List-coloring graphs without $K_{4,k}$-minors

Ken-ichi Kawarabayashi
The National Institute of Informatics, 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan

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1. Introduction

In this note, it is shown that every graph with no $K_{4,k}$-minor is 4k-list-colorable. We also give an extremal function for the existence for a $K_{4,k}$-minor. Our proof implies that there is a linear time algorithm for showing that either $G$ has a $K_{4,k}$-minor or $G$ is 4k-choosable. In fact, if the latter holds, then the algorithm gives rise to a 4k-list-coloring.

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Abstract

In this note, it is shown that every graph with no $K_{4,k}$-minor is 4k-list-colorable. We also give an extremal function for the existence for a $K_{4,k}$-minor. Our proof implies that there is a linear time algorithm for showing that either $G$ has a $K_{4,k}$-minor or $G$ is 4k-choosable. In fact, if the latter holds, then the algorithm gives rise to a 4k-list-coloring.

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Conjecture 1.1 (Hadwiger [6]). For all $k \geq 1$, every $k$-chromatic graph has the complete graph $K_k$ on $k$ vertices as a minor.

For $k = 1, 2, 3$, it is easy to prove, and for $k = 4$, Hadwiger himself [6] and Dirac [5] proved it. For $k = 5$, however, it seems extremely difficult. In 1937, Wagner [29] proved that the case $k = 5$ is equivalent to the Four-Color Theorem. So, assuming the Four-Color Theorem, the case $k = 5$ of Hadwiger’s conjecture holds. In 1993, Robertson, Seymour and Thomas [23] proved that a minimal counterexample to the case $k = 6$ is a graph $G$ which has a vertex $v$ such that $G - v$ is planar. By the Four-Color Theorem, the case $k = 6$ of Hadwiger’s conjecture holds. This result is the deepest in this research area. Hence the cases $k = 5, 6$ are each equivalent to the Four-Color Theorem [1, 2, 22]. So far, the conjecture is open for every $k \geq 7$.

For the case $k = 7$, Toft and the author [14] proved that any 7-chromatic graph has $K_7$ or $K_{3,4}$ as a minor. Recently, the author [10] proved that any 7-chromatic graph has $K_7$ or $K_{3,5}$ as a minor.

It is even not known whether there exists an absolute constant $c$ such that every $ck$-chromatic graph has a $K_c$-minor. So far, it is known that there exists a constant $c$ such that every $c \sqrt{k \log k}$-chromatic graph has a $K_c$-minor. This follows from the results of Kostochka [17, 16] or Thomason [24, 25]. This was proved 25 years ago, but nobody can improve the superlinear order $k \sqrt{k \log k}$.

So it would be of great interest to prove that a linear function of the chromatic number is sufficient to force a $K_c$-minor. From an algorithmic view, we can “decide” this problem in polynomial time. This was proved in [11, 13]. We refer the reader to [27] for further information on Hadwiger’s conjecture.

Let $G$ be a graph. A list-assignment is a function $L$ which assigns to every vertex $v \in V(G)$ a set $L(v)$ of natural numbers, which are called admissible colors for that vertex. An $L$-coloring of the graph $G$ is an assignment of admissible colors to all vertices of $G$, i.e., a function $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for every $v \in V(G)$, and $c(u) \neq c(v)$ for every edge $uv$. If $k$ is an integer and $|L(v)| \geq k$ for every $v \in V(G)$, then $L$ is a $k$-list-assignment. A graph is $k$-choosable if it admits an $L$-coloring for every $k$-list-assignment $L$. 

E-mail address: kkeniti@nii.ac.jp.
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When relaxing Hadwiger's conjecture to allow $ck$ colors, the following conjecture involving list-colorings may also be true.

**Conjecture 1.2.** There exists a constant $c$ such that every graph without $K_k$-minors is $ck$-choosable.

This was conjectured in [12]. Also, the following weaker version of the choosability analog of Hadwiger's conjecture was given by Woodall [32,33].

**Conjecture 1.3 (Woodall [32]).** Every graph with no $K_{r,s}$-minor is $(r + s - 1)$-choosable.

Note that the choosability analog of Hadwiger's conjecture is false, because there is a planar graph which is not 4-choosable [28] (but every planar graph is 5-choosable [26]).

Again, it is even not known that Conjecture 1.2 holds, so it would be of interest to know even the cases where $r$ is small. For the usual graph coloring, Woodall [31] made a related conjecture. Woodall [32,33] proved Conjecture 1.3 when $r = 1, 2$.

The main purpose of this note concerns the cases $r = 3, 4$. We shall prove the following results.

**Theorem 1.4.** If $G$ does not contain a $K_{4,k}$-minor, then $G$ is $4k$-choosable.

This immediately implies that every graph without a $K_{4,k}$-minor is $4k$-choosable.

Actually, Theorem 1.4 follows from the following extremal function for the existence of a $K_{4,k}$-minor, which is of independent interest.

**Theorem 1.5.** Let $G$ be a graph such that $|V(G)| > 2k + 2$ and $|E(G)| > 2k(|V(G)| - k - 1) + 1$, where $k \geq 2$. Then $G$ has a $K_{4,k}$-minor.

Let us remark that the extremal function “$2k(|V(G)| - k - 1) + 1$” is perhaps not best possible, and we conjecture that the factor 2 is not necessary. If true, this, together with our proof, would imply that every graph without a $K_{4,k}$-minor is $2k$-choosable. Let us observe that the cases $k \leq 4$ are already proved in [7], and are best possible.

There are several results [3,15,18–20] concerning the extremal function for the existence of complete bipartite graph minors. These results say that if $k$ is large enough compared to $s$, then an average degree just over $k$ suffices to ensure the existence of a $K_{k,s}$-minor. In particular, the result in [15] implies that if $k$ is sufficiently large, then every graph with no $K_{4,k}$-minor is $(k + 13)$-choosable. But these results give no information on the chromatic number of a graph with no $K_{4,k}$-minor when $k$ is not sufficiently large. Hence it is useful to have our result, which applies to every graph with no $K_{4,k}$-minor. Note that when $k = 4$, the result in [7] implies that every graph with no $K_{4,4}$-minor is 8-choosable.

To see that Theorem 1.5 implies Theorem 1.4, consider a minimal counterexample $G$ to Theorem 1.4. When $k = 1$, then the result easily holds. So suppose $k \geq 2$. Then it is easy to prove that the minimum degree is at least $4k$. For suppose not and that $v$ has degree at most $4k - 1$. Then, by induction, $G - v$ has a desired list-coloring. Since $v$ has at least $4k$ colors available, we can easily extend the list-coloring of $G - v$ to $G$. Hence every vertex has degree at least $4k$, and this implies that $G$ has at least $4k + 1$ vertices and at least $2k|V(G)|$ edges. On the other hand, by Theorem 1.5, if $G$ has at least $2k|V(G)|$ edges for $k \geq 2$ and $|V(G)| \geq 2k + 2$, then $G$ has a $K_{4,k}$-minor. Thus we can conclude that $G$ is not a counterexample, and hence Theorem 1.4 holds.

Let us point out that this proof implies a polynomial time algorithm to $4k$-list-color a graph with no $K_{4,k}$-minor. More precisely, there is a linear time algorithm (linear in the number of vertices plus the number of edges of $G$) which shows either

1. $G$ is $4k$-choosable, or
2. $G$ has a $K_{4,k}$-minor.

Actually, when (1) holds, the algorithm gives a $4k$-list-coloring. To see this, if $G$ has a vertex $v$ of degree at most $4k - 1$, then we just delete $v$ from $G$. We keep doing this procedure until there is no vertex of degree at most $4k - 1$. If this procedure continues until the resulting graph is empty, then clearly we can 4k-list-color $G$ recursively in linear time. If the resulting graph is not empty, this graph contains a $K_{4,k}$-minor by Theorem 1.5. This can be clearly found in linear time.

In order to prove our main result, we need the definition of “rooted minor”.

If $v_1, \ldots, v_k$ are $k$ distinct vertices in a graph $G$, then we say that $G$ has a $K_{a,k}$-minor rooted at $v_1, \ldots, v_k$ if there are disjoint connected subgraphs $H_1, \ldots, H_k, K_1, \ldots, K_k$ such that each $K_i$ contains $v_i$ and is adjacent to all of $H_1, \ldots, H_k$. We say that $G$ has every rooted $K_{a,k}$-minor if, for every choice of $k$ distinct vertices $v_1, \ldots, v_k$ of $G$, $G$ has a $K_{a,k}$-minor rooted at $v_1, \ldots, v_k$.

Rooted minor problems are studied by many researchers. For example, the following result is known.

**Theorem 1.6 (Robertson–Seymour [21]).** Suppose $G$ is 3-connected. Then $G$ has every rooted $K_{2,3}$-minor unless $G$ is planar, in which case it has a $K_{2,3}$-minor rooted at $v_1, v_2, v_3$ if and only if $v_1, v_2, v_3$ are not all on the boundary of the same face.
Theorem 1.6 follows from (2.4) in [21]. See also (3.5) in [23].
Actually, this result is used to prove more interesting results on Hadwiger's conjecture; see [14,23].
Furthermore, every rooted $K_{2,4}$-minor problem and every rooted $K_{3,4}$-minor problem are discussed in [7,8], respectively. Moreover, the general connectivity function for the existence of every rooted minor is obtained by the author [9].
Recently, the following extremal function for the existence of every rooted $K_{2,k}$-minor was obtained by Wollan [30].

**Theorem 1.7.** Every $k$-connected graph $G$ on $n$ vertices with at least $kn - \left(\frac{k+1}{2}\right) + 1$ edges contains every rooted $K_{2,k}$-minor.

The lower bound on the number of edges in Theorem 1.7 is best possible. To see this, let $G$ be the join of a maximal planar graph $H$ of order $n - k + 3$, and a complete graph $K_{k-3}$. Then $|E(G)| = 3(n - k + 3) - 6 + \left(\frac{k-3}{2}\right) + (n - k + 3)(k - 3) = kn - \left(\frac{k+1}{2}\right)$. If $v_1, v_2, v_3$ are in one face of $H$ and $v_4, \ldots, v_k$ are the vertices of $K_{k-3}$, then just as in Theorem 1.6, there is no $K_{2,k}$-minor rooted at $v_1, \ldots, v_k$.

We will use Theorem 1.7 in our main proof.

2. **Proof of the main theorem**

We prove Theorem 1.5 by induction on the number of vertices. We first prove the case $|V(G)| = 2k + 2$. Then it is easy to see that $|E(G)| \geq \binom{2k+2}{2} - k$. Therefore, there are at most $k$ missing edges in $G$. Suppose $k \geq 4$. Then clearly there are four vertices that have at least $n - 6 = 2k - 4 \geq k$ common neighbors. Therefore, $G$ has $K_{4,k}$ as a subgraph. If $k \leq 3$, then $G$ has at most eight vertices. Since there are at most three missing edges, we can easily find $K_{4,k}$ as a subgraph in $G$.

Thus we may assume $|V(G)| \geq 2k + 3$. If $G$ has a vertex $v$ of degree at most $2k$, then

$$|E(G - v)| \geq |E(G)| - 2k \geq 2k(n - k - 1) + 1 - 2k = 2k(|V(G) - v| - k - 1) + 1.$$  

Thus by induction, $G - v$ has a $K_{4,k}$-minor. So, we may assume that every vertex in $G$ has degree at least $2k + 1$.

We now claim that $G$ is $(k + 2)$-connected. Suppose not. Then $G$ has subgraphs $A$ and $B$ such that $G = A \cup B$, both $A - B$ and $B - A$ are not empty, and $|V(A) \cap V(B)| \leq k + 1$. Since every vertex has degree at least $2k + 1$, it follows that $|A|, |B| \geq 2k + 2$.

We may assume that neither $A$ nor $B$ satisfies the induction hypothesis of Theorem 1.5, so otherwise there is a $K_{4,k}$-minor in either $A$ or $B$ by the induction hypothesis. Then

$$|E(G)| \leq |E(A)| + |E(B)| \leq 2k(|V(A)| - k - 1) + 2k(|V(B)| - k - 1) \leq 2k(|V(G)| + k + 1 - 2k - 2) \leq 2k(|V(G)| - k - 1).$$

This contradicts the hypothesis of Theorem 1.5. Thus $G$ is $(k + 2)$-connected, as claimed.

We also claim that every edge in $E(G)$ is contained in at least $2k$ triangles. Suppose not, and there is an edge $e \in E(G)$ such that $e$ is contained in at most $2k - 1$ triangles. Let $G'$ be the graph obtained from $G$ by contracting $e$. Then it follows that

$$|E(G')| \geq |E(G)| - 2k \geq 2k(|V(G)| - k - 1) + 1.$$  

Thus $G'$ satisfies the induction hypothesis of Theorem 1.5, and so $G'$ contains a $K_{4,k}$-minor. Hence we may assume that every edge is contained in at least $2k$ triangles.

Let $e = xy$ be any edge of $E(G)$, and let $v_1, \ldots, v_k$ be $k$ vertices that are adjacent to both $x$ and $y$. Let $G'' = G - \{x, y\}$. Clearly $G''$ is $k$-connected since $G$ is $(k + 2)$-connected. Note that $G''$ omits at most $2|V(G)| - 3$ edges of $G$. Suppose $k \geq 3$.

Since $G$ has at least $2k + 3$ vertices,

$$|E(G'')| \geq |E(G)| - 2|V(G)| + 3 \geq 2k(|V(G)| - k - 1) + 1 - 2|V(G)| + 3 = k(|V(G)| - 2) + (k - 2)|V(G)| - 2k^2 + 4 \geq k(|V(G)| - 2) - k - 2 \geq k|V(G'')| - \left(\frac{k + 1}{2}\right) + 1.$$  

Thus $G''$ has a $K_{2,k}$-minor rooted at $v_1, \ldots, v_k$ by Theorem 1.7. Together with $x, y$, this gives rise to a $K_{4,k}$-minor.

It remains to consider the case $k = 2$. If $G$ is a complete graph, clearly it contains $K_{4,2}$ as a subgraph. Take two nonadjacent vertices $x', y'$. Since $G$ is now $4$-connected, there are four disjoint paths between $x'$ and $y'$, and clearly this gives rise to a $K_{4,2}$-minor.

This completes the proof of Theorem 1.5.
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