x, y \in S_2\}$ must be used. The maximal element of $W_{S_2}$ is $(a + n) + (a + n - 1) = 2a + 2n - 1$ then the corresponding edge label must be 1. Thus the magic constant is $k = 2a + 2n$. Since the maximal edge label in this block is $a$ then the minimal element of $W_{S_1}$ is $a + 2n$.

Let $z^* = \max\{z \mid z \in S_1\}$, then $z^* = a + n - p$, $p \geq 2$. Let $z_1$ be a vertex in $N(z^*)$. The consequence is that the edge label must be minimum, that is $\beta(z^*z_1) = a + n + 1$. Since the magic constant is $k = 2a + 2n$ then $\beta(z_1) = p - 1$. We know that $p - 1 \geq a + 1$, since $a + 1$ is the smallest vertex label. Consequently, $p \geq a + 2$, that is $z^* \leq n - 2$. Thus $n - 1, n, n + 1$ will all be in $S_2$.

Let $\beta(x_1) = n - 1$, $\beta(x_2) = n$ and $\beta(x_3) = 2n - 1$. Then $v_1 \in N(x_1)$, $v_2 \in N(x_2)$, $v_3 \in N(x_3)$. Then the only possibilities are $\beta(x_2y_1) = a$ and $\beta(x_1y_1) = a - 1$, but then we cannot find the value of $\beta(x_3y_3)$. Thus $G$ cannot be a union of three trees. □

A subgraph is called non-trivial if it contains more than one vertex. If $G$ is the union of three non-trivial component subgraphs then the only possibility is $G = T_1 \cup T_2 \cup T_3$, where $T_1, T_2$ and $T_3$ are trees, since the maximal number of edges are $n - 3$. However, the previous theorem states that this is not possible for union of non-trivial trees. Thus, if $G$ has more than two connected non-trivial components, then $G$ has to have at least one isolated vertex.

If $G$ has an $a$-vertex consecutive edge magic labeling, $a \neq 0$ and $a \neq e$, and the maximum number of edges, then in every case the graph needs to have at least one isolated vertex. By counting the maximum number of edges and comparing this to the number of vertices in the graph, we derive the number of isolated vertices that are needed in the following observation (see also Fig. 1). Fig. 1 gives some examples of $a$-vertex consecutive edge magic graphs.

Observation 1. Let $G$ be an $a$-vertex consecutive edge magic graph, $a \neq 0$ and $a \neq e$. If $G$ consists of

- two trees then the number of isolated vertices is one;
- one tree and one unicyclic graph then the number of isolated vertices is two;
- two unicyclic graphs then the number of isolated vertices is three;
- otherwise the number of isolated vertices is at least three.

Theorem 2.4. There is an $a$-vertex consecutive edge magic graph for every $a$ and $n$.

Proof. Let $G$ be the union of two stars, $S_1$ and $S_2$, and one isolated vertex $x$. Let $v_{1i}, i = 1, \ldots, t_1, t_1 = e - a$, denote the leaves of $S_1$ and let $v_{2j}, j = 1, \ldots, t_2, t_2 = a$, denote the leaves of $S_2$. Label the vertices of $G$ as follows.

$$
\beta(v) = \begin{cases}
    n - 1 & \text{if } v = x, \\
n + 1 & \text{if } v \text{ is a centre of } S_1, \\
    a + 1 + i & \text{if } v = v_{1i}, \quad i = 1, \ldots, t_1, \\
n + a & \text{if } v \text{ is a centre of } S_2, \\
    a + n - j & \text{if } v = v_{2j}, \quad j = 1, \ldots, t_2.
\end{cases}
$$
Complete the edge labels \( \{1, 2, \ldots, a\} \cup \{a + n + 1, \ldots, e + n\} \). Then we have an \( a \)-vertex consecutive edge magic labeling for \( G \).

**Theorem 2.5.** Every union of \( r \) stars with \( r - 1 \) isolated vertices has an \( s \)-vertex consecutive edge magic labeling, where \( s = \min\{|V(S_i)|, \ldots, |V(S_r)|\} \).

**Proof.** Let \( G = S_1 \cup \cdots \cup S_r \), where \( S_i \) is a star for \( i = 1, \ldots, n \). Let \( n_i = |V(S_i)| \). Order the stars \( S_i, i = 1, 2, \ldots, r \), so that \( n_1 \leq n_2 \leq \cdots \leq n_r \). Let \( c_1, c_2, \ldots, c_r \) be a centre of star \( S_i, i = 1, \ldots, r \), and \( \{x_1, \ldots, x_{r-1}\} \) be a set of isolated vertices. Define a labeling \( \beta \) as follows.

- \( \beta(c_1x) = i \) if \( x \) is a leaf of \( S_1, 1, 2, \ldots, s = n_1 \);
- Assign the labels
  - \( s + 1 \) to the \( c_2, s + 2 \) to the \( c_3, \ldots, s + r - 1 \) to the \( c_r \),
  - \( s + r, \ldots, s + r + n_r - 2 \) to the leaves of the star \( S_r \),
  - \( s + r + n_r - 1 \) to \( x_1 \),
  - \( s + r + n_r, \ldots, s + r + n_r + n_r - 1 \) to the leaves of the star \( S_{r-1} \),
  - \( s + r + n_r + n_r - 1 \) to \( x_2 \).
- Use the same algorithm to label the rest of the vertices until all the vertices in \( G \) are labeled. For \( S_1 \), use the largest vertex label for the \( c_1 \).
- Complete the labeling of edges, starting from the edges of \( S_r \) to \( S_2 \), using integers \( \sum_{i=1}^{r} n_i + 1, \ldots, \sum_{i=1}^{r} n_i + e \).

It easy to see that \( \beta \) is an \( s \)-vertex consecutive edge labeling. \( \square \)
3. On $b$-edge consecutive edge magic graphs

Let $G$ be a $b$-edge consecutive edge magic graph and let $\gamma$ be a $b$-edge consecutive edge magic labeling of $G$. Then $\gamma(xy) \in \{b + 1, b + 2, \ldots, b + e\}$. The super edge magic labeling is a special case of $b$-edge consecutive edge magic labeling, when $b = n$. Since the vertex labels in a $b$-edge consecutive edge magic labeling, $1 \leq b \leq n - 1$, do not form a set of consecutive integers, it follows that the row/column of the adjacency matrix of $b$-edge consecutive edge magic graph are labeled according to the vertex labels of $G$, not consecutively as $1, 2, \ldots, n$.

**Theorem 3.1.** Every $b$-edge consecutive edge magic graph has edge antimagic vertex labeling.

Considering the dual labeling property, if a graph $G$ has a $b$-edge consecutive edge magic labeling, a similar result in the dual also holds.

**Theorem 3.2.** The dual of a $b$-edge consecutive edge magic labeling for a graph $G$ is an $(n - b)$-edge consecutive edge magic labeling.

A caterpillar is a graph derived from a path by hanging any number of leaves from the vertices of the path. The caterpillar can be considered as a sequence of stars $S_1 \cup S_2 \cup \cdots \cup S_r$, where each $S_i$ is a star with centre $c_i$ and $n_i$ leaves, $i = 1, 2, \ldots, r$, and the leaves of $S_i$ include $c_{i - 1}$ and $c_{i + 1}, i = 2, \ldots, r - 1$.

**Theorem 3.3.** Every caterpillar has a $b$-edge consecutive edge magic graph for every $b$.

**Proof.** Let $r$ be $b = \frac{r}{2}$, for $r$ even; and $b = \frac{r+1}{2}$, for $r$ odd. Let $G$ be a caterpillar $S_{n_1, n_2, \ldots, n_r}$, with centre $c_1, c_2, \ldots, c_r$, such that every centre $c_i$ with $i$ even has degree 2. Note that a star can be regarded as caterpillar $S_{n_1, n_2}$, with $n_2 = 1$.

Let $\gamma$ be a $b$-edge consecutive edge magic labeling for $G$. Label the odd centres as

$$\gamma(c_i) = \frac{i + 1}{2}, \text{ for } i \text{ odd.}$$

Let $v^j_i$ be the $j$th leaf of the centre $i$. Label the leaves of the odd centre by

$$\gamma(v^j_i) = b + e + 1 + j + \left( \sum_{k=1}^{i-1} (n_k + 1) \right).$$

If $i - 1 \leq 1$ then for $k$ odd, $\sum_{k=1}^{i-1} (n_k - 1) = 0$ and the even centre is treated as a leaf of the previous odd centre and is given the largest labels among the leaves.

Thus for every $b, b = \frac{r}{2}$, for $r$ even; and $b = \frac{r+1}{2}$, for $r$ odd, we have constructed a $b$-edge consecutive edge magic graph.

We have an example of a $b$-edge consecutive edge magic labeling for every $b$. Fig. 2 gives examples of labelings for some value of $b$. In general, we have

**Theorem 3.4.** If a connected graph $G$ has a $b$-edge consecutive edge magic labeling, where $b \in \{1, \ldots, n - 1\}$, then $G$ is a tree.

**Proof.** Suppose that $G$ has a $b$-edge consecutive edge magic labeling $\gamma$. Then $\gamma(V) = V_1 \cup V_2$, where $V_1 = \{1, 2, \ldots, b\}$ and $V_2 = \{b + e + 1, b + e + 2, \ldots, n + e\}$. Let $b \in \{1, \ldots, n - 1\}$.

Let $\gamma'$ be the restriction of $\gamma$ under $V$. Thus $\gamma'$ is a VAE labeling.

Let $A$ be the adjacency matrix of $G$. Since the set of vertex labels is a union of two disjoint subsets $V_1$ and $V_2$, then the adjacency matrix of $G$ consists of four blocks as follows.

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}.$$
Since $A$ is a symmetric matrix, it follows that $A^3$ is the transpose of $A^2$. The entries of $A^1$ represent all the edges between the vertices inside $V_1$, the entries of $A^2$ represent all the edges between the vertices in $V_1$ and vertices in $V_2$, and the entries of $A^4$ represent all the edges between the vertices inside $V_2$.

Suppose that $A^1$ is a nonzero submatrix. Then there is at least one edge $xy$ between the vertices in $V_1$ with $\gamma'(xy) = \gamma(x) + \gamma(y) \leq 2b - 1$. Let $x'y'$ be an edge between a vertex in $V_1$ and a vertex in $V_2$. Then $\gamma(x'y') = \gamma(x') + \gamma(y') \geq b + e + 2$. Since $\gamma'$ is a VAE labeling, the edge-weights under $\gamma'$ must be a set of consecutive integers. This means that $(b + e + 2) < (2b - 1)$ or $b > e + 3$. We know that $b \leq n - 1$, whence $e < n - 4$. This means $G$ is disconnected. Similarly, if $A^4$ is nonzero.

Suppose that $A^2$ is a zero submatrix. Then $A^1$ and $A^4$ cannot be zero submatrices of $A$. Obviously, $G$ will then be a disconnected graph.

If $G$ is connected then $A^1$ and $A^4$ must be zero submatrices of $A$. Consider the submatrix $A^2$. The maximum edge-weight under $\gamma'$ is $n + e + b$ and the minimum edge-weight is $b + e + 2$. Thus the maximum number of edges will be $(n + e + b) - (b + e + 2) - 1 = n - 1$. Then $G$ is a tree. □

**Corollary 2.** A double star $S_{n_1,n_2}$ has a $b$-edge consecutive edge magic labeling for some $b \in \{1, 2, \ldots, n\}$ and

- if $b = 1$ then $S_{n_1,n_2}$ is a star;
- if $b > 1$ then $b = n_2 + 1$.

The previous result can be generalised as follows.

**Theorem 3.5.** Every caterpillar has a $b$-edge consecutive edge magic labeling, where

$$b = \begin{cases} \left\lceil \frac{r + 1}{2} \right\rceil + \sum_{i \text{ even}} n_i - 2 & \text{if } i \text{ is odd,} \\ \left\lceil \frac{r + 1}{2} \right\rceil + \sum_{i \text{ even, } i < r} n_i - 2 + (n_r - 1) & \text{if } i \text{ is even.} \end{cases}$$

**Proof.** Considering caterpillar $G$ as a bipartite graph, we can draw $G$ in two rows, each row containing vertices from one partite set. Clearly, it is possible to make the drawing so that there are no edge crossings. Let $a_1, a_2, \ldots, a_b$ be the
vertices in the first row ordered from left to right and let \( b_1, b_2, \ldots, b_{n-b} \) be the vertices in the second row ordered from left to right.

Define the vertex labeling \( \lambda : V \rightarrow \{1, 2, \ldots, n\} \) in the following way.

\[
\lambda(a_i) = i \quad \text{for } 1 \leq i \leq b, \\
\lambda(b_j) = b + e + j \quad \text{for } 1 \leq j \leq n - b.
\]

It is an easy exercise to check that the set of the edge weights is \{\( b + e + 2, b + e + 3, \ldots, b + e + n \)\}. If we assign as edge labels the values \( b + 1, b + 2, \ldots, b + e \) then the resulting labeling will form a \( b \)-edge consecutive edge labeling. □

**Theorem 3.6.** Every \( \bigcup_{i=1}^{r} S_i \cup \{x_1, \ldots, x_{r-1}\} \), where \( S_i \), \( i = 1, \ldots, r \) is a star and \( \{x_1, \ldots, x_{r-1}\} \) are isolated vertices, has an \( r \)-edge consecutive edge magic.

**Proof.** Let \( G = S_1 \cup \cdots \cup S_r \), where \( S_i \) is a star, for \( i = 1, \ldots, n \). Let \( c_1, c_2, \ldots, c_r \) be a centre of star \( S_i \), \( i = 1, \ldots, r \) and \( \{x_1, \ldots, x_{r-1}\} \) be a set of isolated vertices. Define a labeling \( \gamma \) as follows.

- \( \gamma(x) = i \) if \( x \) is a centre \( c_i \), \( i = 1, 2, \ldots, r \);
- Assign the labels
  * \( e + r + 1, \ldots, e + r + n_r \) to the leaves of the star \( S_r \),
  * \( e + r + n_r + 2, \ldots, e + r + n_r + n_{r-1} + 1 \) to the leaves of the star \( S_{r-1} \),
  * \( e + r + n_r + n_{r-1} + 3, \ldots, e + r + n_r + n_{r-1} + n_{r-2} + 2 \) to the leaves of the star \( S_{r-2} \) and so on, until all the leaves of stars \( S_1, \ldots, S_r \) are labeled,
  * \( e + r + n_r + 1, e + r + n_r + n_{r-1} + 2, \ldots, e + r + n_1 + \cdots + n_{r-1} + r - 1 \) to \( \{x_1, \ldots, x_{r-1}\} \).
- Complete the labeling of edges, starting from the edges of \( S_1 \) to \( S_r \), using integers \( r + 1, \ldots, r + e \).

It easy to see that \( \gamma \) is an \( r \)-edge consecutive edge labeling. □

**Acknowledgement**

We wish to thank Professor Michal Tkáč for his suggestions and valuable discussions.

**References**

[4] K.A. Sugeng, M. Miller, Relationships between adjacency matrices and super \( (a, d) \)-edge-antimagic total graphs, JCMCC, in press.