The Minimal Matching Energy of \((n, m)\)-Graphs with a Given Matching Number

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Abstract

The matching energy of a graph is defined as the sum of the absolute values of the zeros of its matching polynomial. Let \(G_{n,m}\) be the set of connected graphs of order \(n\) and with \(m\) edges. In this note we determined the extremal graphs from \(G_{n,m}\) with \(n \leq m \leq 2n - 4\) minimizing the matching energy. Also we determined the minimal matching energy of graphs from \(G_{n,m}\) where \(m = n - 1 + t\) and \(1 \leq t \leq \beta - 1\) and with a given matching number \(\beta\). Moreover, the above extremal graphs have been completely characterized.
1 Introduction

We only consider finite, undirected and simple graphs throughout this paper. Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The cardinality of $E(G)$ is usually denoted by $m(G)$. The degree of $v_i \in V(G)$, denoted by $d_G(v_i)$ or $d_i$ for short, is the number of vertices in $G$ adjacent to $v_i$. In particular, $\Delta(G)$ denotes the maximum degree of vertices in $G$, and $\Delta_2(G)$ is the second maximum degree of vertices in $G$. For each $v_i \in V(G)$, the set of neighbors of the vertex $v$ is denoted by $N_G(v_i)$. For a subset $W$ of $V(G)$, let $G - W$ be the subgraph of $G$ obtained by deleting the vertices of $W$ and the edges incident with them. Similarly, for a subset $E'$ of $E(G)$, we denote by $G - E'$ the subgraph of $G$ obtained by deleting the edges of $E'$. If $W = \{v\}$ and $E' = \{xy\}$, the subgraphs $G - W$ and $G - E'$ will be written as $G - v$ and $G - xy$ for short, respectively. For any two nonadjacent vertices $x$ and $y$ of graph $G$, we let $G + xy$ be the graph obtained from $G$ by adding an edge $xy$. In the following we always denote by $S_n$ the star graph of order $n$, and by $K_n$ the complete graph of order $n$. Other undefined notations and terminology on the graph theory can be found in [1].

For any graph $G$ with edge set $E(G)$, if any two edges of $e_1, e_2, \ldots, e_k \in E(G)$ have no common vertices, we say that $\{e_1, e_2, \ldots, e_k\}$ is a $k$-matching of graph $G$. Moreover, we denote by $m(G, k)$ the number of $k$-matchings in $G$. In particular, $m(G, 1) = m(G)$ and $m(G, k) = 0$ when $k > \frac{n}{2}$ for any graph $G$ of order $n$. For the sake of convenience, we set $m(G, 0) = 1$ for any graph $G$. Recall that the Hosoya index $z(G)$ [16] of a graph $G$ is the sum of total number of all matchings, including the empty edge subset, in $G$. Thus, for a graph $G$ of order $n$, we have

$$z(G) = \sum_{k=0}^{\left\lceil \frac{n}{2} \right\rceil} m(G, k).$$

For some details of the results on the Hosoya index, please refer to [22, 23, 24, 26, 28, 29].

The matching polynomial of a graph $G$ of order $n$ is defined as

$$\alpha(G, \lambda) = \sum_{k \geq 0} (-1)^k m(G, k) \lambda^{n-2k}. \quad (1)$$

Moreover, the theory of matching polynomial of a graph $G$ is well elaborated in [3, 9, 10, 14]. From the expression of matching polynomial (1) of graph $G$, a quasi-order on the set of graphs of order $n$ can be deduced as follows:

$$G \succeq H \iff m(G, k) \geq m(H, k) \text{ for } k = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor.$$
If $G \succeq H$ and there exists at least one integer $k$ such that $m(G, k) > m(H, k)$, then we write $G \succ H$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a graph $G$, i.e., the eigenvalues of its $(0, 1)$-adjacency matrix [4]. The energy of the graph $G$ is defined as

$$\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i|.$$  

Nowadays the theory of energy of graphs have been well developed. Some details on the energy of graphs can be found in the book [21].

Recently Gutamm and Wagner [15] have first introduced the definition of matching energy of a graph $G$ as follows:

$$ME(G) = \frac{2}{\pi} \int_{0}^{+\infty} \frac{1}{x^2} \ln \left( \sum_{k \geq 0} m(G, k) x^{2k} \right) dx.$$  

(2)

Also in [15] they pointed out that, for any forest $G$, its matching energy coincides its energy, and the following formula:

$$ME(G) = \sum_{i=1}^{n} |\mu_i|,$$  

(3)

where $\mu_1, \mu_2, \ldots, \mu_n$ are the zeros of matching polynomial of graph $G$. Very recently the matching energy has attracted the attention of some researchers. Ji, Li and Shi [19] determined the extremal matching energies of all bicyclic graphs of order $n$. Li and Yan [20] characterized the maximal matching energy of some graphs with given parameters, including chromatic number and connectivity. The maximal matching energy of tricyclic graphs of order $n$ have been determined in [2].

From Formula (2) and the monotony of the function logarithm, the following two relations between the quasi-order defined as above and the matching energy, Hosoya index, respectively, can be deduced [15]:

$$G \succeq H \implies ME(G) \geq ME(H),$$  

(4)

$$G \succeq H \implies z(G) \geq z(H).$$  

(5)

Let $G_{n,m}$ be the set of connected graphs of order $n$ and with $m$ edges. Denote by $G_{n,m}(\beta)$ the set of connected graph from $G_{n,m}$ and with matching number $\beta$ where $2 \leq \beta \leq \lfloor \frac{m}{2} \rfloor$. In this note we characterized the extremal graphs from $G_{n,m}$ where $n + 1 \leq m \leq 2n - 3$ minimizing the matching energy. Moreover, we determined the extremal graph from $G_{n,n-1+t}(\beta)$ (where $1 \leq t \leq \beta - 1$) minimizing the matching energy.
2 Some lemmas

Before stating our main results, we will list or prove some lemmas as preliminaries, which will play an important role in the next proofs.

Lemma 2.1. ([9, 14]) For any graph $G$ with $v_q \in V(G)$ and $e = v_i v_j \in E(G)$, we have

(i) $m(G, k) = m(G - e, k) + m(G - \{v_i, v_j\}, k - 1)$;

(ii) $m(G, k) = m(G - v_q, k) + \sum_{v_r \in N_{G}(v_q)} m(G - v_q - v_r, k - 1)$.

Lemma 2.2. ([15]) Let $G$ be a graph with $e \in E(G)$. Then we have

$$ME(G - e) < ME(G).$$

Recall that the first Zagreb index of a graph $G$ is defined ([12, 13]) as $M_1(G) = \sum_{v_i \in V(G)} d_i^2$. Some results of first Zagreb index can be seen in [5, 6, 7, 25]. For convenience, we let \( \binom{a}{b} = 0 \) for two positive integers $a$ and $b$ with $a < b$.

Lemma 2.3. ([15]) For any connected graph $G$ with $m$ edges, we have

$$m(G, 2) = \binom{m}{2} + m - \frac{1}{2} M_1(G).$$

Proof. Note that $\binom{1}{2} = 0$ for any pendent vertex $v_p$ in the graph $G$. From the result in [15], we have

$$m(G, 2) = \binom{m}{2} - \sum_{v_i \in V(G)} \binom{d_i}{2} = \binom{m}{2} - \frac{1}{2} \sum_{v_i \in V(G)} d_i + \frac{1}{2} \sum_{v_i \in V(G)} d_i = \binom{m}{2} + m - \frac{1}{2} M_1(G).$$

For any integer $m$ satisfying $n + 1 \leq m \leq 2n - 4$, we denote by $G_{n,m}$ a graph of order $n$ and with $m$ edges in which maximum degree is $n - 1$ and the second maximum degree is $m - n + 2$. The structure of graph $G_{n,m}$ can be seen in Fig. 1. Moreover, a graph $G'_{n,n+2}$ is shown in Fig. 2.
Lemma 2.4. ([7, 27]) For any graph $G \in \mathcal{G}_{n,m}$ with $n + 1 \leq m \leq 2n - 4$, we have
\[
M_1(G) \leq n(n - 1) + (m - n + 1)(m - n + 6)
\]
with equality holding if and only if $G \cong G_{n,m}$ for $m = n + 1$ or $n + 3 \leq m \leq 2n - 4$; and $G \cong G_{n,m}$ or $G'_{n,n+2}$ for $m = n + 2$.

3 Main results

In [15] the extremal $(n, n)$-graph maximizing the matching energy has been determined, which is just $S^+_n$ obtained by adding a new edge in a star $S_n$. In the next theorem we determine the extremal graph from $\mathcal{G}_{n,m}$ with $n+1 \leq m \leq 2n-3$ maximizing the matching energy, which can be viewed as a more general one of the above result for $(n, n)$-graphs.
Theorem 3.1. For any graph $G \in \mathcal{G}_{n,m}$ with $n + 1 \leq m \leq 2n - 4$, we have

$$ME(G) \geq 2 \left[ \sqrt{\frac{m + \sqrt{m^2 - 4(m - n + 1)(n - 3)}}{2}} + \sqrt{\frac{m - \sqrt{m^2 - 4(m - n + 1)(n - 3)}}{2}} \right]$$

with equality holding if and only if $G \cong G_{n,m}$ for $n + 3 \leq m \leq 2n - 4$ or $m = n + 1$; and $G \cong G_{n,m}$ or $G'_{n,n+2}$ for $m = n + 2$.

Proof. For any graph $G \in \mathcal{G}_{n,m}$, we have $m(G,0) = 1 = m(G_{n,m},0)$, $m(G,1) = m = m(G_{n,m},1)$ and $m(G,k) \geq 0 = m(G_{n,m},k)$ for $k \geq 3$. Next we should prove that, for any graph $G \in \mathcal{G}_{n,m}$,

$$m(G,2) \geq m(G_{n,m},2)$$

for $n + 3 \leq m \leq 2n - 4$ or $m = n + 1$, and

$$m(G,2) \geq m(G_{n,m},2) = m(G'_{n,n+2},2)$$

for $m = n + 2$.

By Lemmas 2.3 and 2.4, Eq.s (6) and (7) hold immediately.

Now the only task for proving this theorem is to compute the value of $ME(G_{n,m})$ for $n + 1 \leq m \leq 2n - 3$ and $ME(G'_{n,n+2})$. Thanks to Lemmas 2.3 and 2.4, again, we have

$$m(G_{n,m},2) = \binom{m}{2} + m - \frac{1}{2} \left[n(n - 1) + (m - n + 1)(m - n + 6)\right] = (m - n + 1)(n - 3).$$

Then the matching polynomial of $G_{n,m}$ is

$$\alpha(G_{n,m}, \lambda) = \lambda^n - m\lambda^{n-2} + (m - n + 1)(n - 3)\lambda^{n-4}.$$

Thus the non-zero roots of $\alpha(G_{n,m}, \lambda)$ are $\sqrt{\frac{m + \sqrt{m^2 - 4(m - n + 1)(n - 3)}}{2}}$ with twice and $\sqrt{\frac{m - \sqrt{m^2 - 4(m - n + 1)(n - 3)}}{2}}$ with twice. Therefore our result in this theorem follows.

From Theorem 3.1, the following corollary can be easily deduced.

Corollary 3.2. ([19]) Let $G$ be an $(n, n + 1)$-graph. Then we have

$$ME(G) \geq 2 \left[ \sqrt{\frac{n + 1 + \sqrt{n^2 - 6n + 25}}{2}} + \sqrt{\frac{n + 1 - \sqrt{n^2 - 6n + 25}}{2}} \right]$$

with equality holding if and only if $G \cong G_{n,n+1}$. 
Based on the relation (5), by a very similar reasoning as that in the proof of Theorem 3.1, the following corollary is straightforward.

**Corollary 3.3.** ([8, 22]) For any graph $G \in \mathcal{G}_{n,m}$ with $n + 2 \leq m \leq 2n - 4$, we have

$$z(G) \geq m(n - 2) - (n - 1)(n - 3) + 1$$

with equality holding if and only if $G \cong G_{n,m}$ for $n + 3 \leq m \leq 2n - 4$; and $G \cong G'_{n,n+2}$ for $m = n + 2$.

After obtaining the result in Theorem 3.1, we naturally ask the following problem: if matching number of graphs from $\mathcal{G}_{n,m}$ are given, what are the extremal graphs maximizing the matching energy under this condition? Equivalently, which graph has the maximal matching energy among all graphs from $\mathcal{G}_{n,m}(\beta)$?

![Figure 3: The graph $F_t(n, \beta)$](image)

Before dealing with this problem, we first introduce some notations. Recall that friendship graph $F_k$ is a graph of order $2k + 1$ obtained from $k$ triangles intersecting in a single vertex. An edge $e$ in $F_k$ is called linking edge if $e$ is incident with the vertex of degree $2k + 1$ in it. Denote by $F_t(n, \beta)$ (see Fig. 3) a graph obtained by attaching $n - 2\beta + 1$ pendent edges and $\beta - t - 1$ paths of length 2 to the vertex of degree $2t + 1$ in $F_t$. Clearly, we have $F_t(n, \beta) \in \mathcal{G}_{n,n-1+t}(\beta)$ with $1 \leq t \leq \beta - 1$. A vertex $v$ of a tree $T$ is called a branching point if $d(v) \geq 3$. Let $T_n(n_1, n_2, \ldots, n_m)$ be the tree of order $n$ obtained by inserting, respectively, $n_1 - 1, \ldots, n_m - 1$ vertices into the $m$ edges of the star $S_{m+1}$, where $n_1 + \ldots + n_m = n - 1$. For convenience, when considering the trees $T_n(n_1, n_2, \ldots, n_k, \ldots, n_m)$ we use the symbols $n^l_k$ to indicate that the number of $n_k$ is $l_k > 1$ in the following. For example, $T_{16}(2, 2, 3, 3, 5)$ will be written as $T_{16}(2^2, 3^2, 5)$. 
Lemma 3.4. ([18]) Let $T$ be a tree of order $n$ and with matching number $\beta$. Then

$$m(T,k) \geq m(T_n(2^{\beta - 1}, 1^{n-2\beta+1}), k)$$

for $k = 0, 1, \ldots, \beta$

with all equalities holding if and only if $T \cong T_n(2^{\beta - 1}, 1^{n-2\beta+1})$.

From the definition of quasi-order introduced in Section 1 and Formula (4), the following corollaries can be obtained immediately.

Corollary 3.5. For any tree $T \in G_{n,n-1}(\beta)$, we have

$$ME(T) \geq ME(T_n(2^{\beta - 1}, 1^{n-2\beta+1}))$$

with equality holding if and only if $T \cong T_n(2^{\beta - 1}, 1^{n-2\beta+1})$.

Corollary 3.6. Let $G$ be a graph of order $n$ and with matching number $\beta$. Then we have

$$m(G,k) \geq m(\beta K_2 \cup (n - 2\beta)K_1, k)$$

for $k = 0, 1, \ldots, \beta$

with all equalities holding if and only if $T \cong \beta K_2 \cup (n - 2\beta)K_1$.

In the following we will prove a generalized result of Lemma 3.4.

Theorem 3.7. For any graph $G \in G_{n,n-1+t}(\beta)$ with $1 \leq t \leq \beta - 1$, we have $ME(G) \geq ME(F_t(n, \beta))$ with equality holding if and only if $G \cong F_t(n, \beta)$.

Proof. We prove this result by induction on $t$. Firstly we deal with the case when $t = 1$. From the definition of the set $G_{n,n-1+t}(\beta)$, we find that, for any graph $G \in G_{n,n-1+t}(\beta)$ with $t = 1$, there exists an edge $e = v_iv_j$ in a unique cycle of $G$ such that $e \notin M$ where $M$ is a maximum matching of $G$. Note that $G - e \in G_{n,n-1}(\beta)$ and $G - \{v_i, v_j\}$ is with matching number $\beta - 2$. In view of Lemma 2.1, for $k = 0, 1, 2, \ldots, \beta$, we have

$$m(G,k) = m(G - e, k) + m(G - \{v_i, v_j\}, k - 1)$$

$$\geq m(T_n(2^{\beta - 1}, 1^{n-2\beta+1}), k) + m((\beta - 2)K_2 \cup (n - 2\beta + 2)K_1, k - 1)$$

by Lemma 3.4 and Corollary 3.6.

Similarly, by choosing $e = v'_iv'_j$ as an edge in the triangle incident with the vertex of maximum degree in $F_t(n, \beta)$, for $k = 0, 1, 2, \ldots, \beta$, we have

$$m(F_t(n, \beta), k) = m(F_t(n, \beta) - e, k) + m(F_t(n, \beta) - \{v'_i, v'_j\}, k - 1)$$

$$= m(T_n(2^{\beta - 1}, 1^{n-2\beta+1}), k) + m((\beta - 2)K_2 \cup (n - 2\beta + 2)K_1, k - 1).$$
By the definition of quasi-order, we have $G \succeq F_1(n, \beta)$ for any graph $G \in \mathcal{G}_{n,n}(\beta)$ with equality holding if and only if $G \cong F_1(n, \beta)$. Thanks to Formula (4), again, our result holds for $t = 1$.

Assume that our result holds for any graph $G \in \mathcal{G}_{n,n-1+k}(\beta)$ with $k$ fewer than $t \leq \beta - 1$. For any graph $G \in \mathcal{G}_{n,n-1+k}(\beta)$ with $M$ as its $\beta$-matching, we choose an edge $e = v_iv_j \in E(G)$ in a cycle of $G$ but not in $M$. By Lemma 2.1, Corollary 3.6 and induction hypothesis, for $k = 0, 1, 2 \ldots, \beta$, we have

$$m(G,k) = m(G-e,k) + m(G-\{v_i,v_j\},k-1) \geq m(F_{t-1}(n,\beta),k) + m((\beta - 2)K_2 \cup (n - 2\beta + 2)K_1, k - 1)$$

since $G-e \in \mathcal{G}_{n,n-2+t}(\beta)$

$$= m(F_t(n,\beta) - e',k) + m(F_t(n,\beta) - \{v'_i,v'_j\},k-1)$$

where $e' = v'_iv'_j \in E(F_t(n,\beta))$ is a linking edge of $F_t$ in it

$$= m(F_t(n,\beta),k).$$

Moreover, the above equality holds if and only if $G-v_iv_j \cong F_{t-1}(n,\beta)$ and $G-\{v_i,v_j\} \cong (\beta - 2)K_2 \cup (n - 2\beta + 2)K_1$, that is, $G \cong F_t(n,\beta)$. Therefore our result holds for $k = t$, finishing the proof of this theorem.

In view of the definition of Hosoya index and an efficient tool [11] to it: $z(G) = z(G-v_iv_j) + z(G-\{v_i,v_j\})$, we can obtain $z(T_n(2^{\beta-1},1^{n-2\beta+1})) = 2^{\beta-2}(2n-3m+3)$ [17] (by induction on $\beta$) and

$$z(F_t(n,\beta)) = z(F_t(n,\beta) - v_iv_j) + z(G-\{v_i,v_j\})$$

where $e = v_iv_j$ is a linking edge of $F_t$ in $F_t(n,\beta)$

$$= z(F_{t-1}(n,\beta)) + z((\beta - 2)K_2 \cup (n - 2\beta + 2)K_1)$$

$$= z(F_{t-1}(n,\beta) - v_kv_j) + z(G-\{v_k,v_j\}) + 2^{\beta-2}$$

where $e = v_kv_j$ is a linking edge of $F_t$ in $F_{t-1}(n,\beta)$

$$= z(F_{t-2}(n,\beta)) + z((\beta - 2)K_2 \cup (n - 2\beta + 2)K_1) + 2^{\beta-2}$$

$$= z(F_{t-2}(n,\beta)) + 2 \times 2^{\beta-2}$$

$$\ldots$$

$$= z(T_n(2^{\beta-1},1^{n-2\beta+1}))) + t2^{\beta-2}$$
\[ = 2^{\beta-2}(2n - 3m + 3) + t2^{\beta-2} \]
\[ = 2^{\beta-2}(2n - 3m + t + 3) \]

Based on Lemma 2.3 and quasi-order with Formula (4), respectively, the following two corollaries can be deduced immediately.

**Corollary 3.8.** Let \(1 \leq t \leq \beta - 1\) be an integer and \(G \in \mathcal{G}_{n,n-1+t}(\beta)\). Then we have

\[ M_1(G) \leq (n - \beta + t)^2 + 3(\beta + t) + n - 4 \]

with equality holding if and only if \(G \cong F_t(n, \beta)\).

**Corollary 3.9.** Let \(1 \leq t \leq \beta - 1\) be an integer and \(G \in \mathcal{G}_{n,n-1+t}(\beta)\). Then we have

\[ z(G) \geq 2^{\beta-2}(2n - 3m + t + 3) \]

with equality holding if and only if \(G \cong F_t(n, \beta)\).

By now we have completely determined the extremal graphs from \(\mathcal{G}_{n,m}\) with \(n \leq m \leq 2n - 4\) and \(\mathcal{G}_{n,n-1+t}(\beta)\) with \(1 \leq t \leq \beta - 1\), respectively, minimizing the matching energy. Naturally we will ask: what graphs from these two sets have the maximal matching energy, respectively? Furthermore, how about this problem when only limiting the order \(n\) and matching number \(\beta\) for the connected graphs? These problems are unknown to us, maybe they will be our research task in the future.

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**References**


