An $f$-chromatic spanning forest of edge-colored complete bipartite graphs

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Abstract

In 2001, Brualdi and Hollingsworth proved that an edge-colored balanced complete bipartite graph $K_{n,n}$ with a color set $\mathbb{C} = \{1, 2, 3, \ldots, 2n - 1\}$ has a heterochromatic spanning tree if the number of edges colored with colors in $R$ is more than $|R|^2/4$ for any non-empty subset $R \subseteq \mathbb{C}$, where a heterochromatic spanning tree is a spanning tree whose edges have distinct colors, namely, any color appears at most once. In 2010, Suzuki generalized heterochromatic graphs to $f$-chromatic graphs, where any color $c$ appears at most $f(c)$. Moreover, he presented a necessary and sufficient condition for graphs to have an $f$-chromatic spanning forest with exactly $w$ components. In this paper, using this necessary and sufficient condition, we generalize the Brualdi-Hollingsworth theorem above.

Keyword(s): $f$-chromatic, heterochromatic, rainbow, multicolored, totally multicolored, polychromatic, colorful, edge-coloring, spanning tree, spanning forest.

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1 Introduction

We consider finite undirected graphs without loops or multiple edges. For a graph $G$, we denote by $V(G)$ and $E(G)$ its vertex and edge sets, respectively. An edge-coloring of a graph $G$ is a mapping $\text{color} : E(G) \to \mathbb{C}$, where $\mathbb{C}$ is a set of colors. An edge-colored graph $(G, \mathbb{C}, \text{color})$ is a graph $G$ with an edge-coloring $\text{color}$ on a color set $\mathbb{C}$. We often abbreviate an edge-colored graph $(G, \mathbb{C}, \text{color})$ as $G$.

An edge-colored graph $G$ is said to be heterochromatic if no two edges of $G$ have the same color, that is, $\text{color}(e_i) \neq \text{color}(e_j)$ for any two distinct

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edges $e_i$ and $e_j$ of $G$. A heterochromatic graph is also said to be *rainbow*, *multicolored*, *totally multicolored*, *polychromatic*, or *colorful*. Heterochromatic subgraphs of edge-colored graphs have been studied in many papers. (See the survey by Kano and Li [4].)

Akbari & Alipour [1], and Suzuki [5] independently presented a necessary and sufficient condition for edge-colored graphs to have a heterochromatic spanning tree, and they proved some results by applying the condition. Here, we denote by $\omega(G)$ the number of components of a graph $G$. Given an edge-colored graph $G$ and a color set $R$, we define $E_R(G) = \{ e \in E(G) \mid \text{color}(e) \in R \}$. Similarly, for a color $c$, we define $E_c(G) = E_{\{c\}}(G)$. We denote the graph $(V(G), E(G) \setminus E_R(G))$ by $G - E_R(G)$.

**Theorem 1.1** (Akbari and Alipour, (2006) [1], Suzuki, (2006) [5]). An edge-colored graph $G$ has a heterochromatic spanning tree if and only if

$$\omega(G - E_R(G)) \leq |R| + 1$$

for any $R \subseteq C$.

Note that if $R = \emptyset$ then the condition is $\omega(G) \leq 1$. Thus, this condition includes a necessary and sufficient condition for graphs to have a spanning tree, namely, to be connected. Suzuki [5] proved the following theorem by applying Theorem 1.1.

**Theorem 1.2** (Suzuki, (2006) [5]). An edge-colored complete graph $K_n$ has a heterochromatic spanning tree if $|E_c(G)| \leq n/2$ for any color $c \in C$.

Jin and Li [3] generalized Theorem 1.1 to the following theorem, from which we can obtain Theorem 1.1 by taking $k = n - 1$.

**Theorem 1.3** (Jin and Li, (2006) [3]). An edge-colored connected graph $G$ of order $n$ has a spanning tree with at least $k$ ($1 \leq k \leq n - 1$) colors if and only if

$$\omega(G - E_R(G)) \leq n - k + |R|$$

for any $R \subseteq C$.

If an edge-colored connected graph $G$ of order $n$ has a spanning tree with at least $k$ colors, then $G$ has a heterochromatic spanning forest with $k$ edges, that is, $G$ has a heterochromatic spanning forest with exactly $n - k$ components. On the other hand, if an edge-colored connected graph $G$ of order $n$ has a heterochromatic spanning forest with exactly $n-k$ components, then we can construct a spanning tree with at least $k$ colors by adding some $n-k-1$ edges to the forest. Hence, we can rephrase Theorem 1.3 as the following.
Theorem 1.4 (3). An edge-colored connected graph $G$ of order $n$ has a heterochromatic spanning forest with exactly $n - k$ components ($1 \leq k \leq n - 1$) if and only if
\[ \omega(G - E_R(G)) \leq n - k + |R| \quad \text{for any } R \subseteq C. \]

Heterochromatic means that any color appears at most once. Suzuki [6] generalized once to a mapping $f$ from a given color set $C$ to the set of non-negative integers, and introduced the following definition as a generalization of heterochromatic graphs.

definition 1.5. Let $f$ be a mapping from a given color set $C$ to the set of non-negative integers. An edge-colored graph $(G, C, \text{color})$ is said to be $f$-chromatic if $|E_c(G)| \leq f(c)$ for any color $c \in C$.

Fig. 1 shows an example of an $f$-chromatic spanning tree of an edge-colored graph. Let $C = \{1, 2, 3, 4, 5, 6, 7\}$ be a given color set of 7 colors, and a mapping $f$ is given as follows: $f(1) = 3$, $f(2) = 1$, $f(3) = 3$, $f(4) = 0$, $f(5) = 0$, $f(6) = 1$, $f(7) = 2$. Then, the left edge-colored graph in Fig. 1 has the right graph as a subgraph. It is a spanning tree where each color $c$ appears at most $f(c)$ times. Thus, it is an $f$-chromatic spanning tree.

Fig. 1: An $f$-chromatic spanning tree of an edge-colored graph.

If $f(c) = 1$ for any color $c$, then all $f$-chromatic graphs are heterochromatic and also all heterochromatic graphs are $f$-chromatic. It is expected many previous studies and results for heterochromatic subgraphs will be generalized.

Let $C$ be a color set, and $f$ be a mapping from $C$ to the set of non-negative integers. Suzuki [6] presented the following necessary and sufficient condition for graphs to have an $f$-chromatic spanning forest with exactly $w$ components. This is a generalization of Theorem 1.1 and Theorem 1.4.

Theorem 1.6 (Suzuki, (2010) [6]). An edge-colored graph $(G, C, \text{color})$ of order at least $w$ has an $f$-chromatic spanning forest with exactly $w$ components if and only if
\[ \omega(G - E_R(G)) \leq w + \sum_{c \in R} f(c) \quad \text{for any } R \subseteq C. \]
By applying Theorem 1.6, he generalized Theorem 1.2 as follows.

**Theorem 1.7** (Suzuki, 2010 [6]). A $g$-chromatic graph $G$ of order $n$ with $|E(G)| > \binom{n}{2}$ has an $f$-chromatic spanning forest with exactly $w$ ($1 \leq w \leq n - 1$) components if $g(c) \leq \frac{|E(G)|}{n-w} f(c)$ for any color $c$.

In this paper, we will generalize the following theorem for edge-colored complete bipartite graphs.

**Theorem 1.8** (Brualdi and Hollingsworth, 2001 [2]). Let $G$ be an edge-colored balanced complete bipartite graph $K_{n,n}$ with a color set $\mathbb{C} = \{1, 2, 3, \ldots, 2n-1\}$. Let $e_c$ be the number of edges with a color $c$, namely, $e_c = |E_c(G)|$, and assume that $1 \leq e_1 \leq e_2 \leq \cdots \leq e_{2n-1}$.

If $\sum_{r=1}^r e_r > \frac{r^2}{4}$ for any color $r \in \mathbb{C}$, then $G$ has a heterochromatic spanning tree.

Fig. 2: An example of Theorem 1.8

Fig. 2 shows an example of Theorem 1.8. The sum of numbers of edges with $1, 2, \ldots, r$ is more than $r^2/4$ for any color $r$, thus, this graph has a heterochromatic spanning tree.

In the next sections, we show a generalization of this theorem and prove it by applying Theorem 1.6.

2 A generalization of Brualdi-Hollingsworth Theorem

Under the conditions of Theorem 1.8, for any non-empty subset $R \subseteq \mathbb{C}$, $|E_R(G)| \geq \sum_{i=1}^r e_i > |R|^2/4$. On the other hand, if $|E_R(G)| > |R|^2/4$ for any non-empty subset $R \subseteq \mathbb{C}$, then

$$\sum_{i=1}^r e_i = \sum_{i=1}^r |E_i(G)| = |E_Q(G)| > |Q|^2/4 = r^2/4,$$

for any color $r$ and color subset $Q = \{1, 2, 3, \ldots, r\} \subseteq \mathbb{C}$. Thus, $\sum_{i=1}^r e_i > r^2/4$ for any color $r \in \mathbb{C}$ if and only if $|E_R(G)| > |R|^2/4$ for any non-empty subset $R \subseteq \mathbb{C}$. Hence, Theorem 1.8 implies as follows.
Theorem 2.1 (Brualdi and Hollingsworth, 2001 [2]). Let $G$ be an edge-colored balanced complete bipartite graph $K_{n,n}$ with a color set $\mathbb{C} = \{1, 2, 3, \ldots, 2n - 1\}$. If $|E_R(G)| > |R|^2/4$ for any non-empty subset $R \subseteq \mathbb{C}$, then $G$ has a heterochromatic spanning tree.

In this paper, we generalize this to the following.

Theorem 2.2. Let $G$ be an edge-colored complete bipartite graph $K_{n,m}$ with a color set $\mathbb{C}$. Let $w$ be a positive integer with $1 \leq w \leq n + m$, and $f$ be a function from $\mathbb{C}$ to the set of non-negative integers such that $\sum_{c \in \mathbb{C}} f(c) \geq n + m - w$. If $|E_R(G)| > (n + m - w - \sum_{c \in \mathbb{C}\setminus R} f(c))^2/4$ for any non-empty subset $R \subseteq \mathbb{C}$, then $G$ has an $f$-chromatic spanning forest with $w$ components.

Theorem 2.1 is a special case of Theorem 2.2 with $m = n$, $w = 1$, $f(c) = 1$ for any color $c$, and $|\mathbb{C}| = 2n - 1$.

The number of edges of a spanning forest with $w$ components of $K_{n,m}$ is $n + m - w$. Thus, in Theorem 2.2, the condition $\sum_{c \in \mathbb{C}} f(c) \geq n + m - w$ is necessary for existence of an $f$-chromatic spanning forest with $w$ components.

In Theorem 2.2, the lower bound of $|E_R(G)|$ is sharp as follows: Let $R \subseteq \mathbb{C}$ be a color subset and $p = n + m - w - \sum_{c \in \mathbb{C}\setminus R} f(c)$. Let $H$ be a complete bipartite subgraph $K_{\frac{p}{2}, \frac{p}{2}}$ of $G$. Color the edges in $E(H)$ with colors in $R$, and the edges in $E(G) \setminus E(H)$ with colors in $\mathbb{C} \setminus R$. Then, $|E_R(G)| = p^2/4 = (n + m - w - \sum_{c \in \mathbb{C}\setminus R} f(c))^2/4$. (See Figure 3.)

Fig. 3: A graph $G$ and $R \subseteq \mathbb{C}$ with $|E_R(G)| = (n + m - w - \sum_{c \in \mathbb{C}\setminus R} f(c))^2/4$.

Recall that $n + m - w$ is the number of edges of a spanning forest with $w$ components of $G$, and $\sum_{c \in \mathbb{C}\setminus R} f(c)$ is the maximum number of edges with colors in $\mathbb{C} \setminus R$ of a desired forest. Thus, $p$ is the number of edges with colors in $R$ needed in a desired forest. However, any $p$ edges of $H$ include a cycle because $|V(H)| = p$. Hence, $G$ has no $f$-chromatic spanning forests with $w$ components, which implies the lower bound of $|E_R(G)|$ is sharp.
In the next section, we prove Theorem 2.2 by applying Theorem 1.6. In order to prove it, we need the following lemma.

**Lemma 2.3.** Let \( G \) be a bipartite graph of order \( N \) that consists of \( s \) components. Then \( |E(G)| \leq (N - (s - 1))^2 / 4 \).

**Proof.** Take a bipartite graph \( G^* \) of order \( N \) that consists of \( s \) components so that

1. \( |E(G^*)| \) is maximum, and
2. subject to (1), for the maximum component \( D_s \) of \( G^* \), \( |V(D_s)| \) is maximum.

By the maximality (1) of \( G^* \), each component of \( G^* \) is a complete bipartite graph. Let \( A_s \) and \( B_s \) be the partite sets of \( D_s \). We assume \( |A_s| \leq |B_s| \).

Suppose that some component \( D \) except \( D_s \) has at least two vertices. Let \( A \) and \( B \) be the partite sets of \( D \). We assume \( |A| \leq |B| \). If \( |A| > |B_s| \) then \( |A_s| \leq |B_s| < |A| \leq |B| \), which contradicts that \( D_s \) is a maximum component of \( G^* \). Thus, we have \( |A| \leq |B_s| \).

Let \( x \) be a vertex of \( B \), where \( \deg_G(x) = |A| \). Let \( D' = D - \{x\}, A' = A, B' = B - x, A'_s = A_s \cup \{x\}, B'_s = B_s \), and \( D'_s = (A'_s \cup B'_s, E(D_s) \cup \{xz \mid z \in B'_s\}) \). Let \( G'^s \) be the resultant graph. Then, we have

\[
|E(D')| + |E(D'_s)| = |E(D)| - \deg_G(x) + |E(D_s)| + |B'_s|
\]

\[
= |E(D)| + |E(D_s)| + |B_s| - |A|
\]

\[
\geq |E(D)| + |E(D_s)|,
\]

which implies \( |E(G'^s)| = |E(G^*)| \) by the condition (1). However, that contradicts the maximality (2) because \( |D'_s| \geq |D_s| + 1 \). Hence, every component except \( D_s \) has exactly one vertex, which implies that \( |V(D_s)| = N - (s - 1) \).

Suppose that \( |B_s| - |A_s| \geq 2 \). Let \( x \) be a vertex of \( B_s \), where \( \deg_G(x) = |A_s| \). Let \( D'_s = (V(D_s), E(D_s) - x) \cup \{xz \mid z \in B_s - x\} \). Then, \( D'_s \) is a complete bipartite graph, and we have

\[
|E(D'_s)| = |E(D_s)| - \deg_G(x) + |B_s - x|
\]

\[
= |E(D_s)| - |A_s| + |B_s| - 1
\]

\[
\geq |E(D_s)| + 1,
\]

which contradicts the maximality (1). Hence, \( |B_s| - |A_s| \leq 1 \).

Therefore,

\[
|E(G)| \leq |E(G^*)| = |E(D_s)| = |A_s||B_s|
\]

\[
= \left( (N - (s - 1) / 2 \right) \left( (N - (s - 1) / 2 \right)
\]

\[
\leq (N - (s - 1))^2 / 4.
\]

\[ \square \]
3 Proof of Theorem 2.2

Suppose that $G$ has no $f$-chromatic spanning forests with $w$ components. By Theorem 1.6, there exists a color set $R \subseteq C$ such that

$$\omega(G - E_R(G)) > w + \sum_{c \in R} f(c). \quad (1)$$

Let $s = \omega(G - E_R(G))$. Let $D_1, D_2, \ldots, D_s$ be the components of $G - E_R(G)$, and $q$ be the number of edges of $G$ between these distinct components. Note that, the colors of these $q$ edges are only in $R$.

If $R = C$ then

$$s = \omega(G - E_C(G)) > w + \sum_{c \in C} f(c) \geq w + n + m - w = n + m = |V(G)|,$$

by the assumption of Theorem 2.2. This contradicts that $s \leq |V(G)|$. Thus, we can assume $R \neq C$, namely, $C \setminus R \neq \emptyset$. Hence, by the assumption of Theorem 2.2

$$|E_{C \setminus R}(G)| > (n + m - w - \sum_{c \in C \setminus (C \setminus R)} f(c))^2/4 = (n + m - w - \sum_{c \in R} f(c))^2/4.$$

Therefore, we have

$$q \leq |E_R(G)| = |E(G)| - |E_{C \setminus R}(G)| < |E(G)| - (n + m - w - \sum_{c \in R} f(c))^2/4. \quad (2)$$

On the other hand,

$$q = |E(G)| - |E(D_1) \cup E(D_2) \cup \cdots \cup E(D_s)|.$$

By Lemma 2.3, $|E(D_1) \cup E(D_2) \cup \cdots \cup E(D_s)| \leq (n + m - (s - 1))^2/4$. Thus, since $s = \omega(G - E_R(G)) \geq w + 1 + \sum_{c \in R} f(c)$ by (1), we have

$$q = |E(G)| - |E(D_1) \cup E(D_2) \cup \cdots \cup E(D_s)|$$
$$\geq |E(G)| - (n + m - (s - 1))^2/4$$
$$\geq |E(G)| - (n + m - (w + 1 + \sum_{c \in R} f(c) - 1))^2/4$$
$$= |E(G)| - (n + m - w - \sum_{c \in R} f(c))^2/4,$$

which contradicts (2). Consequently, the graph $G$ has an $f$-chromatic spanning forest with $w$ components.
References


