Single period stochastic inventory problems with ordering or returns policies

Zhong Yaoa,⇑, Ke Liub, Stephen C.H. Leungc, K.K. Laic

a School of Economics and Management, BeiHang University, 100191 Beijing, China
b Institute of Applied Mathematics, AMSS, The Chinese Academy of Science, 100080 Beijing, China
c Department of Management Sciences, City University of Hong Kong, Hong Kong, China

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A B S T R A C T

In recent years, there has been an increasing adoption of returns policies in the coordination of the supply chain, where market demand is always assumed to be satisfied by manufacturing or by ordering from suppliers. However, many industries face the important decision of how to balance their inventory level. This problem has long been studied in financial institutions such as banks. This study presents an optimal inventory policy under a given stochastic demand such as a uniformly distributed demand, single-item, and single period review inventory system. The optimal inventory control policy obtained in this study is called a four-point policy: that is, when the entity's inventory level is below a reorder point, the entity must increase his stock level by ordering and order up-to a fixed level (second point); when the entity's inventory level is over a return point (third point); the stock level must be decreased by returns and decreased to a fixed level (fourth point); otherwise, nothing should be done. We also analyze the (K, S)-convex properties of the inventory cost function.

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1. Introduction

1.1. General description

With the booming of E-commerce, online retailing has received considerable attention from researchers and practitioners. One of the key features of online retailing is e-tailers' flexibility in managing their inventories. For example, an e-tailer can adopt either the pure-play online retailing with a zero-inventory or click-mortar retailing with a positive-inventory. When we provide management consultative service for a click-mortar e-tailer, they face an inventory decision problem. If the on-hand goods are too few, out-of-stock often occurs; but if storing more on-hand goods, it occupies too much floating capitals and they want reducing the on-hand inventory level. This is typical inventory decision problem. From the perspective of one-entity inventory decision making, the question of how to control ordering and returning quantities to optimize stock levels was concerned in the 60s of last century by the operational management researchers. Studies in classical stochastic cash balance problems is to adjust their inventory level through increasing their inventory level by transferring money to his savings or decreasing it by withdrawing money from his savings (Girgis, 1968). There is lots of literature to report this kind of research. See reviewed in next subsection.

Another related application field is that one inventory system needs to proposal the returned item while simultaneously satisfying some fixed demand rate. Heyman (1977) investigated optimal disposal policies for a single-item inventory system with returns, where the inventory system was described as single-server queue model when the demand and return processes are both Poisson processes. However, due to the sufficient effort for the exactly optimal policy, they designed an approximate model that requires only mean and variance data to solve the problem. Later, Heyman (1978) considered a two-point policy for returned goods, which is similar to the returns policy part of our model. They continued to use queuing theory to investigate the problem. Muckstadt and Isaac (1981) studied a continuous review procurement policy used to control a single-item at a single location under return rate less than the demand rate and to develop a solution method for finding the policy parameter values. In addition, Muckstadt and Isaac (1981) also adopted the normal distribution to model the demand and repairing processes. However, to obtain optimal solutions in model of Muckstadt and Isaac (1981), search algorithm need to be designed.

Analyzing these literature we have known that all of these researches were conducted using pure mathematical methods except for Heyman (1977, 1978) and Muckstadt and Isaac (1981). Also, these papers had not tried to using the special distribution to solve the optimal policies, but used the approximate model or the searching algorithm to obtain the optimal policies. That is, under assumption with a special demand distribution situation the optimal returning and ordering policy has not been investigated with...
one-entity decision maker. This is our aim to investigate a similar inventory problem, that is, under a uniformly distributed stochastic demand (a special distribution) what the optimal policies will be exhibited. This paper has analytically proven that the four-point policy exists with the uniformly distributed demand. Specifically, we analyze an optimal control policy for the retailer under three situations: ordering decision, return decision or doing nothing decision.

Following a review of the literature related more with our study, the reminder of this paper is organized as follows. Section 2 models the decision problem under single period returns and ordering polices. Then the proof of an optimal four-point policy and the properties of decision functions are presented. In Section 3 we present a numerical study to investigate the optimal four-point policy with normally distributed demand. A summary of findings and extensions of the research is found in the last section.

1.2. More relevant literature

It is well-known that traditional inventory theory for one-entity decision mainly focuses on the quantities ordered and when to release an order. However, recent research has moved on investigating the adoption of returns policies in coordinating the supply chain. Several papers in supply chain literature have reported that manufacturers accept that the return of goods can be profitable for both manufacturers and retailers. Examples are Pasternack (1985), Marvel and Peck (1995) and Emmons and Gilbert (1998), Padmanabhan and Png (1997), Webster and Weng (2000), Choi, Li, and Yan (2004), Flath and Nairu (1989), Yao, Yue, Wang, and Liu (2005), etc. In addition, most analyses of returns policies in supply chain fields have concerned on the manufacturer’s advantage, see Pasternack (1985), Padmanabhan and Png (1997). Another research stream focusing on the returns for used goods is the reverse logistics literature. This deals with how to integrate the returned (used) products into the logistics systems via remanufacturing or recovering the returned (used) products. A lot of research has been reported in this research field, such as Mitra (2009), Teunter (2004), and Kelle and Miller (2001). A comprehensive review of this research area can be found in Fleischmann et al. (1997).

However, more relevant literature studying above mentioned question is focus on the cash balance management in the 60s of last century. Eppen and Fama (1969) studied the optimal policies for stochastic cash balance problem under the situation that transaction costs are strictly proportional to the amount of funds transferred. They concluded that the optimal policies for infinite horizon cash balance problems were characterized by the return points T (this point is equivalent to ordering up-to point in our paper) and Q. That is, if the cash balance is in one state less than T at the beginning of any period, the optimal policy is to move it up-to state T, while if the cash balance is in one state more than Q, the optimal policy is to move it down to state Q. Similar conclusion had been reported by Whisler (1967), where Whisler (1967) had analyzed the optimal inventory policies for rented equipment (such as cars) with the ordering or returning decision for balancing the inventory level. The optimal policy for the finite-periods problem has the form of a two-point policy, that is, (T, Q)-policies, where T and Q called the critical ordering number and returning number. This means that when the inventory level x is below the critical number T, then the retailer increases the inventory up-to T; when the inventory level x is over the critical number Q, the retailer must reduce the inventory level to Q; otherwise, leave it unchanged. A similar result is valid in an infinite horizon problem and the two-point policies is expressed by (Tn, Qn)-policies, where n refers to the number of the period. However, both Eppen and Fama (1969) and Whisler (1967) had not considered the fixed ordering and returning set-up cost, that is, K = S = 0.

On the other hand, research in inventory field considering the positive set-up cost has interested in a lot of studies. Following the K-convex function definition by Scarf, Gilford, and Shelly (1963), Girgis (1968) discussed optimal cash balance levels in the cases where K > 0, S > 0 and K = 0, S > 0, but they had not proven the K > 0 and K > 0 case. He found that the optimal policy for the first case takes the form of an approximate three-point policy, that is (T0, Tn, Qn) where Qn ≤ Tn ≤ Q0 and n refers to the number of the period. This means that when the initial inventory level x is below the critical number T0, then the inventory is increased to Tn; when x is above the critical number Qn, then the inventory is decreased to Qn; otherwise, the inventory is kept on-hand quantity. An analogous form of policy is optimal for the second case: (Tn, Qn, qn), where Tn ≤ Qn ≤ qn which means that when the initial inventory level x is below the critical number Tn, then the inventory is increased to Tn; when the x is above the critical number Qn, then the inventory decreased to Qn; otherwise, the inventory level is kept on-hand quantity.

Neave (1970) discussed the case of K = S > 0, ordering cost c > 0 and returning cost r > 0 and demonstrated that the optimal policy has a six-points policy form for n = 1, 2, ..., N. However, when K > 0, S > 0, c > 0 and r > 0 (in Neave’s, 1970 Section 3.3), they did not give the optimal policy, although their problem setting had considered a multiple period situation.

Recently, Archibald, Thomas, and Possani (2007) investigated the keeping or returning policy for a start-up company. They used the dynamic programming recursion for the various Markov decision processes both for profit and long-term survival objectives. They find an optimal returning and ordering polices and the value of the average expected profit. Optimal policy will depend on the different scenarios of holding cost. They also analyzed the properties of best policies. The difference between our studies with Archibald et al. (2007) is that we focus on optimal policy with returning and ordering decision-making for minimizing the cost of firm, but the latter concerned on the survival decisions of start-up firm.

In this study, we extend the Neave (1970) study by two folds: One is we have derived the close-form optimal policy with the assumption of uniformly distributed demand. Previous research has not reported the close-form solution for optimal decisions with related to the specific demand information. Also we have given the optimal policies under K > 0, S > 0, c > 0 and r > 0 situation with uniformly distributed demand assumption. Although the uniform distribution has some limitation in practice, however, it has been used widely for investigating the management insights in operational management studies. For example, recent study by Wanke (2008) has shown that the uniform distribution as the first practical approach in new product inventory management, which is based on the fact that the uniform distribution is defined by two parameters that are easy to estimate. Examples using the uniform distribution to insight into the management are: Emmons and Gilbert (1998), Emmons and Gilbert (1998), Lau, Lau, and Wang (2008), Lariiviere and Porteus (2001), Yan, Liu, and Hsu (2003), Song (2000), Granot and Yin (2005) etc.

Another one is we have proven that the ordering function is of (K, +)-convex property, returning function is of (S, -)-convex property, and the without ordering and returning decision the cost function of the system is of (K, S)-convex property. The (K, +)-convex and the (S, -)-convex are equal to those left-hand K-convex and right-hand K-convex described in Neave (1970) and Girgis (1968), respectively. Also it is the same as Scarf et al. (1963) definition of a K-convex function. Left-hand or right-hand K-convex function requests the existence of first left or right derivative of the function with respect to demand, while the (K, S)-convexity is no this limitation. Our (K, +)-convex, (S, -)-convex and (K, S)-convex definitions are in fact the strong version in quasi-K-convex
defined in Porteus (1971). In the single-period inventory problem, if the ordering cost is linear with respect to the order quantity, there is a simple optimal decision. One simple policy is when stock level below a critical number, then orders the difference between the critical number and the stock level. When the ordering or returning cost is non-linear, this simple policy is not optimal. However, if the ordering cost functions are K-convex, the optimal policy is existed, see Porteus (Scarf et al., 1963). In addition, we also extend the application situation of K-convex into the returning cost function with (S, −) convex and both the ordering and returning cost function existing situation with (K, S)-convex.

In this study, we conclude that under uniform distribution demand the optimal policy in a single period takes the form of a four-point policy, that is, (t, T, Q, q) where (t ≤ T ≤ q). This means when the initial inventory level x is below the critical number t, then the inventory must be increased by ordering and optimal order quantities are T − t; when x is above the critical number q; then the inventory must be decreased by returns and optimal returning quantities are q − Q; otherwise, leave it unchanged. In addition, the optimal inventory decision functions have the (K, S)-convex properties.

2. Single-period problem

In this study, all values of variables and random variables are real numbers. We assume that the demand z is a continuous random variable with the probability density function \( g(z) \) and cumulative distribution function \( G(z) \). Let

- \( x \) = inventory position on-hand before making a decision of ordering or returning,
- \( y \) = inventory position on-hand after making a decision of ordering or returning,
- \( c \) = unit ordering cost,
- \( r \) = unit returning cost (0 < r < c),
- \( h \) = holding cost,
- \( p \) = penalty cost if a shortage occurs,
- \( K \) = fixed ordering set-up cost for one order,
- \( S \) = fixed returning set-up cost for one order,
- \( \mu \) = demand mean and \( \sigma \) = standard deviation of demand.

2.1. The definition of decision functions

For a one-period problem, the central problem is to decide when to change inventory levels and how to change them. For the given initial inventory level \( x \), an ordering decision cost function is expressed by Neave (1970), etc., as follows:

\[
 f_1(x) = \min_{y > x} \left\{ L(y), \quad y > x \right\},
\]

(1)

A returning decision cost function can be expressed by

\[
 f_2(x) = \min_{y < x} \left\{ L(y), \quad S + r(x - y), \quad L(y), \quad y < x \right\}.
\]

(2)

A nothing-to-do (neither ordering nor returning occurs) cost function can be expressed by

\[
 f_3(x) = L(y).
\]

(3)

where \( L(y) = h \int_{0}^{y} (y-z)f(z)dz + p \int_{y}^{\infty} (z-y)f(z)dz \) in the above functions.

Based on (1), (2) and (3), the retailer can minimize their inventory cost with the following decision function:

\[
 f(x) = \min \left\{ f_1(x), \quad f_2(x), \quad f_3(x) \right\}.
\]

(4)

The function \( f_1(x) \) is the ordering decision function, that is when the retailer is only considering ordering or the nothing-to-do situation. The function \( f_2(x) \) is the returns decision function, that is when the retailer is only considering returning or the nothing-to-do situation. The function \( f_3(x) \) is the nothing-to-do function. The decisions (ordering, returning, or nothing-to-do) made by the retailer are dependent on the minimum cost of the relative decision functions (ordering, returning, or nothing-to-do functions), and these kinds of decisions constitute the optimal policy for their stock level.

2.2. The optimal solutions of the decision function

As mentioned in the previous section, this study focuses on a special demand distribution – uniform distribution demand – to investigate the optimal stock level decision. In this section we first investigate the optimal solutions of the decision function in detail and then we sketch the structure properties of the cost function.

Let the demand be uniformly distributed within intervals of width \( a \) centered at \( s \) so that the probability density function and the cumulative distribution function and inverse cumulative distribution functions are:

\[
 g(z) = \frac{1}{a}, \quad G(z) = \frac{z - s + \frac{a}{2}}{a}, \quad G^{-1}(z) = az + s - \frac{a}{2},
\]

where \( z \in \left[ s - \frac{a}{2}, s + \frac{a}{2} \right] \).

(5)

respectively. With this assumption, \( \mu = s, \sigma = \frac{a}{\sqrt{3}} \) and \( L(y) \) in (1), (2) and (3) can be expressed as follows:

\[
 L(y) = h \int_{\frac{s-a}{2}}^{y} \frac{(y-z)^2}{a}dz + p \int_{y}^{s+\frac{a}{2}} \frac{(z-y)^2}{a}dz
\]

(6)

Next, in order to obtain the optimal policy, we need to solve (4). Following the Neave (1970) notation, we define four real numbers \( t, T, Q, q \) by

\[
 ct + L(T) = \min_{y} \{ cy + L(y) \}, \quad (7)
\]

\[
 t = \min \{ y : cy + L(y) = K + ct + L(T) \}, \quad (8)
\]

\[
 -rQ + L(Q) = \min_{y} \{ -ry + L(y) \}, \quad (9)
\]

\[
 q = \max \{ y : -ry + L(y) = S - rQ + L(Q) \} \quad (10)
\]

It is clear from the above that \( t \leq T \leq q \) (for proof, see Neave (1970)).

Now we derive the analytical solution for above problem. Substituting (6) into (7), we have:

\[
 \min_{y} \{ L(y) + cy \} = \min_{y} \left\{ h \int_{\frac{s-a}{2}}^{y} \frac{(y-z)^2}{a}dz + p \int_{y}^{s+\frac{a}{2}} \frac{(z-y)^2}{a}dz + cy \right\}
\]

\[
 = \min_{y} \left\{ h \int_{\frac{s-a}{2}}^{y} \frac{(y-z)^2}{a}dz + p \int_{y}^{s+\frac{a}{2}} \frac{(z-y)^2}{a}dz + cy \right\},
\]

By the standard newsboy solution and (5), we have:

\[
 t = s - \frac{a}{2} \sqrt{\frac{c}{p} - \frac{1}{2}}.
\]

(11)

Similarly, substituting (6) into (9), we obtain:

\[
 Q = s - \frac{a}{2} \sqrt{\frac{c}{p} + \frac{1}{2}}.
\]

(12)

To obtain \( t \), from (8), we have:
\( t = \min \{y : cy + L(y) = K + cT + L(T)\} \)
\[ = \min \left\{y : cy + h \int_0^y (y-z)g(z)dz + p \int_y^\infty (z-y)g(z)dz\right\} \]
\[ = K + cT + h \int_0^T (T-z)g(z) + p \int_T^\infty (z-T)g(z)dz. \]

Substituting (10) and (5) into above formula and solving the equation yields two roots. Taking the smaller as our solution, we have:
\[ t = s - \frac{a}{2} + \frac{a(p-c)}{p+h} - \sqrt{\frac{2ak}{p+h}}. \]

Similarly, we have
\[ q = s - \frac{a}{2} + \frac{a(p+r)}{p+h} + \sqrt{\frac{2as}{p+h}}. \]

Now we analyze the retailer’s optimal decision. With the close-form four-point policy, we can easily calculate the cost to the retailer. The four-point cost is calculated as follows. At the nothing-to-do point, order size \( m = 0 \). The cost function in one-period is:
\[ f(x) = h \int_0^x (x-z)dG(z) + p \int_x^\infty (x-z)dG(z) \]
\[ = h \int_{-s}^x x-z \frac{a}{a}dz + p \int_x^\infty \frac{x-s}{a}dz \]
\[ = \frac{1}{8a} \alpha^2(h+p) - 4a(h-p)(s-x) + 4(h+p)(s-x)^2 \]

For any \( x \), if there is an order and the order quantity is \( m \), then by
\[ \min(m : K + cm + L(m + x)) \]
\[ = \min \left\{m : K + cm + h \int_0^x (x-z)g(z)dz + p \int_x^\infty (x-z)g(z)dz\right\} \]
\[ = \min \left\{m : K + cm + h \int_0^x (x-z)g(z)dz + p \int_{x-m}^\infty (z-x-m)g(z)dz\right\} \]

With the standard newsvendor solution, we have:
\[ m = s - \frac{a}{2} + \frac{a(p-c)}{p+h} - x \]

Therefore, the cost is:
\[ f(x + m) = L(x + m) + c(x + m - x) + K \delta(x + m - x) \]
\[ = \frac{a(p+h+c^2)}{2(p+h)} + cm + K \]
\[ = K + c(s-x) - \frac{a(c+h)(c-p)}{2(h+p)}. \]

In the following, we demonstrate the optimal decisions taken by the decision maker.

1. \( x > T \). We first check costs at \( T \). If there is nothing-to-do at \( T \), the cost is
\[ f(T) = L(T) = h \int_0^T (T-z)dG(z) + p \int_T^\infty (z-T)dG(z) \]
\[ = h \int_{-s}^T (T-z) \frac{a}{a}dz + p \int_T^\infty \frac{T-z}{a}dz = \frac{a(hp+c^2)}{2(p+h)}. \]

When \( x = T \), if an order is made and optimal order quantity is \( m \), then by (16), \( m = 0 \); therefore, at \( T \) nothing-to-do is the optimal inventory level.

Now consider \( T < x \). If an order is made and the order quantity is \( m \), then by (17), the costs will be:
\[ f(m + x) = K + cm + L(m + x) \]
\[ = K + c(s-x) - \frac{a(c+h)(c-p)}{2(h+p)}. \]

Taking (19)–(15), we have
\[ f(x + m) - f(x) = K + c(s-x) - \frac{a(c+h)(c-p)}{2(h+p)} - \frac{1}{8a} \alpha^2(h+p) \]
\[ - 4a(h-p)(s-x) + 4(h+p)(s-x)^2 > 0 \]

because of \( x > s - \frac{a(p-c)}{8a} > \frac{\sqrt{2a}}{\sqrt{h+p}} > T. \) Therefore, when \( x > T \), if an order is made, the cost will be more than that of nothing-to-do, which means that nothing-to-do will be optimal inventory level at \( x > T \).

2. When \( t < x < T \), at \( t \), the cost is:
\[ f(t) = L(t) = K + \frac{a(hp+c^2)}{2(h+p)} + cT + L(T) \]

When an order is made at \( t \) and the order quantity is \( m \) (that is order up-to \( t + m \)), then the optimal order size \( m \) will be the solution of:
\[ \min\{K + cm + L(t + m)\} \]

By (16), we can see that \( m = T - x = T - x / \sqrt{a(h+p)} \), and the costs will be:
\[ f(t + m) = K + cm + L(t + m) = K + cm + L(t) \]
\[ = K + c \left( - \frac{a(p-c)}{2(h+p)} - x \right) + \frac{a(hp+c^2)}{2(h+p)}. \]

By taking (23) = (21), at \( t \), the decision maker chooses both ordering up-to \( T \) and nothing-to-do are optimal policies. When \( t < x < T \), if an order is made and the optimal order quantity is \( m \), by (16), the optimal order quantity will be defined by (16). The costs will be similar to (19), that is:
\[ f(x + m) = K + cm + L(x + m) \]
\[ = K + c \left( - \frac{a(p-c)}{2(h+p)} - x \right) + \frac{a(hp+c^2)}{2(h+p)}. \]

Here \( t < x < T \). By taking (24)–(15), we have
\[ f(x + m) - f(x) = K + c(s-x) - \frac{a(c+h)(c-p)}{2(h+p)} \]
\[ - \frac{1}{8a} \alpha^2(h+p) - 4a(h-p)(s-x) \]
\[ + 4(h+p)(s-x)^2 > 0 \]

because of \( x > s - \frac{a(p-c)}{8a} > \frac{\sqrt{2a}}{\sqrt{h+p}} > T. \) Therefore, when \( T > x > t \), if an order is made, the cost will be more than that of nothing-to-do, which means that nothing-to-do will be the optimal inventory level at \( T > x \).

3. \( x < t \), if an order is made at \( x < t \), the order quantity is \( m_T \) and order up-to \( x + m_T \). By (15), the optimal order size will be:
\[ m_T = s - \frac{a}{2} + \frac{a(p-c)}{p+h} - x. \]

Therefore, the cost after ordering \( m_T \) is:
\[ f(x + m_T) = L(x + m_T) + c(x + m_T - x) + K \delta(x + m_T - x) \]
\[ = K + c(s-x) - \frac{a(c+h)(c-p)}{2(h+p)}. \]

By (27)–(15), we have
\[ f(x + m_T) - f(x) = K + c(s - x) - \frac{a(c + h)(c - p)}{2(h + p)} \]
\[ - \frac{1}{8a} a^2(h + p) - 4a(h - p)(s - x) \]
\[ + 4(h + p)(s - x)^2 < 0 \]

because of \( x < s - \frac{\sqrt{hp}}{\sqrt{m_T}} < \sqrt{\frac{\sqrt{hp}}{m_T}} \). Therefore, at \( x < t \), costs of ordering and the optimal order size \( m_T \) will be less than that of nothing-to-do.

Note that this proof procedure also finds that the optimal order quantity is \( m_T \) and order up-to point \( T = m_T + x \), see (26).

On the other hand, by (18) and (21) we have:

\[ f(t) - f(T) = K + c \sqrt{\frac{2aK}{p + h}} \]

(28)

A similar proof can be conducted for (4) \( x \in Q \), (5) \( Q < x \leq q \), and (6) \( x > q \).

Now for any \( x \), if there are returns and the return quantity is \( m > 0 \), then:

\[
\min\{m : S + rm + L(x - m)\} = \min \left\{ m : S + rm + h \int_{m}^{\infty} (x - m - z) \frac{dz}{a} + p \int_{m}^{\infty} (2 - x + m) \frac{dz}{a} \right\}.
\]

With the standard newsboy solution, we have:

\[
m = \left[ -s - \frac{a}{2} + \frac{a(r + p)}{p + h} - x \right] = x - Q. \tag{16'}
\]

So, the cost is:

\[
f(x - m) = L(x - m) + rm + S = \frac{a(ph + r^2)}{2(p + h)} + rm + S
\]
\[
= S + r(s - x) - \frac{a(r + p)(r - h)}{2(h + p)} \tag{17'}
\]

(4) \( x \in Q \). We first check costs at \( Q \). If there is nothing-to-do at \( Q \), the cost is:

\[
f(Q) = L(Q) = h \int_{0}^{Q} (Q - z) dG(z) + p \int_{Q}^{\infty} (z - Q) dG(z)
\]
\[
= h \int_{0}^{Q} \frac{(Q - z) dz}{a} + p \int_{Q}^{\infty} \frac{(z - Q) dz}{a} = \frac{a(ph + r^2)}{2(p + h)}. \tag{18'}
\]

When \( x > Q \), if there are returns and the optimal return quantity is \( m \), then by (16'), \( m = 0 \), therefore, at \( Q \) nothing-to-do is the optimal inventory level. Moreover, we check \( x < Q \). If there are returns and the optimal return quantity is \( m \), then by (17'), the cost will be:

\[
f(x - m) = S + rm + L(m) = S - r(s - x) - \frac{a(r + p)(r - h)}{2(h + p)} \tag{19'}
\]

By (19')-(15), we have

\[
f(x - m) - f(x) = S - r(s - x) - \frac{a(r + p)(r - h)}{2(h + p)}
\]
\[
- \frac{1}{8a} a^2(h + p) - 4a(h - p)(s - x)
\]
\[
+ 4(h + p)(s - x)^2 < 0 \tag{20'}
\]

because of \( x < s - \frac{s^2 + \sqrt{hp}}{\sqrt{m_T}} < \sqrt{\frac{h}{m_T}} \). Therefore, when \( x < Q \), if there are returns, the cost will be more than that of nothing-to-do, which means that nothing-to-do will be the optimal inventory level at \( x < Q \).

(5) \( Q < x \leq q \). We first check at \( q \). If there is nothing-to-do at \( q \), the cost is

\[
f(q) = L(q) = S + a(hp + r^2) + r \sqrt{\frac{2aS}{h + p}} \tag{21'}
\]

When there are returns at \( q \) and the return quantity is \( m \) (that is, returns up-to \( q - m \)) then the optimal order size \( m \) will be the solution of

\[
\min \{S + rm + L(q - m)\} \tag{22'}
\]

By (16'), we can see that \( m = x - Q = q - Q = \sqrt{\frac{2aS}{h + p}} \), so the cost will be:

\[
f(q - m) = S + rm + L(q - m) = S + rm + L(Q)
\]
\[
= S + r \sqrt{\frac{2aS}{p + h}} + a(hp + r^2) \tag{23'}
\]

By (23')-(21'), at \( q \), both returns and return up-to \( Q \) and nothing-to-do are optimal policies. Therefore, when \( Q < x < q \), if there are returns and the optimal return quantity is \( m \), the optimal return quantity will be defined by (16'). The cost will be similar to (19'), that is:

\[
f(x - m) = S + rm + L(x - m)
\]
\[
= S - r(s - x) - \frac{a(r + p)(r - h)}{2(h + p)} \tag{24'}
\]

Here \( Q < x < q \). By (24')-(15), we have

\[
f(x - m) - f(x) = S - r(s - x) - \frac{a(r + p)(r - h)}{2(h + p)}
\]
\[
- \frac{1}{8a} a^2(h + p) - 4a(h - p)(s - x)
\]
\[
+ 4(h + p)(s - x^2) > 0 \tag{25'}
\]

because of \( x < s - \frac{s^2 + \sqrt{hp}}{\sqrt{m_T}} < \sqrt{\frac{h}{m_T}} \). Therefore, when \( Q < x < q \), if there are returns, the cost will be more than that of nothing-to-do, which means nothing-to-do will be the optimal inventory level at \( Q < x < q \).

(6) \( x < t \). If there are returns at \( x < t \), the returned quantity is \( m_0 \), and return up-to \( x - m_T \) by (15), the optimal return size will be

\[
m_0 = x - \left[ -s - \frac{a}{2} + \frac{a(r + p)}{p + h} \right]. \tag{26'}
\]

Therefore, the cost after returning \( m_T \) is:

\[
f(x - m_T) = L(x - m_T) + r(s - m_T - x) + S(0 + m_T - x)
\]
\[
= S - r(s - x) - \frac{a(r + p)(r - h)}{2(h + p)} \tag{27'}
\]

By (27')-(15'), we have

\[
f(x - m_T) - f(x) = S - r(s - x) - \frac{a(r + p)(r - h)}{2(h + p)}
\]
\[
- \frac{1}{8a} a^2(h + p) - 4a(h - p)(s - x)
\]
\[
+ 4(h + p)(s - x^2) < 0 \tag{28'}
\]

because of \( x > q = s - \frac{s^2 + \sqrt{hp}}{\sqrt{m_T}} < \sqrt{\frac{h}{m_T}} \). Therefore, at \( x < t \), costs of ordering and optimal order size \( m_0 \) will be less than that of nothing-to-do.

Note that this proof procedure also finds that the optimal return quantity is \( m_0 \) and return up-to \( Q = x - m_0 \), see (26').

On the other hand, by (18') and (21'), we have

\[
f(Q) - f(q) = S + r \sqrt{\frac{2aS}{p + h}} \tag{28'}
\]

Now we summarize above conclusions as the following theorem.
Theorem 1. Given the initial inventory level \( x \), the optimal policy of a one-period problem under uniform distribution has the four-point form of \((t, T, Q, q)\), where

\[
T = s - \frac{a(p - c)}{p + h}, \quad Q = s - \frac{a(p + r)}{p + h},
\]

\[
t = T - \frac{2aK}{p + h} \quad q = Q + \frac{2aS}{p + h},
\]

and optimal policies are:

1. when \( x < t \), the optimal solution is to order, and the optimal ordering quantity is \( m_T \) where \( m_T = T - x \);
2. when \( x > q \), the optimal option is to return, and the optimal quantity to be returned is \( m_Q \) where \( m_Q = x - Q \);
3. otherwise, the optimal option is for the stock level not to be adjusted (nothing-to-do).

Theorem 1 also shows that the unit ordering cost has the same effect on \( T \) and \( t \) because \( \partial T / \partial c = \partial t / \partial c = -a/(p + h) \), but the order quantities \( m_T \) are not affected by the ordering cost. Also, \( t \) is affected by the ordering set-up cost, but \( T \) is not. Similar analysis can be conducted for the returns situation.

Fig. 1 shows how the four-point policy can be used to control optimal stock levels for the decision-making retailer. The data used to plot Fig. 1 are \( K = 30, S = 50, s = 100, a = 35, r = 0.15, c = 0.45, h = 0.25, p = 0.75 \), and the optimal points are \( t = 47, T = 93, Q = 114, \) and \( q = 173 \).

Remark 2.1

For the special cases, we have

1. When \( K = 0 \) and \( S > 0 \), the optimal policy is of the type \((T, Q, q)\), which means that: when \( x < t \), the optimal option is to order and the order quantity is \( T - x \); when \( x > q \), the optimal option is to return and the returned quantity is \( x - Q \); otherwise, the optimal option is to do nothing.
2. When \( K > 0 \) and \( S = 0 \), the optimal policy is of the type \((t, T, Q)\) which means that: when \( x < t \), the optimal option is to order and the order quantity is \( T - x \); when \( x > Q \), the optimal option is to return and the returned quantity is \( x - Q \); otherwise, the optimal option is to do nothing.
3. When \( K = S = 0 \), the optimal policy is of the type \((T, Q)\) which means that: when \( x < T \), the optimal option is to order and the order quantity is \( T - x \); when \( x > Q \), the optimal option is to return and the returned quantity is \( x - Q \); otherwise, the optimal option is to do nothing.

Remark 2.2

Ordering size

\[
m_T > T - t = \frac{2aK}{p + h} \quad \text{and returning size} \quad m_Q > Q - q = \frac{2aS}{p + h}.
\]

Proof. For \( m_T > T - t \), by Theorem 1, \( m_T = T - x \), optimal ordering point is \( x > t \), then the result follows. The second part can similarly be obtained. □

Remark 2.3

Although the order or returned quantity in the single period is relevant to the initial stock level \( x \), the minimum quantity ordered or returned can be determined from Theorem 1.

From Theorem 1, we can obtain the following Corollary 2.1

Corollary 2.1

\[
f(t) - f(T) = K + c\sqrt{\frac{2aK}{p + h}} \quad \text{and} \quad f(q) - f(Q) = S + r\sqrt{\frac{2aS}{h + p}}.
\]

Corollary 2.1 claims that if ordering occurs at \( t \), the optimal order quantity is \( \sqrt{\frac{2aK}{p + h}} \). That is, the cost saving equals the set-up cost plus ordering cost. If returns occur at \( q \) and the optimal quantity returned is \( \sqrt{\frac{2aS}{h + p}} \), the cost saving equals the returns set-up cost plus the returning cost.

2.3. The cost properties of optimal policy

Now let us analyze the convexity of function (4).
Definition 1. A function \( f(x) \) is \((K, S)\)-convex in \( x \) if:
\[
f(x') \leq \min \left\{ \frac{K}{2}f(x_1) + \frac{1 - \lambda}{2}f(x_2) : \lambda \in [0, 1], \right\}
\]
for all \( x_1 < x_2, K \geq 0, S \geq 0, x' = \lambda x_1 + (1 - \lambda)x_2, 0 \leq \lambda \leq 1 \).

Definition 2. A function \( f(x) \) is \((K, +)\)-convex in \( x \) if:
\[
f(x') \leq \frac{K}{2}f(x_1) + \frac{1 - \lambda}{2}f(x_2) : \lambda \in [0, 1],
\]
for all \( x_1 < x_2, K \geq 0, x' = \lambda x_1 + (1 - \lambda)x_2, 0 \leq \lambda \leq 1 \).

Definition 3. A function \( f(x) \) is \((S, -)\)-convex in \( x \) if:
\[
f(x') \leq \frac{S}{2}f(x_1) + \frac{1 - \lambda}{2}f(x_2) : \lambda \in [0, 1],
\]
for all \( x_1 < x_2, S \geq 0, x' = \lambda x_1 + (1 - \lambda)x_2, 0 \leq \lambda \leq 1 \).

Above definitions are in fact the strong versions for the quasi-
\( K \)-convex defined in Porteous (1971) and is equivalent to the \( K \)-convex
defined in Scarf et al. (1963). We just extend them into \( S \)-convex and
\((K, S)\)-convex. The equivalence between the \( K \)-convex and the
\((K, +)\)-convex has been demonstrated by the Porteous (1971).

Theorem 2. (a) Ordering function (1) is \((K, +)\)-convex, (b) returning
function (2) is \((S, -)\)-convex, and (c) the nothing-to-do function (3) is
\((K, S)\)-convex.

Proof. See Appendix A.

With the \((K, +)\)-convex property of ordering function, the single-
period inventory decision has an optimal decision. Similar conclusions
may be presented for the \((S, -)\)-convex, \((K, S)\)-convex.

3. Numerical analysis

In this section, we numerically analyze the impact of cost
parameters and the demand variability on the optimal control policy.
In Section 2, we use uniformly distributed demand to obtain
easily operational close-form control policy and conclude some
insights into management. However, the uniform distribution is kind
of limitation for general cases. Therefore, in this section, we as-
sume that the demand distribution is normally distributed given
that it is a frequently used demand distribution in the inventory
literature. The situation of adopting uniform distribution to derive
the close-form solutions and using normal distribution to conduct
the numerical analysis is often used in literature, see, for example,
Lau et al. (2008), Granot and Yin (2005), Yan et al. (2003), Song
(2000), etc. Under normally distributed demand, we show that
both the variation of cost parameters the demand variability im-
)pacts on the critical points. Cost parameters used in numerical
analysis are: \( K = 50, S = 30, c = 0.25, r = 0.15, b = 0.45, p = 1 \) (if these
parameters are fixed) and demand parameters \( \mu = 100 \) and \( \sigma = 10, 20 \)
and 30 (denoted by SD in figures).

3.1. The effect of unit ordering cost and demand variability on the
optimal policy

It is clear that the unit ordering cost \( c \) only affects the ordering
point \( t \) and order up-to point \( T \) but exerts no effect on the returns
point \( q \) and returning up-to point \( Q \) (see Fig. 2) because the unit
ordering cost is the main component in ordering stock. The higher
of the ordering cost, the lower of ordering point and ordering up-to
point will be. These conclusions are consistent with those results
described in Theorem 1 under uniform distribution assumption.
Moreover, the higher the unit ordering cost is, the lower of the
ordering point \( t \) will be. This is because a higher unit ordering cost
will force the retailer to keep lower inventory levels in order to
lower holding costs. Note that when \( c \) is very big (in this situation,
\( c > 0.63 \) when SD = 10 = \( \sigma \), the order point \( t \) will become negative:
this implies that the retailer should always order stock and should
order up-to \( T \). For the effect of demand variability (SD, here we use
the Standard Deviation to stand for the demand variability) on the
control points, the increase of the demand variability, i.e., SD
increasing from 10 to 20, 30, both returning point \( q \) and returning
up-to point \( Q \) increase and the ordering point \( t \) decreases. From the
Theorem 1, one can easily derive that \( \frac{\partial Q}{\partial \sigma} > 0, \frac{\partial q}{\partial \sigma} > 0 \) and
\( \frac{\partial T}{\partial \sigma} < 0 \), thus our numerical solutions with normal distribution
assumption in a reasonable parameter scope are consistent with
the analytical solutions under uniform distribution assumption.
From the viewpoint of practice, the increase of demand variability
will lead to the decrease of the ordering point. This observation can
be used to guide the management decision that the optimal inven-
tory control may have a low ordering point under high demand
variability. In other word, in order to avoid to be faced with a risk
caused due to the high demand variability, the optimal stock level
should be ordering at lower stock level. Also denote that increase
of the unit ordering cost with the enlargement of demand variability
increases the ordering up-to point decreasing amount. However,
the effect of demand variability on the ordering up-to point

![Fig. 2. The effect of unit ordering cost and demand variability on the t, T, Q, and q.](image-url)
has a complex trend with variable of unit ordering cost. If the unit ordering cost is lower, the increase of SD will increase the returning up-to point; but if the unit ordering cost is higher, the conclusion is reversed. This observation can explained as follows: when the unit ordering cost is high, both the ordering up-to point and the ordering point decrease with the increase of demand variability, because with these policy can lower the ordering cost. When the unit ordering cost is low, the ordering point is still decreasing with the increase of demand variability, but the ordering up-to point will increase. This may be because the unit ordering cost has less effect on the inventory holding cost than the unit holding cost. Therefore, it has a higher ordering up-to point.

For the effect of SD on the returning point and returning up-to point, similar analysis can be conducted.

3.2. The effect of unit returns cost and demand variability on the optimal policy

For the impact of the unit returns cost on the optimal control policy (see Fig. 3) we find that the higher the unit returns cost is, the higher the returns point will be. This conclusion is clear in that with a high unit returns cost the retailer will keep stock in order to reduce the returns loss. Of course, the unit returns cost will not affect the order point and ordering up-to point, but will slightly affect the returns up-to point. This is consistent with the uniformly distributed demand situation. For the demand variability effect on the optimal control points of \( t, Q, q \), we don’t discuss it in detail, because it can be analyzed with similar method used in Section 3.1. Here we only point out that ordering up-to point \( T \) is almost no affected by the demand variability, although there in fact is a slightly increase with of the increase of demand variability.

3.3. The effect of the holding cost and demand variability on the optimal policy

For the impact of the holding cost on the optimal control policy (see Fig. 4) we find that the holding cost affects all of optimal control points \( (t, T, Q, q) \). This is different with those of unit ordering cost and unit returning cost. However, the returns point has been significantly affected by the holding cost compared with other control points. The lower the holding cost is, the higher the returns point will be, because the stock level has a high returning point can still keep minimum inventory cost with a lower holding cost. These conclusions also are consistent with the results described in Theorem 1.
in which it assumes the uniform distribution demand. For the demand variability effect on the optimal control points of \( t, T, Q \) and \( q \), the basic effects are similar with that described in Section 3.1. We still do not discuss it in detail, because it can be analyzed with similar method used in Section 3.1. Here we point out that the effect of the SD on the returning up-to point decreases with the increase of the unit holding cost. And the demand variability effect on the ordering up-to point has a similar pattern with the situation of unit ordering cost described in Fig. 2. Therefore, similar analysis can be conducted for the SD effect on the ordering up-to point.

3.4. The effect the penalty cost and demand variability on the optimal policy

For the penalty cost (see Fig. 5) we find that the unit penalty cost affects on all optimal control point (\( t, T, Q \), and \( q \)). But the difference with the unit holding cost is that the ordering point is affected significantly by the unit penalty cost. Specifically, if the unit penalty cost is lower, the retailer’s ordering point and ordering up-to point will be lower. However, when the penalty cost is very low \((p < 0.60 \text{ in the situation of SD } = 10)\) the retailer will always order its stock in order to reduce any goodwill loss. These observations also are consistent with those concluded in Theorem 1 in which it assumes the uniform distribution demand. For the demand variability effect on the optimal control points of \( t, T, Q \) and \( q \), the basic effects are similar with that described in Section 3.1, we still don’t discuss it in detail, because it can be analyzed with similar method used in Section 3.1. Here we only point out that the effect of demand variability will decrease with the increase of the unit penalty cost. This may be because when the unit penalty cost is higher, the effect of unit penalty cost is more than that of demand variability. However, both the returning point and returning up-to point increase with the increase of demand variability when the unit penalty cost is increasing.

3.5. The effect of \( K \) and demand variability on the optimal Policy

For the effect of ordering set-up cost on the optimal control policy (see Fig. 6), we find that the ordering set-up cost only affects on the ordering point and has no effect on other optimal control points. The greater the ordering set-up cost is, the lower of the ordering point will be. This is reasonable because under higher ordering set-up cost, the retailer will have a high inventory cost. For the uncertain demand effect on the optimal control points, the increase of SD leads to the increase of both returning point and returning up-to point and decrease of both ordering point and ordering up-to point. However, the ordering up-to point has been affected less than other control points. This observation has an insight into management that when the ordering set-up cost changes the effect of demand variability on the ordering up-to point is little.

3.6. The effect of \( S \) and demand variability on optimal policy

Finally, considering the returning set-up cost effect on the optimal control points (see Fig. 7), we find that only the returning point \( q \) has been affected by the returning set-up cost: the increase of the returning set-up cost will increase the returning point. Because the higher returning set-up cost leads to increase the returning cost, the retailer may keep the inventory up till holding cost is very high. In that case, the retailer will lower the stock level to \( Q \). For the demand variability effect on the optimal control points, changing patterns of optimal control points are similar to the effect of ordering set-up cost. We omit to analyze it. Interested readers can refer to situation analyzed in ordering set-up cost (in Section 3.5).

4. Conclusions

In this paper, we study the problem of optimal order and returns policies with a stochastic demand, single-item, single periodic review inventory system. We obtain a four-point control policy \((t, T, Q, q)\), where \( t < T < Q < q \) for a single period, under the assumption of uniformly distributed demand. The four-point policy means that when the inventory level is below the order point \( t \), the retailer should order stock levels up-to \( T \); when the inventory level is greater than the return point \( q \), the retailer should return the stock level up-to \( Q \); otherwise, nothing should be done. Also, we find that the ordering cost functions have \((K, +)\)-convex property, returning cost function have \((S, -)\)-convex property and the nothing-to-do cost function has \((K, S)\)-convex property. A numerical study shows that the optimal control policy takes different parameter points under different cost parameters.
Considering the effect of demand uncertainty, both the order and returns quantities increased with an increase in standard deviation.

For the finite or infinite period inventory decision problem, even with a uniform distribution to model the problem with interval [0, 1], it needs complex calculation and so it will be delayed to further study. Another research interests may be use the empirical distribution to model the problem.

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Appendix A

A.1. Proof of Theorem 2

We first prove (c). Apparently, the fact that \( L(y) \) is a \((K, S)\)-convex function means function (3) is \((K, S)\)-convex. By the \((K, S)\)-convex definition,

\[
L(y) \leq \min \left\{ \begin{array}{ll}
(a) & \lambda L(y_1) + (1 - \lambda)[K + L(y_2)] \\
(b) & \lambda[S + L(y_1)] + (1 - \lambda)L(y_2)
\end{array} \right.
\]

for \( y_1 < y_2 \), \( K \geq 0 \), \( S \geq 0 \), \( y' = \lambda y_1 + (1 - \lambda)y_2 \), \( 0 \leq \lambda \leq 1 \), then \( L(y) \) is \((K, S)\)-convex.
Substituting \( y' = y_1x + (1 - \lambda)y_2 \) into \( L(y) \), we have:

\[
L(y') = \int_0^{y'/\alpha} \left( \frac{y - z}{a} \right)^2 dz + p \int_0^{\alpha} \left( \frac{z - y'}{a} \right)^2 dz
\]

\[
= \int_0^{y_1/(1 - \lambda\alpha)} \left( \frac{y_1 - \alpha z}{\alpha} \right)^2 dz + \int_0^{y_2/(1 - \lambda\alpha)} \left( \frac{y_2 - \alpha z}{\alpha} \right)^2 dz
\]

\[
+ p \int_0^{\alpha} \left( \frac{z - y_1}{\alpha} \right)^2 dz + p \int_0^{\alpha} \left( \frac{z - y_2}{\alpha} \right)^2 dz
\]

\[
= \frac{1}{8a} \left( a^2(h + p) - 4a(h - p)(s - (1 - \lambda)y_2 - \lambda y_1) + 4(h + p)(s - (1 - \lambda)y_2 - \lambda y_1)^2 \right)
\]

Part (a):

\[
\frac{\partial}{\partial y} \left( \frac{L(y')}{(1 - \lambda)|K + L(y_2)|} \right) = \frac{1}{2a} \left( 1 - \lambda \right) \left[ \int_0^{y_1/(1 - \lambda\alpha)} \frac{(y_1 - z)^2}{\alpha} dz + \int_0^{y_2/(1 - \lambda\alpha)} \frac{(y_2 - z)^2}{\alpha} dz \right]
\]

\[
+ (1 - \lambda) \left[ K + \frac{1}{2a} \left( z(h + p) \right) + \frac{1}{8a} \left( a^2(h + p) - 4a(h - p)(s - y_2) \right) \right]
\]

\[
+ \frac{1}{8a} \left( a^2(h + p) + 4a(h - p)(s - y_1) \right) + \frac{1}{2a} \left( 1 - \lambda \right) \left( a^2(h + p) - 4a(h - p)(s - y_2) \right) + 4(h + p)(s - y_2)^2 \right]
\]

Part (b):

\[
\frac{\partial}{\partial (y_1)} \left( \frac{L(y_1) + (1 - \lambda)L(y_2) - L(y')}{(1 - \lambda)|K + L(y_2)|} \right)
\]

\[
= \frac{1}{2a} \left( 1 - \lambda \right) \left[ \int_0^{y_1/(1 - \lambda\alpha)} \frac{(y_1 - z)^2}{\alpha} dz + \int_0^{y_2/(1 - \lambda\alpha)} \frac{(y_2 - z)^2}{\alpha} dz \right]
\]

\[
+ (1 - \lambda) \left( h + \frac{1}{2a} \left( z(h + p) \right) \right)
\]

\[
+ \frac{1}{8a} \left( a^2(h + p) - 4a(h - p)(s - y_2) \right) + 4(h + p)(s - y_2)^2 \right]
\]

\[
+ \frac{1}{8a} \left( a^2(h + p) + 4a(h - p)(s - y_1) \right) + \frac{1}{2a} \left( 1 - \lambda \right) \left( a^2(h + p) - 4a(h - p)(s - y_2) \right) + 4(h + p)(s - y_2)^2 \right]
\]

By (A.2)–(A.1), we have

\[
\frac{\partial}{\partial y} \left( \frac{L(y_1) + (1 - \lambda)L(y_2) - L(y')}{(1 - \lambda)|K + L(y_2)|} \right)
\]

\[
= \frac{1}{2a} \left( 1 - \lambda \right) \left[ 2aK + \lambda(h + p)(y_1 - y_2)^2 \right] \geq 0.
\]

Therefore,

\[
L(y') \leq L(y_1) + (1 - \lambda)L(y_2),
\]

that is,

\[
f_2(y') \leq f_2(y_1) + (1 - \lambda)f_2(y_2)
\]

By part (a) and part (b) we conclude that function (3) is \((K, S)\)-convex. Now for (a), by \((K, +)\)-convex definition (Definition 2),

\[
f_1(y') = K + c(y' - x) + L(y')
\]

\[
= K + c(y_1x + (1 - \lambda)y_2 - x) + L(y_1 + (1 - \lambda)y_2)
\]

\[
= K + c(y_1 + (1 - \lambda)y_2 - x) + L(y_1 + (1 - \lambda)y_2 - z)
\]

\[
+ h \int_{y_1/(1 - \lambda\alpha)}^{y_1/(1 - \lambda\alpha)} \left( \frac{y_1 - \alpha z}{\alpha} \right)^2 dz
\]

\[
+ p \int_0^\alpha \left( \frac{z - y_1}{\alpha} \right)^2 dz
\]

\[
= \frac{1}{8a} \left( a^2(h + p) + 4a(h - p)(s - y_2) \right) + 4(h + p)(s - y_2)^2
\]

\[
+ 4a(h - p)(s - y_2)^2 + 4a(2K + p - 2cx + 2cy_2 - py_2 + 2cy_1
\]

\[
- \lambda py_1 - 2c\lambda y_2 + \lambda py_2 + h(y_2 - s - (1 - \lambda)y_2 - \lambda y_2), \quad \text{(A.4)}
\]

\[
\frac{\partial}{\partial y} \left( \frac{L(y_1) + (1 - \lambda)L(y_2) - L(y')}{(1 - \lambda)|K + f_2(y_2)|} \right)
\]

\[
= \frac{1}{2a} \left( 1 - \lambda \right) \left[ 2aK + \lambda(h + p)(y_1 - y_2)^2 \right] \geq 0
\]

Therefore,

\[
f_1(y') \leq f_1(y_1) + (1 - \lambda)f_1(y_2).
\]

Last for (b), by \((S, -)\)-convex definition (Definition 3),

\[
f_2(y') = S + r(x - y') + L(y')
\]

\[
= S + r(y_1x + (1 - \lambda)y_2 - x) + L(y_1 + (1 - \lambda)y_2)
\]

\[
= S + \frac{1}{2a} \left( 1 - \lambda \right) \left[ 2aK + \lambda(h + p)(y_1 - y_2)^2 \right] \geq 0
\]

Therefore,

\[
f_1(y') \leq f_1(y_1) + (1 - \lambda)f_1(y_2).
\]

\[
\frac{\partial}{\partial y} \left( \frac{L(y_1) + (1 - \lambda)L(y_2) - L(y')}{(1 - \lambda)|S + f_2(y_2)|} \right)
\]

\[
= \frac{1}{2a} \left( 1 - \lambda \right) \left[ 2aK + \lambda(h + p)(y_1 - y_2)^2 \right] \geq 0
\]

\[
+ \frac{1}{8a} \left( a^2(h + p) + 4a(h - p)(s - y_2)^2 \right) + 4(h + p)(s - y_2)^2
\]

\[
+ 4a(h - p)(s - y_2)^2 + 4a(2S + 25y - hs + ps + 2rx - 2ry_1
\]

\[
+ hy + py_1 - 2c\lambda y_2\right), \quad \text{(A.5)}
\]
Here $S_1 > 0$ and $S_1$ is only distinguished from $S$ in the returning functions.

By (A.7)–(A.6), we have:

$$i[S_1 + f_2(y_1)] + (1 - i)f_2(y_2) - f_2(y')$$

$$= \frac{1}{2a}i[2aS_1 + (1 - i)(h + p)(y_1 - y_2)^2] \geq 0$$

Therefore,

$$f_2(y') \leq \frac{1}{2a}[S_1 + f_2(y_1)] + (1 - i)f_2(y_2).$$

References


