Discrete Optimization

A location-routing problem for the conversion to the “click-and-mortar” retailing: The static case

Deniz Aksen a,*, Kemal Altinkemer b

a Koç University, College of Administrative Sciences and Economics, Rumeli Feneri Yolu, Sarıyer, Istanbul 34450, Turkey
b Purdue University, Krannert Graduate School of Management, West Lafayette, IN 47907, United States

Received 8 March 2005; accepted 29 January 2007
Available online 16 March 2007

Abstract

The static conversion from brick-and-mortar retailing to the hybrid click-and-mortar business model is studied from the perspective of distribution logistics. Retailers run warehouses and brick-and-mortar stores to meet the demand of their walk-in customers. When they decide to operate on the Web as an e-tailer, also click-and-mortar stores are needed which can serve both walk-in and online customers. While the distance between home and the nearest open store is used as a proxy measure for walk-in customers, a quality of service (QoS) guarantee for online customers is timely delivery of their orders. We describe and solve a static location-routing based problem for companies that embrace the clicks-and-bricks strategy in their retail operations. An augmented Lagrangian relaxation method embedded in a subgradient optimization procedure generates lower bounds, whereas a heuristic method finds feasible solutions. The performance of the Lagrangian-based solution method is tested on a number of randomly generated test problems.

© 2007 Elsevier B.V. All rights reserved.

Keywords: Click-and-mortar; Location routing; Augmented Lagrangian relaxation; Distribution

1. Introduction

Most national retailers in the USA are selling today their products and goods online, and the term e-tailing has been now widely accepted as a synonym for online retailing. It all started in the first era of e-commerce (1995–2000) where we saw an infusion of pure-play (or Internet-only) businesses. These businesses offered Web ordering and delivery, and operated with one or more warehouses, but without retail storefronts. In the second era of e-commerce, which is said to commence in January 2001, many firms began to use a mixed “clicks-and-bricks” strategy, combining traditional sales channels such as physical stores, order-by-phone and printed catalogs with online efforts. This strategy led to the birth of a hybrid store type called click-and-mortar. Consumers’ desire for companies’ real-world presence with physical facilities was the main motive behind this transformation. They wanted to touch-and-feel, try and talk before they agreed to pay the merchants. The
shift to a click-and-mortar strategy is often found in the grocery segment of retailing since late 1990’s (Hays et al., 2005). Mentioned in Hays et al. as the most prominent brand-name grocery stores now online are Albertsons, Publix and Safeway in the USA, Fairprice and Cold Storage in Singapore, and Wellcome Supermarkets and Park’n Shop in Hong Kong. Van Mieghem (2001) considers UK’s biggest Internet grocer Tesco.com a thriving click-and-mortar business outside the USA. Tesco’s successful Internet shopping services eventually steered the company to the USA, and led to a partnership with Safeway in June 2001.

In this paper, we study a static location-routing based optimization problem to model the conversion of a traditional retailer that embraces the clicks-and-bricks strategy. The solution of the problem under investigation, which we call CMBP-S, reveals a location–allocation profile for the stores of the retailer, and a routing plan for its delivery vehicles. These entities of the retailer serve two customer types, namely walk-in and online customers. We propose a Lagrangian relaxation method embedded in a subgradient optimization procedure to solve the problem CMBP-S. The paper is organized as follows. Section 1.1 dwells on the conversion to click-and-mortar retailing operations while Section 1.2 gives the complete description and underlying assumptions of the CMBP-S. Section 2 provides an extensive review of location routing and distribution network design problems in the literature, and stresses the characteristics of the CMBP-S. In Section 3 we present a mixed integer programming (MIP) formulation of our problem. This is followed by the discussion of the Lagrangian relaxation and the solution of the resulting subproblems in Section 4. Section 5 features computer experiments with the proposed solution methodology and the results thereof. The paper concludes with future research directions in Section 6.

1.1. The conversion to click-and-mortar retailing operations

We consider a traditional brick-and-mortar retailer that operates two types of facilities and serves only one type of customers. Its facilities comprise warehouses (WH’s) and physical stores, where goods are transferred from the former to the latter in direct shipments. Goods are then sold to walk-in type customers who are assumed to go to the nearest physical store for shopping. When the retailer opens a website for the online shopping convenience, it will need the capability of receiving, processing and then delivering orders placed by online customers at that website. Some of its present brick-and-mortar stores (BM’s) might have to be reconfigured, or several new stores might be opened with that capability. A physical store is designated as a click-and-mortar store (CM) if it is Internet-enabled, equipped with the necessary hardware, software and personnel such that it can effectively handle online orders. A physical store can serve online customers only then if it is opened as a CM or if it is a BM that is reconfigured as (converted to) a CM. Online customers have different expectations and service requirements than walk-in customers. Most of the time, their orders are delivered to their residences while walk-in customers visit the stores in person. Sometimes, these two types might exchange their roles. An online customer places his or her order on the website of the retailer, but prefers going to the nearest store to pick up the order. On the other hand, a walk-in customer can buy a bulky good in the store, and might need it delivered home. Our criterion in determining the customer type is whether or not the purchased good is to be delivered subject to a time restriction. In this case, we consider him or her an online customer.

1.2. The static click-and-mortar business problem

Customers and facilities of the retailer in the static click-and-mortar business problem (CMBP-S) are given symbolically in Fig. 1 along with the direction of travel between these. Any CM can serve both customer types as shown in Fig. 1. BM’s, on the other hand, can only serve walk-in customers. The orders of online customers can be delivered only from CM’s. The retailer may decide to open one or more new stores. These will be built as CM’s by default. There are three options regarding any of the already present physical stores (BM’s):

(i) It can be preserved as is. In this case, it cannot deliver orders to online customers.
(ii) It can be reconfigured as a CM, which would enable it to serve online customers as well.
(iii) It can be closed, as a result of which some walk-in customers in its vicinity could be lost.
The CMBP-S is applicable especially in the retailing of non-durable goods. Customers who order on the Web such items as flowers or groceries usually expect fast and prompt delivery. Typical temporal constraints upon the deliveries are time windows. In the CMBP-S, however, we assume that deliveries to online customer addresses are restricted by time deadlines only. Depending on the limited capacity of the homogeneous vehicle fleet and on the time deadlines, customers are visited on one or more arc-disjoint tours. A single tour consolidating all orders at a CM will most likely be infeasible. Instead, multiple tours should be constructed. The conversion is as a whole a stochastic and dynamic problem. However, the following simplifying assumptions reduce the problem into a single period static problem.

- Facilities bear unlimited storage space, which implies that they can supply/transship any volume of demand. Moreover, stores act as no-inventory transit-depots.
- The reconfiguration of a BM as a CM is associated with some fixed cost. The alternative of shipping orders from a WH is studied in the scenario analyses in Aksen and Altinkemer (2003).
- In order for a store to serve a walk-in customer, it must be located within the maximum driving (walking) distance of that customer. If there is no such store, the retailer loses his or her demand. Walk-in customers will always choose the nearest among multiple open stores within the maximum distance. This is only a proxy measure since in reality a customer may visit multiple stores in one shopping trip.
Any BM can be closed at some fixed cost or gain. Likewise, opening a new CM incurs a fixed cost. All fixed costs and gains are known a priori. They are amortized with a discount factor to a daily basis so that they match with the variable costs of operating the stores.

At each store delivering online orders, special vehicles have to be acquired at a fixed acquisition cost per vehicle in order to carry these orders. Trucks currently used by the retailer for the bulk shipping of goods from WH’s to stores are unsuitable for home deliveries since such deliveries are small-size.

Shipments from WH’s to stores go in trucks on direct replenishment routes. No two stores are visited on the same replenishment route.

Each vehicle should return to its own store. No open route is permitted between stores and customers.

The complete matrix of distances is also known a priori.

The flow of goods from WH’s to stores will have to match the amount of goods sold or delivered from stores to customers. We assume that these two flows are synchronized. Hence, no inventories build up, and inventory holding costs are ignored.

Online orders are pooled before all vehicles are dispatched at the same time.

The last assumption implies that orders are not received on a real-time basis. Stores wait until a certain hour of the day to get all orders. No order is accepted afterward. Each deadline is assumed to be after that time. Therefore, CMBP-S is a single-period static problem. This situation is different than the home delivery problem (HDP) of consumer direct full-line grocery services. First introduced by Campbell and Savelsbergh (2005), the HDP is quite a new problem where the vendor dynamically decides which deliveries to accept or reject and the time slot for the delivery, if it is accepted, to maximize expected profits. The HDP also involves determining cost-effective incentive schemes for customers to choose less stringent time slots for their deliveries. In this aspect, it supports operational decisions of very short terms, but does not make long-term decisions like opening new stores or Internet-enabling existing stores.

2. Literature review and problem characteristics

The problem CMBP-S basically combines a multisource facility location-allocation problem (FLAP) with a network design problem which involves vehicle routing decisions. The newest annotated literature review of the location-routing problem (LRP) and its extensions is currently due to Ahipaşaoğlu et al. (2003). A perfect synthesis and survey of the LRP is accomplished earlier by Min et al. (1998). They write: “The concept of integrated logistics systems recognizes the interdependence among the location of facilities, the allocation of suppliers and customers to the facilities, and the vehicle route structure around depots. As such, it coordinates a broader spectrum of location and routing options available to logistics managers and consequently avoids the suboptimization of distribution solutions.” Min et al. define the location-routing model as solving the joint problem of determining the optimal number, capacity and location of facilities serving more than one customer/supplier (demand node), and finding the optimal set of vehicle schedules and routes. Its main difference from the classical location/allocation problem is that, once the facility is located, LRP requires a visitation of demand nodes through tours, whereas the latter assumes a straight-line or radial trip from the facility to the demand node. Actually, the effect of ignoring routes when locating depots has been stressed earlier by Sallhi and Rand (1989). Min et al. argue that “Significant productivity gains can be achieved through the design of combined location-routing models that may determine true least-cost solutions to a logistic problem in light of both strategic (facility location) and operational (vehicle routing) policy. Increasing popularity of combined location-routing models reflects such a trend.” Sequential methods consisting of decomposition have their limitation. The authors recommend solving the whole LRP concurrently in order to be able to analyze tradeoffs between location and routing factors at the same level of decision hierarchy.

Jayaraman and Ross (2003) formally define supply chain and its management. They present a simulated annealing approach to a multi-echelon distribution network design problem (DNDP). It is a 2-phase methodology solving first the location model, then the replenishment (distribution) model. There is a central manufacturing plant that transports products to distribution centers (DC’s) which in turn transship products to cross-docking sites. Demands of customer zones are then satisfied by direct shipments from cross-docking sites. DC’s and cross-docking sites in Jayaraman and Ross quite resemble the WH’s and stores in our
CMBP-S in the sense that products are directly and continuously transshipped from one layer of the supply chain to a lower layer without ever sitting in inventory. With this practice, retail giants like the department store chain Wal-Mart and the pharmacy chain Walgreens achieve the economies that come with purchasing full truckloads of products while avoiding the usual inventory and handling costs. Jayaraman and Ross’ model is also a two-stage cost minimization model uniting strategic and operational decision-making processes. In the joint objective function of their model, they add fixed costs to operate/open facilities with the variable costs to transport/transship products between the layers of the supply chain. A likely outcome of their model is the reconfiguration of product flows, given a set of facilities already operating, and not necessarily the complete redesign and location of all facilities and product flows in the system. Besides these similarities, nonetheless, the DNDP in Jayaraman and Ross does not address the routing issues in the operational stage.

This deficiency of the DNDP has been sorted out by a recent work of Ambrosino and Scutellà (2005). Along with an extensive literature review of the LRP and DNDP, the authors attribute the significance of strategic decisions like facility locations, transportation and inventory levels in the DNDP to an article by Crainic and Laporte (1997). These decisions are core problems for each company, mainly because they influence also tactical and operational decisions like customer allocations and routes, which subsequently determine the cost of the distribution system and the quality of the customer service. Ambrosino and Scutellà adopt Laporte’s LRP classification (1988). Their proposed integrated DNDP can be formulated as an LRP of category 4/R/T/T involving facility, warehousing and transportation as well as inventory decisions. This category label means distribution networks made up of four layers, with routes of type replenishment (direct shipments) and type tour (vehicle routes.) According to the same labeling, our CMBP-S would be categorized as 3/R/T since it is a 3-layer model with fixed warehouses, present and candidate stores, and known customer nodes. There is state-of-the-art research focused on the standard 2-layer multi-depot LRP in the recent literature. Tuğzun and Burke (1999) propose a two-phase tabu search architecture for the solution of the multi-depot LRP where depots have unlimited throughput capacity. Wu et al. (2002) decompose the standard LRP with capacitated depots into a FLAP and a vehicle routing problem (VRP), and try then to solve both subproblems using simulated annealing. For the same class of the LRP, Albareda-Sambola et al. (2005) apply a method which generates first a lower bound either from the linear relaxation of the given problem or from the solutions of a pair of ad hoc knapsack and asymmetric traveling salesman problems. This lower bound is then used as a starting point of a tabu search heuristic. Lastly, Melechovský et al. (2005) address an LRP with non-linear depot costs that grow with the total demand satisfied by the depots. They present a hybrid metaheuristic consisting of tabu search and variable neighborhood search heuristics.

Tuğzun and Burke (1999) explain that the 2-layer multi-depot LRP with unlimited throughput capacities belongs to the class of \( \mathcal{NP} \)-hard problems. Being a 3/R/T LRP with two types of customers and three modes for stores, also our problem CMBP-S is \( \mathcal{NP} \)-hard. In compliance with Min et al. (1998) classification scheme, we can list the characteristics of CMBP-S as follows:

I. Two stage (transshipment from WH’s to stores, then from stores to customers).
II. Deterministic demand and supply.
III. Multiple facilities.
IV. Multiple homogeneous vehicles.
V. Capacitated vehicles.
VI. Uncapacitated facilities.
VII. 3-layer (WH’s, BM’s and CM’s, walk-in and online customers).
VIII. Single period (static).
IX. Single-sided hard time windows (time deadlines) required by online customers.
X. Single joint objective function.
XI. Hypothetical (randomly generated) model data.

The objectives jointly considered in our problem CMBP-S are total cost and total distance which are interrelated with each other. We can achieve the shorter vehicle routes the more service centers are made available. However, the more service centers are opened the higher fixed costs are incurred.
3. Formulation of the CMBP-S

A mathematical model of the CMBP-S comprises the following index sets, parameters, and decision variables.

Index sets

- $I_o$ set of online customers = \{1, \ldots, N_o\}
- $I_w$ set of walk-in customers = \{N_o + 1, \ldots, N_o + N_w\}
- $I$ set of customers = $I_o \cup I_w$ = \{1, \ldots, N\} where $N = N_o + N_w$
- $J_B$ set of BM’s (locations) = \{1, \ldots, M_B\}
- $J_C$ set of potential CM’s (locations) = \{M_B + 1, \ldots, M_B + M_c\}
- $J_S$ set of all present and potential stores = $J_B \cup J_C$ = \{1, \ldots, M_B + M_c\}
- $J_W$ set of warehouses (locations) = \{M_B + M_C + 1, \ldots, M_B + M_C + M_W\}
- $J$ set of all present and potential facility locations = $J_w \cup J_B \cup J_C$ = \{1, \ldots, M\} where $M = M_B + M_C + M_W$

Parameters

- $d_i$ demand of customer $i$ ($i \in I$)
- $c_{ij}$ distance between locations $i$ and $j$ ($i, j \in I \cup J$, $c_{ii} = \infty$)
- $t_{ij}$ traveling time between locations $i$ and $j$ ($i, j \in I \cup J$, $t_{ii} = \infty$)
- $s_{ti}$ service time (duration of delivery operations) at online customer $i$ ($i \in I_o$)
- $p_i$ cost of losing one unit demand of walk-in customer $i$ ($i \in I_w$)
- $ac_1$ walk-in customers’ average cost per mileage of traveling to any store
- $ac_2$ average cost per mileage of transporting one unit good from warehouses to stores
- $ac_3$ any vehicle’s average cost per mileage of traveling between stores and online customers
- $FC_j$ daily fixed cost/gain of opening (closing) a click-and-mortar (brick-and-mortar) store ($j \in J_S$)
- $CC_j$ daily fixed cost of reconfiguring a BM $j$ as a CM ($j \in J_B$)
- $OC_j$ daily fixed cost of operating a store ($j \in J_S$)
- $FCV_j$ daily sunk cost of operating a vehicle at facility $j$ ($j \in J$)
- $Q$ uniform vehicle capacity of trucks used by stores
- $Q^w$ uniform vehicle capacity of trucks used by warehouses
- $TD_i$ latest delivery time at the online customer location $i$ ($i \in I_o$) including also the service time $s_{ti}$
- $MaxW$ maximum driving (walking) distance of walk-in customers to stores
- $C_{max}$ the maximum of the distances between walk-in customers and stores, i.e., $\max_{(i,j) \in I_o \times J_w} \{c_{ij}\}$

Nonnegative continuous decision variables

- $s_{jk}$ flow of goods from warehouse $j$ ($j \in J_W$) to store $k$ ($k \in J_S$)
- $a_i$ arrival time of a vehicle at the location of online customer $i$ ($i \in I_o$)

Nonnegative integer decision variables

- $v_j$ number of vehicles that should depart from store $j$ ($j \in J_S$)
- $V_{jk}$ number of vehicles that should depart from warehouse $j$ ($j \in J_W$) to store $k$ ($k \in J_S$)

Binary (indicator) decision variables

- $y^b_j = \begin{cases} 1 & \text{if a brick-and-mortar store } j \ (j \in J_B) \text{ is kept at its current location} \\ 0 & \text{o/w} \end{cases}$
- $y^c_j = \begin{cases} 1 & \text{if a click and mortar store is opened at the potential location } j \ (j \in J_C) \\ 0 & \text{o/w} \end{cases}$
- $\theta_j = \begin{cases} 1 & \text{if a brick-and-mortar store } j \ (j \in J_B) \text{ is to be converted to a click and mortar store} \\ 0 & \text{o/w} \end{cases}$
\( \delta_{ij}^w = \begin{cases} 1 & \text{if store } j \ (j \in J^S) \text{ serves walk-in customer } i \ (i \in I_w) \\ 0 & \text{o/w} \end{cases} \)

\( \delta_{i}^{WL} = \begin{cases} 1 & \text{if the demand of walk-in customer } i \text{ is lost } (i \in I_w) \\ 0 & \text{o/w} \end{cases} \)

\( \delta_{ij}^o = \begin{cases} 1 & \text{if store } j \ (j \in J^S) \text{ serves online customer } i \ (i \in I_o) \\ 0 & \text{o/w} \end{cases} \)

\( x_{jk}^i = \begin{cases} 1 & \text{if arc } (i,k) \text{ is traversed by a vehicle sent from store } j, \text{ where } (j \in J^S) \text{ and } (i,k \in J^S \cup I_o) \\ 0 & \text{o/w} \end{cases} \)

### 3.1. The objective function

There are three types of constraints in the CMBP-S. The first type originates from the facility location-allocation problem (FLAP). The second type belongs to the multi-depot vehicle routing problem with time deadlines (MDVRPTD). In addition, there are coupling constraints which ensure the connectivity between the FLAP and MDVRPTD. We shall refer to the CMBP-S as the optimization problem \( \mathbf{P} \) henceforth. The structure of \( \mathbf{P} \) is exposed below with the figures in brackets indicating the number of constraint equations excluding nonnegativity and integrality restrictions. The joint objective function of \( \mathbf{P} \) is a weighted sum of several total cost terms as shown in (1).

Minimize \( Z_p = \sum \text{FLAP Objectives} + \sum \text{MDVRPTD Objectives} \)

Subject to:

(A) Pure FLAP constraints, \([M_B + M_C](2 + M_W + N_w) + M_B + N_o + 3N_w\].

(B) Pure MDVRPTD constraints, \(2(M_B + M_C) + N_o^2 + 2N_o\].

(C) FLAP and MDVRPTD coupling constraints, \(2N_o(M_B + M_C)\].

\[
\begin{align*}
\sum_{j \in J_n} c_j \theta_j + \sum_{j \in J_c} (F_{cj} + O_{cj}) y_j^c + \sum_{j \in J_n} F_{cj}(1 - y_j^c) + \sum_{j \in J_n} O_{cj} y_j^c + \sum_{j \in J_W} \sum_{k \in J^S} FCV_{jk} \cdot V_{jk} \\
+ ac_1 \cdot \sum_{i \in I_w} \sum_{j \in J^S} c_{ij} \delta_{ij}^w + ac_2 \cdot \sum_{j \in J_W} \sum_{k \in J^S} c_{jk} s_{jk} + \sum_{i \in I_w} p_{di} d_{ij}^{WL} + \sum_{j \in J^S} FCV_j \cdot v_j + ac_3 \cdot \sum_{j \in J^S} \sum_{k \in I_o \cup \{j\}} c_{jk} x_{jk}^i.
\end{align*}
\]

The first term in the objective function in (1) shows the cost of reconfiguring BM’s as CM’s. The second term denotes the cost of opening and operating CM’s at potential locations. The third term is the cost of closing currently operating BM’s. The cost of operating open BM’s, and the fixed cost of vehicle acquisition at warehouses for the shipment of goods to stores are given in the fourth and fifth terms, respectively. The sixth and seventh terms are the cost of total distance traveled between stores and walk-in customers, and the cost of transferring goods between warehouses and stores. The eighth term yields the cost of lost sales due to the absence of an open store within the maximum driving (walking) distance. The cost of vehicle acquisition at stores for the delivery of online orders and the cost of delivering goods from stores to online customer locations are given in the last two terms, respectively.

### 3.2. The constraints

\[
\sum_{j \in J^S} \delta_{ij}^w + \delta_{i}^{WL} = 1 \quad \forall i \in I_w, \tag{2}
\]

\[
\sum_{j \in J^S} \delta_{ij}^o = 1 \quad \forall i \in I_o, \tag{3}
\]
\[
\sum_{j \in J_w} s_{jk} = \sum_{i \in I_{w}} \delta_{ik}^w \cdot d_i + \sum_{i \in I_{w}} \delta_{ik}^o \cdot d_i \quad \forall k \in J^S,
\]

(4)

\[
V_{jk} \geq \frac{s_{jk}}{Q^w} \quad \forall j \in J_w, \quad \forall k \in J^S,
\]

(5)

\[
\sum_{i \in I_{w}} \delta_{ij}^o + \sum_{i \in I_{w}} \delta_{ij}^w \leq |I| \cdot y_j^b \quad \forall j \in J_B,
\]

(6)

\[
\sum_{i \in I_{w}} \delta_{ij}^o + \sum_{i \in I_{w}} \delta_{ij}^w \leq |I| \cdot y_j^c \quad \forall j \in J_C,
\]

(7)

\[
\sum_{i \in I_{w}} \delta_{ij}^b \leq |I_o| \cdot \theta_j \quad \forall j \in J_B,
\]

(8)

\[
\sum_{j \in J^S} c_{ij} \cdot \delta_{ij}^w \leq \text{Max} W \quad \forall i \in I_w,
\]

(9)

\[
\sum_{j \in J^S(i), j \neq j_{sh}(i)} \delta_{ij}^w \geq \frac{1}{|J^S|} \left( \sum_{j \in J^S(i)} y_j^c + \sum_{j \in J^S(i)} y_j^b \right) \quad \forall i \in I_w,
\]

(10)

\[
\sum_{j \in J^S} c_{ij} \cdot \delta_{ij}^w \leq C_{\text{max}} (1 - y_k^b) + c_{ik} \quad \forall i \in I_w, \quad \forall k \in J_B,
\]

(11)

\[
\sum_{j \in J^S} c_{ij} \cdot \delta_{ij}^w \leq C_{\text{max}} (1 - y_k^c) + c_{ik} \quad \forall i \in I_w, \quad \forall k \in J_C,
\]

(12)

\[
\sum_{i \in I_{w}} x_{ji}^j = v_j \quad \forall j \in J^S,
\]

(13)

\[
\sum_{i \in I_{w}} x_{ij}^j = \sum_{i \in I_{w}} x_{ij}^i \quad \forall j \in J^S,
\]

(14)

\[
\sum_{j \in J^S} \sum_{i \in I_{w}} x_{ji}^j + \sum_{j \in J^S} \sum_{i \in I_{w}} \sum_{k \in I_{w}} x_{jk}^j = |I_o|,
\]

(15)

\[
\sum_{j \in J^S} \sum_{i \in I_o \setminus S_0} \sum_{k \neq i} x_{ik}^j \leq |S_0| - L_{S_0} \quad \forall S_0 \subseteq I_o \ni |S_0| \geq 2,
\]

(16)

\[
a_i = \sum_{j \in J^S} t_{ij} x_{ji}^j \quad \forall i \in I_o,
\]

(17)

\[
a_k \geq a_i + s_{ti} + t_{ik} - \left( 1 - \sum_{j \in J^S} x_{ij}^j \right) \cdot T \quad \forall (i, k) \in I_o \times I_o, i \neq k,
\]

(18)

\[
a_i + s_{ti} \leq TD_i \quad \forall i \in I_o,
\]

(19)

\[
\sum_{k \in I_{w}(i)} x_{ik}^j = \delta_{ij}^o \quad \forall i \in I_o, \quad \forall j \in J^S,
\]

(20)

\[
\sum_{k \in I_{w}(j)} x_{ik}^j = \delta_{ij}^o \quad \forall i \in I_o, \quad \forall j \in J^S,
\]

(21)

\[
s_{jk} \geq 0; V_{jk} \in \mathbb{Z}^+; \delta_{ij}^o \in \{0, 1\}; \delta_{ij}^w \in \{0, 1\}; \delta_{ij}^{wL} \in \{0, 1\}; y_j^c \in \{0, 1\}; y_j^b \in \{0, 1\}; \theta_j \in \{0, 1\}, \quad a_i \geq 0; x_{ik}^j \in \{0, 1\}; v_j \in \mathbb{Z}^+.
\]

(22)

In the first constraint set in (2), every walk-in customer must either be served by exactly one store, or his demand will be lost while the second set in (3) guarantees the demand of online customers be met. The constraints in (4) maintain the balance between the flow of goods from warehouses to stores and the flow of goods from stores to customers. The number of vehicles leaving a warehouse $j$ for a store $k$ should be large enough to carry shipments between the two as required in (5). The constraints in (6) and (7) forbid service from a closed
BM and from an unopened CM. Similarly, those in (8) forbid service from a BM to online customers if it is not reconfigured as a CM. The constraints in (9) imply that the distance between a walk-in customer and the store he will choose is restricted by the maximum driving (walking) distance parameter. Let \( J_B(i) \) and \( J_C(i) \) denote the set of BM and CM locations within the maximum permissible distance of walk-in customer \( i \), respectively. This is, \( J_B(i) = \{ j \in J_B : c_{ij} \leq \text{MaxW} \} \) and \( J_C(i) = \{ j \in J_C : c_{ij} \leq \text{MaxW} \} \). Then, a walk-in customer for whom there exists an open store within MaxW should get service, and must not be left out as provided in (10). The next two constraint sets in (11) and (12) guarantee that if a walk-in customer \( i \) is served by a store \( j \), then \( j \) has to be the nearest open store for that walk-in customer. Eqs. (13)–(15) set the number of vehicles leaving a store \( i \) to the number of outgoing arcs at each store; and finally match the total number of outgoing arcs be equal to the number of incoming arcs sent from location \( j \) to another online customer \( k \), then the arrival time at \( k \) should be greater than or equal to the sum of arrival time at \( i \), the service time at \( i \) (if any), and the traveling time on the arc \((i, k)\). This arrival time condition is met by the constraints in (17) and (18), where the former apply to the customers directly reached from the stores. The next constraints in (19) impose the deadlines on service completion times at online customers. \( T \) in (18) satisfies \( T \geq \max x_{ijkl} \{ T_{Dk} + st_i + T_k \} \) and serves as a Big-M number. Eqs. (20) and (21) prevent a visit to customer \( i \) by a vehicle sent from store \( j \) if customer \( i \) has not been assigned to store \( j \). If assigned (i.e., if \( \delta_{ij} = 1 \)), then (20), (21) ensure that customer \( i \) is visited exactly once. Finally, nonnegativity and integrality constraints of the FLAP and MDVRPTD are provided in (22) and (23), respectively. Note that no arc in the network of stores and online customers can be traversed by any vehicle sent from location \( j \) unless there is an open and appropriate store at \( j \). This logical condition is enforced by the co-presence of the constraints (6)–(8) and (20), (21).

4. Lagrangian relaxation for CMBP-S

The Lagrangian relaxation (LR) method embedded in a subgradient optimization can be used for a variety of \( NP \)-hard optimization problems to bracket the true optimal solution between a lower and an upper bound \([Z_{lb}, Z_{ub}]\). In minimization problems of the form \( P: Z^* = \min_{\lambda \in \Lambda} f(x) \) the upper bound constitutes a good heuristic feasible solution to the problem. The quality of this solution is measured as \( \frac{Z_{ub} - Z_{lb}}{Z_{lb}} \), the relative percentage gap between the final bound values. Since the true optimal objective value \( Z_p \) of \( P \) is guaranteed to be equal to or greater than \( Z_{lb} \), the final gap between the bounds \( (Z_{ub} - Z_{lb}) \) cannot be higher than the gap \( Z_{lb} - Z_p \). Cordeau et al. (2002) state that LR based methods for the VRP with time windows are marked by the usual trade-off between ease of solving the subproblem(s) in the Lagrangian relaxed problem and the quality of the bound obtained. Geoffrion (1974) and Fisher (1981) explain the LR method and subgradient optimization. Gavish (1985) describes an augmented LR method for the Capacitated Minimum Spanning Tree problem (CMST).

In our problem \( P \) given in (1)–(23), we relax constraints in (20), (21) which connect subproblems FLAP and MDVRPTD. Left-hand sides (LHS) of these coupling constraints are subtracted from their right-hand sides (RHS), then multiplied by the Lagrange multipliers \( \lambda \) and \( \mu \), respectively, and added to the objective function of \( P \). The Lagrangian relaxed problem of CMBP-S is denoted as LR. A heuristic method is proposed in Aksen and Altinkemer (2003) to generate an initial feasible solution to the problem \( P \). The objective value found by this heuristic is plugged in \( Z_{ub} \), the initial upper bound on \( Z_p \). A second method explained in the same working paper tries to improve this initial \( Z_{ub} \) during the subgradient iterations of the main Lagrangian relaxation. The paper also describes an Add-Drop heuristic as a tool of post-Lagrangian relaxation refinement. This heuristic is applied if the gap \( \frac{Z_{ub} - Z_{lb}}{Z_{lb}} \) does not drop below 2% at the end of the Lagrangian relaxation of \( P \).

We first write LR in plain English, and show how it becomes separable into two independent subproblems. \( L \) represents the compound vector of Lagrange multipliers \( \lambda \) and \( \mu \) which are unrestricted in sign. The first subproblem (SubP1) is a special two-stage discrete FLAP and the second one (SubP2) is a special degree- and deadline-constrained Capacitated Minimum Spanning Forest (CMSF)-like problem.
4.1. Lagrangian relaxed problem LR

To mathematically formulate LR we first need to rearrange the terms in the augmented objective function. The summation \( \sum_{i} \sum_{j} \sum_{k} (\lambda_{ij} + \mu_{ij}) \cdot \delta_{ij} \) made of Lagrange multipliers will be added to the FLAP objectives as the pseudo cost of assigning online customers to stores. A 3-dimensional, asymmetric and store dependent cost matrix \( C_{\text{new}} = [(c'_{ik})_{\text{new}}] \) can be derived for the augmented MDVRPTD objectives of LR. The new traveling cost of a vehicle from node \( i \) to node \( k \) is now also dependent on the origin store \( j \) of the vehicle route. The new cost matrix \( C_{\text{new}} \) is eventually asymmetric for a given store \( j \) even if the original distances \([c_{ik}]\) are symmetric. Let \( G(N, A) \) denote the complete weighted and directed graph of online customers and all stores, where \( N = J^S \cup I_0 \) and \( A = \{(i, j) \in (J^S \cup I_0) \times (J^S \cup I_0) : i \neq k \} \). Let \( (c'_{ik})_{\text{new}} \) denote the cost (weight) of arc \((i, k)\) if it is traversed with a vehicle sent from the source node \( j \in J^S \). Arc costs in \( G \) are then defined as follows.

\[
\begin{align*}
(i) \quad & \forall (i, k) \in I_0 \times I_0, i \neq k: (c'_{ik})_{\text{new}} = ac_3c_{ik} - \lambda_{ij} - \mu_{kj} \\
(ii) \quad & \forall (i, j) \in I_0 \times J^S: (c'_{ij})_{\text{new}} = ac_3c_{ij} - \lambda_{ij} \\
(iii) \quad & \forall (j, i) \in J^S \times I_0: (c'_{ji})_{\text{new}} = ac_3c_{ji} + FCV_j - \mu_{ij}
\end{align*}
\]

In (iii) of (24), we have embedded vehicle acquisition costs into the augmented costs of the outgoing arcs \((j, i)\) of stores. This not only integrates acquisition costs into \( C_{\text{new}} \) but also eliminates the first MDVRPTD constraint in (13) from LR. The entries of \( C_{\text{new}} \) for incoming and outgoing arcs of stores must be set to infinity if there is a contradiction between the source node and the head/tail nodes of the arc. For example, let \( h (h \in J^S) \) denote the source node, \( i \) and \( j \) the head/tail nodes of an arc where \( j \in J^S \) too. Then, \( (c'_{ih})_{\text{new}} \) must be \( +\infty \) if \( h \neq j \). Similarly, \( (c'_{ji})_{\text{new}} \) is also \( +\infty \) if \( h \neq j \). The new cost matrix \( C_{\text{new}} \) will dynamically change from one subgradient iteration to another as Lagrange multipliers \( L \) get updated. Using \( C_{\text{new}} \) we can pile up the augmented objective components of the MDVRPTD and rewrite LR as follows.

\[
\begin{align*}
\text{LR} (\lambda, \mu) : \quad \text{Min} \quad & Z_{\text{LR}}(\lambda, \mu) = \sum_{j \in J^S} CC_j \theta_j + \sum_{j \in J^S} (FC_j + OC_j) \gamma_j + \sum_{j \in J^S} FC_j (1 - y^h_j) + \sum_{j \in J^S} OC_j y^b_j \\
& + \sum_{j \in J^S} \sum_{k \in J^S} FCV_j \cdot V_{jk} + ac_1 \cdot \sum_{i \in I_0} c_{ij} \delta_{ij}^w + ac_2 \cdot \sum_{j \in J^S} \sum_{k \in J^S} c_{jk} s_{jk} \\
& + \sum_{i \in I_0} p_i d_i \delta_{ih}^L + \sum_{i \in I_0} \sum_{j \in J^S} (\lambda_{ij} + \mu_{ij}) \cdot \delta_{ij}^o + \sum_{j \in J^S} \sum_{i \in I_0 \cup \{j\}} \sum_{k \in I_0 \cup \{j\}} (c'_{ik})_{\text{new}} x_{ih}^j
\end{align*}
\]

Subject to: (2)–(12), (14)–(19), (22), (23).

4.2. Subgradient optimization in the Lagrangian relaxed problem

Let \( SG^q \) denote the subgradient vector of the problem LR(\( L \)) at iteration \( q \) of the subgradient optimization procedure. Step size \( s^q \) is derived from the norm square of \( SG^q \) and the gap between the current best objective \( Z_{ub} \) (upper bound on \( Z_p \)) and current Lagrangian objective \( Z_{LR} \). It is multiplied also by a scalar \( A^q \) whose first value \( A^1 \) is 2.0 by convention (see Fisher, 1981). This scalar is halved whenever the objective \( Z_{LR} \) does not improve for a specified number of iterations. Formulae of the subgradient optimization routine for the
Lagrangian relaxation of $P$ are given below. At the beginning (when $q = 0$), we set all Lagrange multipliers to the initial value zero.

$$
SG^q_{ij} = (\delta^q_{ij}) - \sum_{k \in I_o \cup \{j\}} (x_{ik}^q), \quad (SG^q_{ij}) = (\delta^q_{ij}) - \sum_{k \in I_o \cup \{j\}} (x_{ik}^q) \quad \forall i \in I_o, \quad \forall j \in J^S;
$$

$$
\|SG\|^2 = \|SG^q\|^2 + \|SG^0\|^2,
$$

$$
s^q = \lambda^q + \frac{Z^q_{LR}(\lambda, \mu)}{\|SG\|^2},
$$

$$
(\lambda^q)^{q+1} = (\lambda^q)^q + s^q \cdot (SG^q_{ij}), \quad (\mu^q)^{q+1} = (\mu^q)^q + s^q \cdot (SG^0_{ij}) \quad \forall i \in I_o, \quad \forall j \in J^S.
$$

### 4.3. FLAP-like Subproblem 1

The first subproblem of LR is a FLAP-like problem SubP1($L$). The objective coefficients of the binary variables $\delta^q_{ij}$ in SubP1 depend on the current values of the Lagrange multipliers $L$, hence they change dynamically from one subgradient iteration to another. The formulation is given below.

SubP1($\lambda, \mu$):

$$
\text{Min } Z^E_{\text{SubP1}}(L) = \sum_{j \in J^S} \sum_{i \in I_o} CCj \beta_j + \sum_{i \in I_o} FCj \cdot OCj \cdot \beta_j + \sum_{i \in I_o} FCj \cdot (1 - \beta_j) + \sum_{i \in I_o} OCj \cdot \beta_j
$$

$$
+ \sum_{j \in J^W} \sum_{k \in J^W} FCV_j \cdot V_{jk} + ac_1 \cdot \sum_{j \in J^W} \sum_{i \in I_o} c_{ij} \delta_{ij} + ac_2 \cdot \sum_{j \in J^W} \sum_{k \in J^W} c_{jk} \delta_{jk}
$$

$$
+ \sum_{i \in I_o} \sum_{j \in J^W} (\lambda_{ij} + \mu_{ij}) \cdot \delta_{ij}
$$

Subject to: (2)–(12), (22).

SubP1($L$) corresponds to a 3-layer FLAP with balanced flows from warehouses to stores and subsequently from stores to customers. A fixed number of warehouses have their own vehicle acquisition and material flow costs, the latter being linearly proportional to the volume of goods and to the distance they are transferred. Fixed costs of opening, closing, and operating stores are incurred in the second layer. There are two types of stores in this FLAP: BM’s with three attributes (open/closed/converted) and CM’s with two attributes (opened/not opened). The attributes are mutually exclusive in the model. The allocation part occurs in the third layer with two types of customers; online and walk-in. Each online customer $i$ must be allocated exactly to one store $j$, either a converted BM or an opened CM. The cost of this allocation is $(\lambda_{ij} + \mu_{ij})$. Each walk-in customer is either lost if there is no store within the maximum distance, or assigned to the nearest open store within that distance. Losing a walk-in customer incurs a cost equal to the total profit from that customer. The assignment of a walk-in customer to some store occurs at a cost proportional to the distance between the two. Although there are no capacity restrictions on facilities, the binary constraints upon the assignment variables $\delta^q$, $\delta^w$ and $\delta^w_l$ are not to be relaxed. Due to 3-layer nature of material flow in the CMBP-S it cannot be guaranteed that assignment variables will each attain the value 0 or 1. SubP1 has been modeled with the mathematical modeling suite GAMS, and solved with the mixed integer linear programming solver Cplex 9.0 to optimality. In at least one instance with randomly generated values for Lagrange multipliers $L$, the optimal solution had noninteger values for several of the relaxed assignment variables. This justifies upholding the binary constraints upon those variables.

When all WH’s, BM’s and walk-in customers are eliminated from SubP1, it simplifies to an uncapacitated fixed charge location problem, also known as the simple plant location problem. Cornuéjols et al. (1990) report that even this type of location problems are $\mathcal{NP}$-hard; hence, SubP1 is $\mathcal{NP}$-hard too. There are $[M_W(M_B + M_o)]$ linear and $[M_B + (M_B + M_o)M_W + N + 1] + N_o$ integer and binary decision variables in an instance of SubP1. Although the dimensions of SubP1 seem huge at first glance, Cplex 9.0 is able to handle large-size instances of SubP1 to optimality within acceptable amounts of time. We solved sample instances having one or two warehouses, 5–75 stores, and 20–500 customers with the relative optimality criterion of GAMS set to 0.01%. Solution times range between 0.2 and 36.5 seconds on an Intel Pentium® 4 HT 3.40 GHz processor.
4.4. Minimum spanning forest-like Subproblem 2

SubP2 is the second independent subproblem of LR. Like in SubP1, also SubP2’s objective coefficients shown by the matrix $C_{\text{new}}$ largely depend on the Lagrange multipliers $L$. We first give a formulation of SubP2. Next we discuss an augmented Lagrangian relaxation method to find lower and upper bounds on the true optimal objective value of SubP2.

$$
\text{SubP2}(\lambda, \mu) :
\begin{align*}
\text{Min} & \quad Z_{\text{SubP2}}^L(L) = \sum_{j \in J^O} \sum_{i \in I_0} \sum_{k \in J^O \setminus \{j\}} (c_{ik})_{\text{new}} x_{ik} \\
\text{Subject to :} & \quad (14)-(19), (23).
\end{align*}
$$

In SubP2 per constraint in (15), online customers in $I_o$ are equivalent to terminal nodes, each of which must be accessible from exactly one center site via a subtree (multipoint line) rooted at that center. A currently open BM or a candidate CM in $J^O$ represents a center site; hence SubP2 involves a directed spanning forest of all nodes. The cost of connecting nodes $i$ and $k$ shown by $(c_{ik})_{\text{new}}$ depends on the center node $j$ from which $i$ and $k$ are visited. There must be a balance between the incoming and outgoing degrees of each and every center node per constraints in (14). Capacity and general subtour elimination constraints are imposed together in (16). A capacitated spanning forest implies that the total weight of nodes (customer demands $d_i$) spanned on the same subtree must not exceed a constant capacity (vehicle capacity $Q$). Integrality and nonnegativity constraints are given by the first two terms in (23). The remaining constraints relate to time deadlines. There are no typical 2-degree constraints upon the terminal nodes in SubP2, because such constraints have been relaxed and put in the objective function of the Lagrangian problem. However, there are still time deadline constraints, which make no physical sense in a spanning forest setting. We next explain the dualization and treatment of time deadline constraints in (17)–(19).

SubP2, the CMSF-like subproblem of $P$, is still hard to solve even after the Lagrangian decomposition. If time deadline and balance of degree constraints are discarded, and if the number of stores in $J^O$ is dropped to one, SubP2 becomes equivalent to the CMST. Papadimitriou (1978) showed that the CMST is an $NP$-hard problem. Hence, SubP2 also belongs to the $NP$-hard class. In order to solve SubP2 we adopt and modify the augmented Lagrangian method of Gavish (1985). A concise overview of Lagrangian lower bounds obtained in an augmented fashion can be found in a study of branch-and-bound algorithms for the vehicle routing problems in Toth and Vigo (2002). We relax the subtour elimination, capacity, and time deadline constraints in SubP2, since this relaxation scheme achieves empirically better lower bounds on the optimal objective value of SubP2. First, the constraint set in (16) is divided into two parts as (16.a) and (16.b), the second of which will be relaxed. Secondly, a trivial constraint which sets the minimum number of vehicles required is added to the original formulation of SubP2 as (16.c). This minimum number is calculated by solving the associated bin-packing problem which embraces demand values $d_i, i \in I_o$.

$$
\begin{align*}
\sum_{j \in J^O} \sum_{i \in S_0} \sum_{k \in J^O \setminus \{j\}} x_{ik} & \leq |S_0| - 1 \quad \forall S_0 \subseteq I_o \ni |S_0| \geq 2, \quad (16.a) \\
\sum_{j \in J^O} \sum_{i \in S_0} \sum_{k \in J^O \setminus \{j\}} x_{ik} & \leq |S_0| - L_{S_0} \quad \forall S_0 \subseteq I_o \ni |S_0| \geq 2 \land L_{S_0} \geq 2, \quad (16.b) \\
\sum_{j \in J^O} \sum_{i \in I_0} x_{ji} & \geq L_{I_0}. \quad (16.c)
\end{align*}
$$

Note that the value of $L_{S_0}$ will always be greater than zero if customer demands are strictly positive, because we need at least one vehicle to load any order no matter how small it is. For unit cardinality subsets of $I_o$, (16.a) will always be satisfied; however, both (16.a) and (16.c) are relevant to the Lagrangian relaxation of SubP2. As we did in problem LR, we subtract the LHS of inequalities (16.b), (17) and (18) from their RHS and multiply by Lagrange multipliers $\lambda$, $\sigma$, and $\omega$ ($\lambda \leq 0, \sigma \geq 0, \omega \geq 0$), respectively. We embed the relaxed constraints into
the objective function of SubP2, and rearrange the summations to end up with the objective function of the augmented Lagrangian problem ALR_{SubP2} in Eq. (30). In rearranging the terms in (30) we exploit the equivalences given in (28), (29).

\[
\sum_{j \in J} \sum_{i \in I \cup \{j\}} \sum_{k \neq i} (c_{ik})_{\text{new}} x_{ik}^j = \sum_{j \in J} \sum_{i \in I_o} (c_{ik})_{\text{new}} x_{ik}^j + \sum_{j \in J} \sum_{i \in I_o} x_{ik}^j + \sum_{j \in J} \sum_{i \in I_o} \sum_{k \neq i} (c_{ik})_{\text{new}} x_{ik}^j \tag{28}
\]

\[
\sum_{i \in I_o} \sum_{k \notin I_o} (a_i - a_k) = \sum_{i \in I_o} \sum_{k \notin I_o} (a_i - a_k) \tag{29}
\]

ALR_{SubP2}(\mathbf{z}, \mathbf{\sigma}, \mathbf{\omega}) :

Minimize \( Z_{\text{ALR}}^{\text{SubP2}(L)}(\mathbf{z}, \mathbf{\sigma}, \mathbf{\omega}) = \sum_{j \in J} \sum_{i \in I_o} \left[ (c_{ik})_{\text{new}} + \sigma_i t_{ji} \right] x_{ik}^j + \sum_{j \in J} \sum_{i \in I_o} (c_{ij})_{\text{new}} x_{ij}^j \)

\[+ \sum_{j \in J} \sum_{i \in I_o} \sum_{k \neq i} \left[ (c_{ik})_{\text{new}} + T a_i - \sum_{r \in G_{ik}} \sigma_r \right] x_{ik} + \sum_{i \in I_o} \left[ \sum_{k \neq i} (a_i - a_k) - \sigma_i \right]
\]

\[+ \sum_{S \in \mathcal{P}} \left| (S_0) \right| - \left| S_0 \right| \mathbf{\omega} S_0 + \sum_{i \in I_o} \sum_{k \neq i} (s_t + t_k - T) a_i \]

Subject to: (14), (15), (16.a), (16.c), (19), (23). \( \tag{30} \)

The last two terms of the objective function in (30) are constant for a given set of Lagrange multipliers \((\mathbf{z}, \mathbf{\sigma}, \mathbf{\omega})\). In the first three terms, the original coefficients \((c_{ik})_{\text{new}} \) of SubP2 can be augmented by the multipliers \((\mathbf{z}, \mathbf{\sigma}, \mathbf{\omega})\) to yield the 3-dimensional cost matrix \( \mathbf{C} = [c_{ik}^j] \). We refer to \( \mathbf{C} \) as the matrix of the objective coefficients of ALR_{SubP2}. The optimal values of the variables \( a_i \) (arrival times of vehicles at online customer nodes) at any optimal solution of ALR_{SubP2} are given by the following formula.

\[
a_i^* = \begin{cases} 
0 & \text{if } \sum_{k \in I_o, k \neq i} (a_{ik} - a_{ki}) - \sigma_i \geq 0 \\
TD_i - s_t & \text{otherwise.} 
\end{cases} \tag{31}
\]

As seen in the above formula, \( a_i^* \) values are determined by the time deadlines \( TD_i \), service times \( s_t \), and current Lagrange multipliers \( \mathbf{\sigma} \) and \( \mathbf{\omega} \). They are independent of the routing variables \( x_{ik}^j \). Therefore, we can discard them when solving the problem ALR_{SubP2}. Observe that \( S_0 \) in (16.b) represents any unordered subset of \( I_o \) with a cardinality greater than one, which requires two or more vehicles. The set of such subsets is denoted by \( \mathcal{P} \). Its cardinality \( |\mathcal{P}| \) is data contingent, but it cannot be higher than \( (2^{n_o} - n_o - 1) \) in any given problem. For each \( S_0 \in \mathcal{P} \) there is an associated Lagrange multiplier \( \sigma_r \geq 0 \). Let \( G_{ik} \) denote the index set of subsets \( S_0 \) in \( \mathcal{P} \) that contain customer nodes \( i \) and \( k \). In other words, \( \{i, k\} \in (S_0), i \in I_o \) for the index \( r \in G_{ik} \).

The first augmented Lagrangian relaxation feature is used here: We do not explicitly generate all \( |\mathcal{P}| \) constraints in (16.b). So we do not compute the entire multiplier vector \( \mathbf{z} \) either. The same relaxation scheme is also applied to the constraints in (17) and (18) with multipliers \( \mathbf{\sigma} \) and \( \mathbf{\omega} \), respectively. This is, initially all three multiplier vectors are zero, and the augmented Lagrangian relaxed problem ALR_{SubP2} is equivalent to an MSF (minimum spanning forest) problem without capacity constraints. The optimal arrival times \( a_i^* \) \((i \in I_o)\) do not interfere with the binary edge variables. The edge costs matrix \( C_{\text{new}} \) is dependent on the center node of departure. However, there are two distinct limitations in this MSF problem:

1. The sum of outgoing degrees of all center nodes (BM and CM nodes) has to be equal to or greater than \( L_{I_o} \) as required by the constraints in (16.c).
2. At each center node, incoming and outgoing degrees should be equal as required by constraints in (14).
The solution of the problem ALR_{SubP2} is checked against the violation of constraints in (16.b), (17) and (18) in SubP2. If any violated constraint is detected, it is added together with its associated Lagrange multiplier to the set of active constraints and multipliers. The objective function $Z_{ALR}^{SubP2(k)}$ is augmented with the product of the difference between the violated constraint’s right- and left-hand side values and the associated Lagrange multiplier’s initial value. This sequence of operations is repeated for every constraint of SubP2 that has been relaxed, but has not been augmented yet to ALR_{SubP2}. We do not remove previously augmented constraints from the set of active constraints in the Lagrangian problem, neither do we generate any such constraint for a second time. Checking the presence of an inequality from the constraint sets in (17) and (18) is easy, because the distinctive attribute of these constraints is a pair of indices $(i, k) \in (I_o \times I_o)$. However, the identification of a constraint in (16.b) is intricate. The attribute unique to a particular constraint in (16.b) is the subset of customers $S_0 \subseteq I_o$, spanned by the same subtree. If we keep a symbolic list of bit strings representing each of such subsets $S_0$, then we can scan this list to see whether a particular constraint was detected and added to problem ALR_{SubP2} before. This way of storing augmented subtour elimination and capacity constraints has been successfully applied in Gavish (1985) for the CMST problem. Gavish explains a further technique to generate a tight Lagrangian objective function by finding an initial multiplier value for every augmented constraint while maintaining the optimality property of the Lagrangian solution before that constraint. We adopted this technique into the augmented Lagrangian relaxation of SubP2. Finally, the degree balance constraints in (14) and the minimum sum constraint in (16.c) on the outgoing degrees of the center nodes should be reckoned with.

The closest version of ALR_{SubP2} is the degree-constrained minimum spanning tree problem (DCMST). Garey and Johnson (1979) prove that the general DCMST with arbitrary degree constraints on nodes other than the center is \textit{NP}-hard. In spite of copious methods and algorithms developed for the DCMST in the literature, we cannot use any of them as is. First of all, ALR_{SubP2} displays a forest structure with center-node dependent costs. Secondly, the degree constraints in ALR_{SubP2} relate to the balance of incoming and outgoing degrees at the center nodes (stores) only. There exists also a lower bound on the sum of outgoing degrees at those centers. From this perspective, ALR_{SubP2} is conceivably easier to solve than a general DCMST problem. We develop a polynomial-time procedure called [MSF-ALR], which is basically an adaptation of Prim’s MST algorithm to the problem ALR_{SubP2}. The reader is referred to the text by Cormen et al. (2001) for an inclusive discussion of optimal MST algorithms. [MSF-ALR] requires a running time of $O(N_o^2(M_B + M_C + N_o))$ on the complete graph of online customers and stores of the comprehensive problem CMBP-S. A detailed description of [MSF-ALR] can be found in a separate document at http://home.ku.edu.tr/~daksen/SubP2.pdf. The generation of an initial upper bound $Z_{ub}^{SubP2}$ for the subproblem SubP2 and attempts to reduce this upper bound in the course of the subgradient iterations of the augmented Lagrangian relaxation are explained in the same document as well.

### 4.5. Subgradient optimization in the augmented Lagrangian relaxation of SubP2

We provide the formulae of subgradient iterations that will be used in the augmented Lagrangian relaxation of SubP2. The subgradient vector does not have a static size. Its cardinality grows as subgradient iterations proceed and as we detect more constraints violated by the Lagrangian solution to ALR_{SubP2}. We designate the compound subgradient vector of the Lagrangian problem as $\Psi$. There are three main subgradient components in this vector $(\Psi^s, \Psi^o, \Psi^{eo})$ each associated with the generated constraints from the relaxed constraint sets (16.b), (17) and (18). Let $G^q$ denote the index set of those constraints in (16.b) which have been violated, therefore generated in the current iteration $q$ or in some previous iteration. Each index $r$ in $G^q$ corresponds to some subtree of customer nodes whose indices comprise a particular subset $S_r$ in $\Psi$ as explained in Section 4.4. In that case, there will be as many as $|G^q|$ constraints from (16.b) relaxed and augmented into ALR_{SubP2}. Similarly, let $I_o^q$ and $I_o^{eo}$ denote the index sets of those node pairs $(i, k)$ for whom the time deadline constraints in (17) and (18), respectively, have been violated, therefore have been generated either in the current iteration $q$ or in some previous iteration. Clearly, $I_o^q$ is a subset of $I_o$ and $I_o^{eo}$ is some subset of $(I_o \times I_o)$. The cardinality of the subgradient vector is equal to $|\Psi| = |G^q| + |I_o^q| + |I_o^{eo}|$ at the $q$th iteration. We give first the formulae of subgradient vector components, and then the formulae of Lagrange multipliers for those constraints which are generated and added to ALR_{SubP2}. $S_o(r)$ in (32) indicates the $r$th subset of online customers in $\Psi$ which
are spanned by the same subtree. \( s_{ALR}^q \) in (34) denotes the step size of the subgradient optimization, \( A_{ALR}^q \) is a scalar with the initial value 2.0, \( Z_{ub}^{ALR,SubP2} \) is an upper bound on the true optimal objective value of SubP2, and finally \( Z_{ub}^{ALR,SubP2} \) is the current augmented Lagrangian objective value. The scalar \( A_{ALR}^q \) is halved whenever the value of \( Z_{ub}^{ALR,SubP2} \) does not increase for a specified number of iterations.

\[
(Y^i_r)^q = (|S_0(r)| - L_{S_0(r)}) - \sum_{j \in J^q} \sum_{i \in S_0(r)} \sum_{k \in S_0(r)} (x^j_{ik})^q \quad \forall r \in G^q, \\
(Y^q_i)^q = \sum_{j \in J^q} t_{j}(x^j_{ik})^q - a_i \quad \forall i \in I^q, \\
(Y^q_{ik})^q = a_i - a_k + s_{ik} - t_{ik} - \left(1 - \sum_{j \in J^q} (x^j_{ik})^q\right) T \quad \forall (i, k) \in \mathbf{I}^q, \\
\|Y\|^2 = \|Y^s\|^2 + \|Y^q\|^2 + \|Y^w\|^2, \\
X = \begin{bmatrix} S_{ub}^{q_{ALR}} - Z_{ub}^{q_{ALR,SubP2}}(\mathbf{z}, \mathbf{\sigma}, \mathbf{\omega}) \end{bmatrix}, \\
(z_r)^{q+1} = \min \left\{0, (z_r)^q + s_{ALR}^q(Y^r_r)^q\right\} \quad \forall r \in G^q, \\
(\sigma_i)^{q+1} = \max \left\{0, (\sigma_i)^q + s_{ALR}^q(T^i_r)^q\right\} \quad \forall i \in I^q, \\
(\omega_{ik})^{q+1} = \max \left\{0, (\omega_{ik})^q + s_{ALR}^q(Y^q_{ik})^q\right\} \quad \forall (i, k) \in \mathbf{I}^q.
\]

5. Computer experiments and results

Our composite Lagrangian relaxation method has been coded in C and compiled with Microsoft Visual C++ 6.0© on an Intel Pentium® 4 HT 3.40 GHz processor with 2 GB RAM. We tested our method’s performance in terms of the Lagrangian gap and solution time on a test bed of randomly generated problems. In Section 5.1, we describe how we generated the problems, and give the parameters used in the implementation of our method. In Section 5.2, we present the test results.

5.1. Data generation and stopping conditions

The customer and facility locations have been randomly generated over a rectangular problem space with its lower left corner at the origin. Possible choices for customer locations are uniformly distributed, uniformly but heterogeneously clustered into rectangular regions of equal size, and uniformly dispersed along a user-specified number of equidistant altitudes and longitudes of the problem space. The second option implies that inside a given rectangular region of the problem space customers are uniformly distributed, but they are unequally allotted among these regions. For facility locations, possible choices are uniformly distributed, discrete uniformly distributed, and user-specified. The second option for a group of facilities (BM’s, candidate CM’s, or WHs) implies that they are located on equidistant longitudes and altitudes on the problem space. However, the distance between longitudes (altitudes) depends on the number of those facilities. In some of the problems, both walk-in and online customers have constant demand. In others, demand values are independent and identically distributed (i.i.d.) derived from three distributions: Uniform \( U[\text{min}, \text{max}] \), Normal \( N(\mu, \sigma^2) \), and Exponential \( \text{Exp}(\mu) \). The service times (\( s_{ik} \)) can either be all zero, or they can attain random values. In the latter case, a customer’s time deadline \( T_D \) is dependent on the average of his shortest and farthest distances from stores. The formula for \( T_D \) uses the average service time \( s_{Avg} \) as an input parameter. It is multiplied by \( \kappa = Q \mu^{-1} \), the ratio of vehicle capacity \( Q \) to the mean customer demand. The formula reads then \( T_D = \left[ \hat{c} \cdot (1.5 + U_i) + k s_{Avg}\right] \quad \forall i \in I_o \). In this formula, \( U_i \) denotes a nonnegative integer random number between zero and some constant \( K \). The fixed and variable costs of stores (OC, CC, and FC) as well as the unit cost of lost demand of walk-in customers (\( p_i \)) can either be set to a constant, or they can be derived from a uniform discrete distribution with user-specified range and increment value.
The solution quality and time of a Lagrangian relaxation algorithm depends on – among many other factors – when to stop the inbound subgradient iterations. The stopping conditions of our main Lagrangian relaxation for the problem \( P \) are as follows.

1. Iteration counter reaches the iteration limit \( \text{ITER\_LIMIT} \).
2. Scalar \( K \) in the computation of the step size \( s \) in (26) drops below the parameter \( \text{MIN\_STEP} \).
3. Gap between upper and lower bounds becomes smaller than a certain \( \text{TOLERANCE} \).
4. Gap does not improve during the last \( \text{MAX\_NONIMP} \) iterations.
5. Increment in each component of the Lagrange multipliers \( L \) in (27) is less than \( \text{EPS} \).

The first four stopping conditions are also used to decide when to stop the inbound subgradient iterations of SubP2. The only difference is that this time we refer to Eqs. (34), (35) in the second and fifth conditions, respectively. The stopping conditions have been detailed in Table 1 for \( P \) and SubP2.

We first tested our method on a small-size test problem with 1 WH, 1 BM, 2 candidate CM’s, 15 walk-in and 15 online customers. The problem space is a 200 \( \times \) 100 rectangular area on which online customers are dispersed uniformly, and walk-in customers are uniformly clustered around the stores. The warehouse and the BM are both located at (100, 50), the center of this rectangular area. The coordinates of candidate

**Table 1**
Parameters for the stopping conditions of subgradient iterations

<table>
<thead>
<tr>
<th>Parameter name</th>
<th>Value in the LR of problem ( P )</th>
<th>Value in the ALR of problem SubP2</th>
</tr>
</thead>
<tbody>
<tr>
<td>ITER_LIMIT</td>
<td>300</td>
<td>400</td>
</tr>
<tr>
<td>MIN_STEP</td>
<td>1.0e(-5)</td>
<td>1.0e(-5)</td>
</tr>
<tr>
<td>TOLERANCE</td>
<td>1.0e(-5)</td>
<td>1.0e(-5)</td>
</tr>
<tr>
<td>MAX_NONIMP</td>
<td>100</td>
<td>75</td>
</tr>
<tr>
<td>EPS</td>
<td>1.0e(-6)</td>
<td>–</td>
</tr>
</tbody>
</table>

**Table 2**
Results of different seeds on \( t-15-15-4 \)

<table>
<thead>
<tr>
<th>Distance measure</th>
<th>Seed</th>
<th>Best store plan</th>
<th>Final ( Z_{lb} )</th>
<th>Final ( Z_{ub} )</th>
<th>Gap (%)</th>
<th>CPU time [seconds]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euclidean</td>
<td>519</td>
<td>C</td>
<td>XX</td>
<td>1894.35</td>
<td>1946.92</td>
<td>2.77</td>
</tr>
<tr>
<td></td>
<td>622</td>
<td>C</td>
<td>XX</td>
<td>1826.07</td>
<td>1845.86</td>
<td>1.08</td>
</tr>
<tr>
<td></td>
<td>2006</td>
<td>C</td>
<td>XX</td>
<td>1837.33</td>
<td>1888.95</td>
<td>2.81</td>
</tr>
<tr>
<td></td>
<td>71</td>
<td>C</td>
<td>XX</td>
<td>2231.00</td>
<td>2310.00</td>
<td>3.54</td>
</tr>
<tr>
<td></td>
<td>519</td>
<td>C</td>
<td>XX</td>
<td>2429.15</td>
<td>2485.00</td>
<td>2.30</td>
</tr>
<tr>
<td></td>
<td>622</td>
<td>C</td>
<td>XX</td>
<td>2124.26</td>
<td>2160.00</td>
<td>1.68</td>
</tr>
<tr>
<td></td>
<td>2006</td>
<td>C</td>
<td>XX</td>
<td>2153.26</td>
<td>2266.00</td>
<td>5.24</td>
</tr>
<tr>
<td>Rectilinear</td>
<td>71</td>
<td>C</td>
<td>XX</td>
<td>2231.00</td>
<td>2310.00</td>
<td>3.54</td>
</tr>
<tr>
<td></td>
<td>519</td>
<td>C</td>
<td>XX</td>
<td>2429.15</td>
<td>2485.00</td>
<td>2.30</td>
</tr>
<tr>
<td></td>
<td>622</td>
<td>C</td>
<td>XX</td>
<td>2124.26</td>
<td>2160.00</td>
<td>1.68</td>
</tr>
<tr>
<td></td>
<td>2006</td>
<td>C</td>
<td>XX</td>
<td>2153.26</td>
<td>2266.00</td>
<td>5.24</td>
</tr>
</tbody>
</table>

**Table 3**
Results of different no. customers with 1 WH, 1 BM and 2 CM’s

<table>
<thead>
<tr>
<th>Distance measure</th>
<th>No. of customers ( (N_w, N_{ew}) )</th>
<th>Best store plan</th>
<th>Final ( Z_{lb} )</th>
<th>Final ( Z_{ub} )</th>
<th>Gap (%)</th>
<th>CPU time [seconds]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euclidean</td>
<td>(15, 15)</td>
<td>C</td>
<td>XX</td>
<td>1826.07</td>
<td>1845.86</td>
<td>1.08</td>
</tr>
<tr>
<td></td>
<td>(20, 20)</td>
<td>C</td>
<td>XX</td>
<td>2319.38</td>
<td>2492.20</td>
<td>7.45</td>
</tr>
<tr>
<td></td>
<td>(25, 25)</td>
<td>C</td>
<td>XX</td>
<td>2888.49</td>
<td>3078.52</td>
<td>6.58</td>
</tr>
<tr>
<td></td>
<td>(30, 30)</td>
<td>C</td>
<td>XX</td>
<td>3539.07</td>
<td>3751.63</td>
<td>6.01</td>
</tr>
<tr>
<td>Rectilinear</td>
<td>(15, 15)</td>
<td>C</td>
<td>XX</td>
<td>2124.26</td>
<td>2160.00</td>
<td>1.68</td>
</tr>
<tr>
<td></td>
<td>(20, 20)</td>
<td>C</td>
<td>XX</td>
<td>2774.18</td>
<td>2936.00</td>
<td>5.83</td>
</tr>
<tr>
<td></td>
<td>(25, 25)</td>
<td>C</td>
<td>XX</td>
<td>3336.56</td>
<td>3596.00</td>
<td>7.78</td>
</tr>
<tr>
<td></td>
<td>(30, 30)</td>
<td>C</td>
<td>XX</td>
<td>4026.81</td>
<td>4330.00</td>
<td>7.53</td>
</tr>
</tbody>
</table>
Table 4
Characteristics and parameters of the test problems in Table 5

<table>
<thead>
<tr>
<th>Problem name</th>
<th>Le01</th>
<th>Le02</th>
<th>Le03</th>
<th>Le04</th>
<th>Le05</th>
<th>Le06</th>
<th>Le07</th>
<th>Le08</th>
</tr>
</thead>
<tbody>
<tr>
<td>((M_w, M_B, M_c))</td>
<td>(1.2, 2)</td>
<td>(1.2, 3)</td>
<td>(1.2, 3)</td>
<td>(1.2, 2)</td>
<td>(1.2, 2)</td>
<td>(1.2, 3)</td>
<td>(1.2, 3)</td>
<td>(1.2, 3)</td>
</tr>
<tr>
<td>Problem space</td>
<td>200 × 200</td>
<td>200 × 200</td>
<td>200 × 200</td>
<td>200 × 200</td>
<td>200 × 200</td>
<td>200 × 200</td>
<td>200 × 200</td>
<td>200 × 200</td>
</tr>
<tr>
<td>Max. walking distance</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Unit traveling costs</td>
<td>(0.01, 1.00, 0.10)</td>
<td>(0.01, 1.00, 0.10)</td>
<td>(1.00, 1.00, 1.00)</td>
<td>(1.00, 1.00, 1.00)</td>
<td>(1.00, 1.00, 1.00)</td>
<td>(0.10, 1.00, 0.25)</td>
<td>(0.10, 1.00, 0.25)</td>
<td>(0.10, 1.00, 0.25)</td>
</tr>
<tr>
<td>Customer demand</td>
<td>(N(25, 10^2))</td>
<td>(N(25, 10^2))</td>
<td>Unit demand</td>
<td>Unit demand</td>
<td>Unit demand</td>
<td>Exp((\mu = 10))</td>
<td>Exp((\mu = 10))</td>
<td>Exp((\mu = 10))</td>
</tr>
<tr>
<td>Gross marginal profits from sales</td>
<td>Const.: 150</td>
<td>Const.: 150</td>
<td>Const.: 750</td>
<td>Const.: 750</td>
<td>Const.: 750</td>
<td>Const.: 150</td>
<td>Const.: 150</td>
<td>Const.: 150</td>
</tr>
<tr>
<td>Time deadline stringency (K)</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>Vehicle capacities</td>
<td>(750, 125)</td>
<td>(750, 125)</td>
<td>(30, 6)</td>
<td>(30, 6)</td>
<td>(30, 6)</td>
<td>(200, 50)</td>
<td>(200, 50)</td>
<td>(200, 50)</td>
</tr>
<tr>
<td>Unit vehicle costs</td>
<td>(250, 75)</td>
<td>(250, 75)</td>
<td>(250, 75)</td>
<td>(250, 75)</td>
<td>(250, 75)</td>
<td>(250, 75)</td>
<td>(250, 75)</td>
<td>(250, 75)</td>
</tr>
<tr>
<td>Operating and fixed costs</td>
<td>((-250, 500, 1500))</td>
<td>((-250, 500, 1500))</td>
<td>((-250, 500, 1500))</td>
<td>((-250, 500, 1500))</td>
<td>((-250, 500, 1500))</td>
<td>((-200, 150, 500))</td>
<td>((-200, 150, 500))</td>
<td>((-200, 150, 500))</td>
</tr>
<tr>
<td></td>
<td>((1750, \ldots, 1250))</td>
<td>((1750, \ldots, 1250))</td>
<td>((1750, \ldots, 1250))</td>
<td>((1750, \ldots, 1250))</td>
<td>((1750, \ldots, 1250))</td>
<td>((1000, \ldots, 250))</td>
<td>((1000, \ldots, 250))</td>
<td>((1000, \ldots, 250))</td>
</tr>
<tr>
<td>Facility locations</td>
<td>WH: (Center of the problem space)</td>
<td>2 BM's: (at the same altitude as the WH equidistant from it on the left and right halves of the problem space)</td>
<td>CM's: (discrete uniformly distributed on equidistant grid lines on the problem space)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Customer locations</td>
<td>Walk-in customers: clustered uniformly around current and candidate store locations</td>
<td>Online customers: rectilinear uniformly distributed on 4 vertical and 4 horizontal equidistant grid lines</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

CM’s are (50, 50) and (150, 50). The maximum driving distance is taken as 250 units. Vehicle capacities and unit vehicle acquisition costs are (15 units, $100) for the warehouse, and (5 units, $75) for the stores. The cost of traveling from stores to online customers and that from walk-in customers’ residences to stores are both $1 per unit distance. Also the cost of transferring goods from the warehouse to stores is $1 per good and per unit distance. All customers have unit demand. The marginal gross profit from sales is $250 for each walk-in customer. Online customers have moderate deadlines with zero service times. The constant $K$ which adjusts the time deadline stringency is set to 4. (FC, CC, OC) values for the BM and candidate CM’s are ($1000, 0, 50) and ($100, -, 50), respectively. Thus, reconfiguring the BM as CM is possible at no cost. This base test problem has been called $t\text{-15-15-4}$.

5.2. Test results

The first tests have been done with several initial random seeds and different types of distance measures. The results are given in Table 2. The symbolic encoding of the best store allocation plans in the tables indicates first the status of BM’s (O for a preserved, X for a closed, and C for a converted BM), then the status of candidate CM’s (O for an opened and X for an unopened CM) separated by a vertical line |. The lower and upper bounds on the optimal objective function value of the problem are denoted by $Z_{lb}$ and $Z_{ub}$, respectively. The final gap between these two at the end of the solution procedure is also provided. The last column shows elapsed CPU times in seconds. The GAMS/Cplex 9.0 solution times of the FLAP-like SubP1 at every subgradient iteration have been added correctly to the respective overall CPU time.

For the set of parameters used in the pilot problem $t\text{-15-15-4}$, our method performs better in the sense of the final gap when the distances are taken as Euclidean, becoming later a general trend for the rest of the problems. Table 3 shows the results of eight test problems. As more and more customers are added to the problem, the solution times deteriorate quickly. All facility numbers kept fixed, the problem’s bottleneck lies in the routing decisions, which become more complicated with the increasing number of online customers. Also note that no new CM is opened even when the total number of customers rises to 60.

We have tested our method on eight medium-size problems each with Euclidean distances. Their characteristics are listed in Table 4, their results are reported in Table 5. In Table 7 we show the results of five large-size test problems also with Euclidean distances. Their details are given in Table 6. Unit traveling costs in Table 4 and Table 6 are, in the order of appearance, unit cost of transferring goods from WH’s to stores per km, vehicle traveling cost per km from stores to online customers, and walk-in customers, traveling cost per km from home to stores. Vehicle capacities and unit vehicle acquisition costs are given first for the WH’s, then for the stores. Operating and fixed cost values appear in the order of (FC, CC, OC) for BM’s followed by CM’s. As a matter of fact, CC value for CM’s is undefined. By convention, service times are taken as zero in all test problems. The problem Ap100-200-5 in Table 7 has the highest number of online customers. In spite of this, it yields a final gap below 10%. From the solution time point of view, problem P becomes intractable when the number of online customers exceeds 100. Only feasible solutions can be obtained for such problems.

We compared our method against the mixed integer programming (MIP) solver Cplex 9.0 of the mathematical modeling suite GAMS 21.7. Both the C code of our method ($ANA\text{-cmbm}$) and the GAMS models have been run on the same platform (Intel Pentium® 4 HT 3.40 GHz processor with 2 GB RAM producing $\sim$3630 MIPS). The maximum solution time of Cplex has been set to five hours. Best feasible objective values,
Table 6
Characteristics and parameters of the test problems in Table 7

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem space</td>
<td>1000 × 500</td>
<td>2000 × 1000</td>
<td>2000 × 1000</td>
<td>4000 × 2000</td>
<td>10000 × 5000</td>
</tr>
<tr>
<td>Max. driving distance</td>
<td>400 Distance units</td>
<td>200 Distance units</td>
<td>200 Distance units</td>
<td>400 Distance units</td>
<td>1000 Distance units</td>
</tr>
<tr>
<td>Unit traveling costs</td>
<td>(0.50, 1.0, 0.50)</td>
<td>(0.01, 1.00, 0.25)</td>
<td>(0.01, 1.00, 0.50)</td>
<td>(0.01, 1.00, 0.25)</td>
<td>(0.01, 1.00, 0.10)</td>
</tr>
<tr>
<td>Customer demand</td>
<td>Normal (30, 100)</td>
<td>Normal (30, 100)</td>
<td>Normal (30, 100)</td>
<td>Normal (30, 100)</td>
<td>Exp(μ = 50)</td>
</tr>
<tr>
<td>Time deadlines</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Operating and fixed costs</td>
<td>(−250, 150, 350), (750, −, 200)</td>
<td>(−250, 150, 350), (750, −, 200)</td>
<td>(−250, 150, 350), (750, −, 200)</td>
<td>(−250, 150, 350), (750, −, 200)</td>
<td>(−100, 100, 150), (1000, −, 150)</td>
</tr>
<tr>
<td>Facility locations</td>
<td>WH: (500, 250)</td>
<td></td>
<td></td>
<td></td>
<td>WH: (30, 261),</td>
</tr>
<tr>
<td>Customer locations</td>
<td>Stores are placed on the centers of the quadrants of the rectangular problem space</td>
<td></td>
<td></td>
<td></td>
<td>All customers are rectilinear uniformly distributed on equidistant grid lines</td>
</tr>
</tbody>
</table>
store allocation plans and solution times on the Euclidean and rectilinear distance versions of six benchmark problems are reported in Table 8. The benchmark problems are relatively small in size, having at most 6 facilities and 115 customers. The number of online customers ($N_o$) ranges from 10 to 15 only. The mathematical model of problem $P$ holds $O(2^{N_o})$ subtour elimination and capacity constraints as implied by Eqs. (15), (16). For example, in the models of problems Eu-W100-O15 and Rec-W100-O15, where $N_o = 15$, there are exactly 32,753 such constraint equations along with 1892 discrete variables. This clearly shows the explosion in the model size with an increase in $N_o$. The CPU seconds given for the solver Cplex exclude the model generation and execution times of GAMS. Those figures for ANA-cmbm do not account for the I/O operations with the data file and with the intermediate GAMS files, the latter of which are created to obtain the objective value of SubP1. Though, cumulative Cplex solution times of SubP1 have been added to the respective overall CPU times of ANA-cmbm in Table 8.

In problem Eu-W50-O10, Cplex is better than ANA-cmbm both in objective value and in solution time. In the rest of the benchmark problems, the solution times of ANA-cmbm are incomparably lower than those of Cplex. Eu-W60-O11 and Rec-W50-O10 are the only problems where Cplex falls behind ANA-cmbm in terms of the objective value. In all others, Cplex slightly outperforms our method. For example, in problem Rec-W90-O14, Cplex finds a total cost value that is by 1.22% lower than the one found by ANA-cmbm. However, Cplex consumes almost 84 minutes to get this solution, while ANA-cmbm converges in only 7.59 seconds.

Being a general purpose solver, Cplex treats each given problem equally with brute-force MIP techniques. We conjecture that our method’s weakness comes from the classical “fast” heuristic that is used to solve the MDVRPTD for a given store allocation plan obtained at the end of SubP1. If a metaheuristic algorithm for MDVRPTD is incorporated into the main Lagrangian relaxation, then one may achieve a lower-cost routing plan for the vehicles dispatched from stores to online customers. This would, in return, improve the final

Table 7
Results of 5 large-size test problems with Euclidean distances and erratic cost data

<table>
<thead>
<tr>
<th>Problem name</th>
<th>Best store plan</th>
<th>Final $Z_{lb}$</th>
<th>Final $Z_{ub}$</th>
<th>Gap (%)</th>
<th>CPU time [seconds]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eu-W150-O50-5</td>
<td>XC</td>
<td>OX</td>
<td>784644.62</td>
<td>884297.20</td>
<td>12.70</td>
</tr>
<tr>
<td>Eu-W60-O60-5</td>
<td>CC</td>
<td>OO</td>
<td>32221.21</td>
<td>32845.38</td>
<td>1.94</td>
</tr>
<tr>
<td>Eu-W120-O60-5</td>
<td>CC</td>
<td>OO</td>
<td>52593.91</td>
<td>52672.07</td>
<td>0.15</td>
</tr>
<tr>
<td>Eu-W80-O80-5</td>
<td>CC</td>
<td>OO</td>
<td>33482.97</td>
<td>34125.39</td>
<td>1.92</td>
</tr>
<tr>
<td>Eu-W200-O100-5</td>
<td>CC</td>
<td>XX</td>
<td>188949.94</td>
<td>205494.98</td>
<td>8.76</td>
</tr>
</tbody>
</table>

Table 8
Comparisons between ANA-cmbm and Cplex 9.0 of GAMS on small-size test problems

<table>
<thead>
<tr>
<th>Problem name</th>
<th>No. facilities</th>
<th>No. customers</th>
<th>ANA-cmbm</th>
<th>Cplex 9.0 solver of GAMS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Best objective</td>
<td>Best alloc. plan</td>
</tr>
<tr>
<td>Euclidean</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eu-W50-O10</td>
<td>(1,1,2)</td>
<td>(10, 50)</td>
<td>23315.43</td>
<td>X</td>
</tr>
<tr>
<td>Eu-W60-O11</td>
<td>(1,1,2)</td>
<td>(11, 60)</td>
<td>25779.28</td>
<td>X</td>
</tr>
<tr>
<td>Eu-W70-O12</td>
<td>(1,2,2)</td>
<td>(12, 70)</td>
<td>26564.39</td>
<td>CX</td>
</tr>
<tr>
<td>Eu-W80-O13</td>
<td>(1,2,2)</td>
<td>(13, 80)</td>
<td>28862.71</td>
<td>CO</td>
</tr>
<tr>
<td>Eu-W90-O14</td>
<td>(1,2,3)</td>
<td>(14, 90)</td>
<td>28533.43</td>
<td>XC</td>
</tr>
<tr>
<td>Eu-W100-O15</td>
<td>(1,2,3)</td>
<td>(15, 100)</td>
<td>31383.11</td>
<td>X</td>
</tr>
<tr>
<td>Rectilinear</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rec-W50-O10</td>
<td>(1,1,2)</td>
<td>(10, 50)</td>
<td>28413.84</td>
<td>C</td>
</tr>
<tr>
<td>Rec-W60-O11</td>
<td>(1,1,2)</td>
<td>(11, 60)</td>
<td>30872.74</td>
<td>C</td>
</tr>
<tr>
<td>Rec-W70-O12</td>
<td>(1,2,2)</td>
<td>(12, 70)</td>
<td>31550.03</td>
<td>CO</td>
</tr>
<tr>
<td>Rec-W80-O13</td>
<td>(1,2,2)</td>
<td>(13, 80)</td>
<td>34281.77</td>
<td>OO</td>
</tr>
<tr>
<td>Rec-W90-O14</td>
<td>(1,2,3)</td>
<td>(14, 90)</td>
<td>33402.71</td>
<td>XC</td>
</tr>
<tr>
<td>Rec-W100-O15</td>
<td>(1,2,3)</td>
<td>(15, 100)</td>
<td>36860.77</td>
<td>OO</td>
</tr>
</tbody>
</table>

Customer and Facility coordinates are uniformly distributed or clustered.
percentage gap. We observe that when the number of online customers in the CMBP-S rises above 16, our LR based composite method turns out to be an advantageous alternative to the commercial MIP solver Cplex, since Cplex generally takes more than five hours of computing time in order to find a solution to such problems.

6. Conclusions and future research

In this paper, we propose a static location-routing model (CMBP-S) for the conversion from the traditional brick-and-mortar to the hybrid click-and-mortar retailing business. The model entities consist of online and walk-in customers, warehouses and stores. The model combines store location and customer allocation decisions with vehicle routing decisions into an integrated problem. The transshipment of goods from warehouses to stores with unlimited storage capacity and from there to customers is assumed to be synchronized. This assumption eliminates any inventory concerns. In the allocation of walk-in customers a customer directly goes to the nearest open store if there is such a store within a specified maximum distance. Otherwise, the retailer loses the demand of this customer. The allocation of online customers, on the other hand, materializes on vehicle tours. Their orders are delivered before a time deadline, which can be viewed as a quality of service (QoS) guarantee. Obviously, customer allocation decisions are contingent on store locations. The CMBP-S is static because the real-time arrival of orders from online customers and the corresponding re-optimization of vehicle routes are left out. It is also a deterministic model, because the exact amounts and deadlines of orders as well as all prices, distances, demands, and costs of operations are known in advance. The CMBP-S disregards the outsourcing option for order delivery. The retailer is assumed to acquire its vehicle fleet at a fixed unit cost. Outsourcing may become an option in the future if the shipping companies can guarantee customer satisfaction by meeting quite stringent deadlines on delivery times as this is crucial in the retailing of nondurable goods.

We describe a Lagrangian relaxation (LR) based solution method for the CMBP-S of which subproblems are known to be \( \mathcal{NP} \)-hard. The LR based method enables us to decompose the overall problem into two independent subproblems. The first one is solved by the optimization package GAMS and its MIP solver Cplex 9.0, the second subproblem by an augmented LR method. Computational experiments with our method show that large-size instances of the CMBP-S with 100 or more online customers have unfavorably long solution times. However, a commercial solver such as Cplex cannot be substituted for our method. Instances of the CMBP-S with as few as 18 online customers proved intractable with Cplex.

A future research extension is the real-time version of the CMBP-S. In fact, vehicle assignment and order delivery should be done on a real time basis. The major difference between the static and real-time versions is the probabilistic nature of customer deadlines and ordering times. In addition, vehicle tours with backhauls should be looked into as a future extension, as backhauls in vehicle routing signify returns or refunds of orders that have been delivered earlier. Finally, QoS guarantees for online customers can be treated as soft constraints. If it is impossible or unprofitable to meet some online customer’s deadline, or yet to deliver the order of that customer, then his order can be forsaken. The gross marginal profit from sales to such customers determines in that case the total cost of lost demand. If skipping orders is allowed, a multi-depot vehicle routing problem with time deadlines and customer selection (MDVRPTD with profits) would be one of the new sub-problems of the updated CMBP-S logistics model.

Acknowledgements

The authors would like to thank the editors of the European Journal of Operational Research, Jacques Teghem and Lorenzo Peccati, and two anonymous referees for their insightful comments which improved the readability and the theoretical soundness of the paper.

References


