A HIERARCHICAL DECOMPOSITION
OF CHOQUET INTEGRAL MODEL

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In this paper we give a necessary and sufficient condition for a Choquet integral model to be decomposable into an equivalent hierarchical Choquet integral model constructed by hierarchical combinations of some ordinary Choquet integral models. The condition is obtained by Inclusion-Exclusion Covering (IEC). Moreover we show some properties on the set of IECs.

Keywords: Hierarchical decomposition, IEC, Irreducible covering, Choquet integral

1. Introduction

A subjective evaluation model related to multi-attribute object using Choquet integral with respect to fuzzy measures have been applied in various fields, with good results. Suppose that \( X \) is the set of all attributes, \( f(x) \) is the partial evaluated value on each attribute \( x \in X \) of an object \( f \), and \( z \) is the overall evaluated value. Choquet integral model as shown in Figure 1 is the model whose input is \( f(x) \) on \( X \), and whose output is represented as \( z = (C) \int_X f d\mu \). It can be regarded as the generalization of a conventional linear model whose input is \( f(x) \) on \( X \), and whose output is represented as a weighted sum \( z = \sum_{x \in X} \omega_x f(x) \). Because if weight \( \omega_x \) is regarded as the measure \( m(\{x\}) \) of a singleton \( \{x\} \), output of linear model can be represented by Lebesgue integral \( z = (L) \int_X f dm \). Choquet integral model is a replacement of the measure \( m \) and Lebesgue integral of linear model by a fuzzy measure \( \mu \) and Choquet integral, respectively.

Clearly, the expressive power of Choquet integral model is higher than that of
linear model. However, Choquet Integral model is difficult to handle because $2^n - 1$ parameters (values of fuzzy measure) are generally required for an $n$-attributes object. Accordingly, as the number of attributes increases, not only does identification of the fuzzy measure become difficult, but the evaluation model obtained becomes quite complex and structure difficult to grasp. These are reasons why we propose here hierarchical Choquet integral model, constructed by hierarchical combinations of more than one ordinary Choquet integral models, as shown in Figure 2(a)(b).

Figure 2(a) shows the separated hierarchical Choquet integral model in which one attribute must belong to only one macro-attribute. Figure 2(b) shows the overlapping hierarchical Choquet integral model in which one attribute can belong to some macro-attributes.

In the remainder of this introduction, we shall indicate the contents of 3 sections that follow. Section 2 gives definitions and basic properties of fuzzy measure and Choquet integral. Section 3 gives theorems for an ordinary Choquet integral model to be decomposable into an equivalent hierarchical Choquet integral model. Section 4 discusses various properties and structures on the set of IECs given in section 3.

2. Fuzzy measure and Choquet integral

In this section we give the definition of (non-)monotonic fuzzy measure and the Choquet integral, and show basic properties of those.

Throughout the paper we assume that $(X, \mathcal{X})$ is a measurable space.
Definition 2.1 (monotonic fuzzy measure)
A **monotonic fuzzy measure** on \((X, \mathcal{X})\) is a real valued set function \(\lambda : \mathcal{X} \to \mathbb{R}^+\) satisfying the following two conditions:

(a) \(\lambda(\emptyset) = 0\),
(b) \(\lambda(A) \leq \lambda(B)\) whenever \(A \subset B\) and \(A, B \in \mathcal{X}\),

where \(\mathbb{R}^+ = [0, \infty)\), the set of non-negative real numbers.

Definition 2.2 (non-monotonic fuzzy measure)
A **non-monotonic fuzzy measure** on \((X, \mathcal{X})\) is a real valued set function \(\mu : \mathcal{X} \to \mathbb{R}\) satisfying \(\mu(\emptyset) = 0\).

Definition 2.3 (Choquet integral)
The **Choquet integral** of a measurable function \(f : X \to \mathbb{R}\) with respect to a non-monotonic fuzzy measure \(\mu\) is defined by

\[
(C) \int_A f d\mu \equiv \int_{-\infty}^{+\infty} \mu_f(r) dr
\]

whenever the integral in the right-hand side exists, where

\[
\mu_f(r) = \begin{cases} 
\mu(\{x|f(x) > r\} \cap A) & \text{if } r \geq 0, \\
\mu(\{x|f(x) > r\} \cap A) - \mu(A) & \text{if } r < 0.
\end{cases}
\]
A measurable function \( f \) is called integrable if the Choquet integral of \( f \) exists and its value is finite.

**Proposition 2.1** Suppose that \( \mu \) is a non-monotonic fuzzy measure on \( (X, \mathcal{X}) \), then there exist two monotonic fuzzy measures \( \mu^+ \) and \( \mu^- \) on \( (X, \mathcal{X}) \) such that

\[
\mu(A) = \mu^+(A) - \mu^-(A)
\]

for every \( A \in \mathcal{X} \). Furthermore

\[
(C) \int_A f d\mu = (C) \int_A f d\mu^+ - (C) \int_A f d\mu^-,
\]

if either \( (C) \int_A f d\mu^+ \) or \( (C) \int_A f d\mu^- \) is finite.

3. Hierarchical Decomposition

3.1. Null set

**Definition 3.1 (null set)**

A set \( N \in \mathcal{X} \) is called a **null set** (with respect to \( \mu \)) if

\[
\mu(A \cup N) = \mu(A)
\]

for every \( A \in \mathcal{X} \).

A **null set**, in the measure theory, is a set \( N \) of measure zero, (i.e. \( \mu(N) = 0 \).)

A **null set**, in the fuzzy measure theory, such as defined above, coincides with a **null set**, in measure theory, when \( \mu \) is additive. A set of fuzzy measure zero need not be a null set in the fuzzy measure theory. Clearly, an empty set is a null set in both theories.

**Proposition 3.1** If \( N \) is a null set, then

\[
(C) \int_X f d\mu = (C) \int_{N^c} f d\mu
\]

for every measurable function \( f \) on \( X \).

The above proposition means that a null set (attribute) is a set (attribute) which does not influence integral value. Hence by removing such a set (attribute) we can get a model which is simpler than, and equivalent to, the original one.
3.2. **Inclusion-Exclusion Covering (IEC)**

**Definition 3.2 (Inter-Additive Partition (IAP))**

A finite measurable partition $\{P_i\}_{i \in \{1, \ldots, n\}}$ of $X$ is called an Inter-Additive Partition of $X$ (with respect to $\mu$) if

$$\mu(A) = \sum_{i \in \{1, \ldots, n\}} \mu(A \cap P_i) \quad (6)$$

for every $A \in X$.

**Definition 3.3 (Inclusion-Exclusion Covering (IEC))**

A finite measurable covering $\{C_i\}_{i \in \{1, \ldots, n\}}$ of $X$ is called an Inclusion-Exclusion Covering of $X$ (with respect to $\mu$) if

$$\mu(A) = \sum_{I \subseteq \{1, \ldots, n\}, I \not= \emptyset} (-1)^{|I|+1} \mu(\bigcap_{i \in I} C_i \cap A) \quad (7)$$

for every $A \in X$.

Clearly if an IEC $\mathcal{C}$ is a partition of $X$, it is an IAP.

**Definition 3.4 (hierarchical Choquet integral model)**

Suppose that $\mathcal{C} = \{C_i\}_{i \in \{1, \ldots, n\}}$ is a measurable covering, then each subalgebra $\mathcal{S}_i$ of $\mathcal{X}$ is defined by

$$\mathcal{S}_i \equiv \{C_i \cap A \mid A \in \mathcal{X}\}.$$  

Let $\mathcal{M} = \{\mu_{C_i}(\cdot)\}_{i \in \{1, \ldots, n\}}$ be an n-ary class of non-monotonic fuzzy measures on $\mathcal{S} = \{\mathcal{S}_i\}_{i \in \{1, \ldots, n\}}$. Then the function $f_{\mathcal{M}}$ on $\mathcal{C}$ is defined by

$$f_{\mathcal{M}}(C_i) \equiv (C) \int_{C_i} f \, d\mu_{C_i} \quad (8)$$

for every measurable function $f$ on $X$.

The hierarchical Choquet integral model is defined by the model whose input is $f(x)$ on $X$, whose output $z$ is represented as $z = (C) \int_C f_{\mathcal{M}} \, d\nu$, where $\nu$ is a non-monotonic fuzzy measure on $2^C$.

In the following, we will give a necessary and sufficient condition for an ordinary Choquet integral model as shown in Figure 1 to be decomposable into an equivalent hierarchical Choquet integral model as shown in Figure 2. In general cases, the condition is to be quite complex, but in case the (non-)monotonic fuzzy measure $\nu$ is additive, it is to be simple. Furthermore, the condition in this case is not so different in essence from one in general cases. Hence, in the paper, we deal only with the case where (non-)monotonic fuzzy measures $\nu$ are additive.
Lemma 3.1 Let \( \{ C_i \}_{i \in \{1, \ldots, n\}} \) be a measurable covering of \( X \). Then \( \{ C_i \}_{i \in \{1, \ldots, n\}} \) is an IEC, if and only if there exists an \( n \)-ary class of non-monotonic fuzzy measures \( Z \equiv \{ \zeta_{C_i(\cdot)} \}_{i \in \{1, \ldots, n\}} \) on \( S \equiv \{ S_i \}_{i \in \{1, \ldots, n\}} \) such that

\[
\mu(A) = \sum_{i \in \{1, \ldots, n\}} \zeta_{C_i}(A \cap C_i) \quad (9)
\]

for every \( A \in \mathcal{X} \).

**proof**

Suppose that \( \mu \) is given by (9), then

\[
\begin{align*}
\sum_{I \subseteq \{1, \ldots, n\}} ( -1 )^{I \setminus 1} & \mu(\bigcap_{i \in I} C_i \cap A) \\
= & \sum_{I \subseteq \{1, \ldots, n\}} ( -1 )^{I \setminus 1} \sum_{j \in \{1, \ldots, n\}} \zeta_{C_j}(\bigcap_{i \in I} C_i \cap A \cap C_j) \\
= & \sum_{j \in \{1, \ldots, n\}} \sum_{I \subseteq \{1, \ldots, n\}} ( -1 )^{I \setminus 1} \zeta_{C_j}(\bigcap_{k \in I \cup \{j\}} C_k \cap A) \\
= & \sum_{j \in \{1, \ldots, n\}} \left\{ \sum_{M \subseteq \{1, \ldots, n\}} ( -1 )^{M \setminus 1} \zeta_{C_j}(\bigcap_{p \in M \cup \{j\}} C_p \cap A) \\
& + \sum_{N \subseteq \{1, \ldots, n\}} ( -1 )^{N \setminus 1} \zeta_{C_j}(\bigcap_{q \in N \cup \{j\}} A \cap C_q) \right\} \\
= & \sum_{j \in \{1, \ldots, n\}} \left\{ \zeta_{C_j}(C_j \cap A) + \sum_{N \subseteq \{1, \ldots, n\}} ( -1 )^{N \setminus 2} \zeta_{C_j}(\bigcap_{p \in N \cup \{j\}} C_p \cap A \cap C_j) \\
& + \sum_{N \subseteq \{1, \ldots, n\}} ( -1 )^{N \setminus 1} \zeta_{C_j}(\bigcap_{q \in N \cup \{j\}} A \cap C_q) \right\} \\
= & \sum_{j \in \{1, \ldots, n\}} \zeta_{C_j}(A \cap C_j) \\
= & \mu(A).
\]

Therefore \( \{ C_i \}_{i \in \{1, \ldots, n\}} \) is an IEC.
Now put
\[ \zeta_{C_i}(A \cap C_i) = \sum_{J \subseteq \{1, \ldots, n\}} (-1)^{|J|+1} \frac{\mu(\bigcap_{j \in J} C_j \cap A)}{|J|}, \]

If \( \{C_i\}_{i \in \{1, \ldots, n\}} \) is an IEC, then
\[
\sum_{i \in \{1, \ldots, n\}} \zeta_{C_i}(C_i \cap A) = \sum_{i \in \{1, \ldots, n\}} \sum_{J \subseteq \{1, \ldots, n\}} (-1)^{|J|+1} \frac{\mu(\bigcap_{j \in J} C_j \cap A)}{|J|}
\]
\[
= \sum_{I \subseteq \{1, \ldots, n\}} (-1)^{|I|+1} \mu(\bigcap_{j \in I} C_j \cap A)
\]
\[
= \mu(A). \quad \Box
\]

**Theorem 3.1** Let \( \mathcal{C} = \{C_i\}_{i \in \{1, \ldots, n\}} \) be a measurable covering of \( X \). Then there exist an n-ary class of non-monotonic fuzzy measures \( \mathcal{M} \equiv \{\mu_{C_i}(\cdot)\}_{i \in \{1, \ldots, n\}} \) on \( S \equiv \{S_i\}_{i \in \{1, \ldots, n\}} \) and an additive non-monotonic fuzzy measure \( \nu \) on \( 2^\mathcal{C} \) such that
\[
(C) \int_X f d\mu = (C) \int_\mathcal{C} f \mathcal{M} d\nu
\]
for every measurable function \( f \) on \( X \), if and only if \( \mathcal{C} \) is an IEC.

**proof**

Suppose that
\[
(C) \int_X f d\mu = (C) \int_\mathcal{C} f \mathcal{M} d\nu,
\]
then by considering the integration of \( f \equiv 1_A \), we obtain that \( \mu(A) = \sum_{i \in \{1, \ldots, n\}} \mu_{C_i}(A \cap C_i) \cdot \nu(\{C_i\}) \) for every \( A \in X \). It follows from lemma 3.1 that \( \mathcal{C} \) is an IEC.

Now put
\[
\mu_{C_i}(A \cap C_i) = \begin{cases} 
\sum_{J \subseteq \{1, \ldots, n\}} (-1)^{|J|+1} \frac{\mu(\bigcap_{j \in J} C_j \cap A)}{|J|} & \text{if } \mu(C_i) \neq 0, \\
0 & \text{if } \mu(C_i) = 0,
\end{cases}
\]
and
\[
\nu(\bigcup_{i \in I} \{C_i\}) = \sum_{i \in I} \mu(C_i)
\]
for every $I \subseteq \{1, \ldots, n\}$. If $C$ is an IEC, then

$$(C) \int_X f d\mu = \int_0^{+\infty} \mu(\{x | f(x) > r\}) dr + \int_{-\infty}^0 [\mu(\{x | f(x) > r\}) - \mu(X)] dr$$

$$= \int_0^{+\infty} \left[ \sum_{i \in \{1, \ldots, n\}} \mu_{C_i}(\{x | f(x) > r\} \cap C_i) \cdot \nu(\{C_i\}) \right] dr \quad (\text{by Lemma 3.1})$$

$$= \sum_{i \in \{1, \ldots, n\}} \nu(\{C_i\}) \cdot \left\{ \int_0^{+\infty} \mu_{C_i}(\{x | f(x) > r\} \cap C_i) dr + \int_{-\infty}^0 [\mu_{C_i}(\{x | f(x) > r\} \cap C_i) - \mu_{C_i}(X \cap C_i)] dr \right\}$$

$$= \sum_{i \in \{1, \ldots, n\}} \nu(\{C_i\}) \cdot f_M(C_i)$$

$$= (C) \int_C f_M d\nu. \quad \square$$

This theorem means that an overlapping hierarchical Choquet integral model as shown in Figure 2(b) can be hierarchically constructed by an IEC from an ordinary Choquet integral model as shown in Figure 1.

As this is the model constructed by linear combinations of some ordinary Choquet integral models, it can be regarded as the model lying between linear model and ordinary Choquet integral model. This is also the reason why we propose hierarchical Choquet integral model.

**Corollary 3.1** Let $P = \{P_i\}_{i \in \{1, \ldots, n\}}$ be a measurable partition of $X$. Then there exists an $n$-ary class of non-monotonic fuzzy measures $M \equiv \{\mu_{P_i}(\cdot)\}_{i \in \{1, \ldots, n\}}$ on $S \equiv \{S_i\}_{i \in \{1, \ldots, n\}}$ and an additive non-monotonic fuzzy measure $\nu$ on $2^P$ such that

$$(C) \int_X f d\mu = (C) \int_P f_M d\nu$$

for every measurable function $f$ on $X$, if and only if $P$ is an IAP.

This corollary means that a separated hierarchical Choquet integral model as shown in Figure 2(a) can be hierarchically constructed by an IAP from an ordinary Choquet integral model as shown in Figure 1.
By applying Proposition 3.1 and 3.1, an ordinary Choquet integral model is decomposed into an equivalent hierarchical Choquet integral model as shown in Figure 3,

where $X = \{x_1, \ldots, x_7\}$ is the set of all attributes, 
$\{\{x_1, x_2\}, \{x_2, x_3, x_4\}, \{x_4, x_5\}, \{x_6, x_7\}\}$ is an IEC, and $\{x_7\}$ is a null set.

4. The Set of IECs

From section 3, we can see that IEC is useful for structural analysis of Choquet integral model. However, if there are more than one CIE, how do we grasp the structure?

In this section we show some properties on the set of all IECs.

4.1. Structures on the set of IECs

Proposition 4.1 Let $\mathcal{C} = \{C_i\}_{i \in \{1, \ldots, n\}}$ be a measurable covering of $X$. If $A$ is a subset of some $C_i \in \mathcal{C}$, then the following holds:

$$
\mu(A) = \sum_{J \subseteq \{1, \ldots, n\}} (-1)^{|J|+1} \mu(\bigcap_{i \in J} C_i \cap A).
$$

Fig. 3. Hierarchical decomposition by an IEC and a null set
proof

Suppose that \( A \subseteq C_i \), then

\[
\sum_{J \subseteq \{1, \ldots, n\}, J \neq \emptyset} (-1)^{|J|+1} \mu(\bigcap_{j \in J} C_j \cap A) = \mu(A)
\]

Moreover if \( X \in C \), then \( C - D \) is an IEC for every \( D \subset C \) such as \( X \notin D \). This shows that there are generally more than one IEC. Hence giving some sort of structure on the set of all IECs is useful.

**Definition 4.1** Let \( C \) and \( D \) be two finite measurable coverings of \( X \). If for every \( D \in D \), there exists \( C \in C \) such that \( D \subseteq C \), we write \( D \subseteq C \). As two important operations on the set of finite measurable coverings, we introduce the inter-refinement of \( C \) and \( D \) and the union of \( C \) and \( D \) denoted by \( C \cap D \) and \( C \cup D \), respectively, and defined by

\[
C \cap D \equiv \{ C \cap D \mid C \in C, D \in D \}
\]

\[
C \cup D \equiv C \cup D
\]

The relation \( \subseteq \) is clearly reflexive and transitive. For two measurable coverings \( C \) and \( D \), \( C \cap D \) and \( C \cup D \) are clearly measurable coverings, and clearly \( C \cap D \subseteq C, D \subseteq C \cup D \).

**Proposition 4.2** Let \( C \) and \( D \) be two finite measurable coverings of \( X \). If \( C \) is an IEC and \( C \subseteq D \), then \( D \) is also an IEC.

**proof**

Assume that \( C = \{C_i\}_{i \in N} \) and \( D = \{D_j\}_{j \in M} \). It suffices to prove that

\[
\sum_{J \subseteq M} (-1)^{|J|} \mu\left( \bigcap_{j \in J} D_j \cap A \right) = 0
\]

for every \( A \in \mathcal{X} \). For each \( I \subset N \), we write \( M_I \equiv \{ j \in M \mid \bigcap_{i \in I} C_i \subset D_j \} \). Note that, by the hypothesis \( C \subseteq D \), \( M_I \neq \emptyset \) whenever \( I \neq \emptyset \). It follows the fact that \( C \)
is an IEC that

\[
\sum_{J \subseteq M} (-1)^{|J|} \mu \left( \bigcap_{j \in J} D_j \cap A \right)
\]

\[
= \sum_{J \subseteq M} (-1)^{|J|} \sum_{I \subseteq N} (-1)^{|I|+1} \mu \left( \bigcap_{i \in I} C_i \cap \bigcap_{j \in J} D_j \cap A \right)
\]

\[
= \sum_{I \subseteq N} \sum_{J_1 \subseteq M_1, J_2 \subseteq M \setminus M_1} (-1)^{|J_1|+|J_2|+|I|+1} \mu \left( \bigcap_{i \in I} C_i \cap \bigcap_{j \in J_2} D_j \cap A \right) \sum_{J_1 \subseteq M_1} (-1)^{|J_1|}
\]

\[
= 0.
\]

The last equality holds since \( \sum_{J \subseteq M_1} (-1)^{|J_1|} = 0 \) whenever \( M_1 \neq \emptyset \). \( \square \)

**Corollary 4.1** Let \( \mathcal{C} \) be an IEC. If there exist \( C, D \in \mathcal{C} \) such that \( C \subseteq D \) then \( \mathcal{C} \setminus \{C\} \) is also an IEC.

Hence all we have to do is focus on the irreducible IEC which is defined as follows.

**Definition 4.2 (irreducible covering)** An irreducible covering \( \mathcal{C} \) of \( X \) is a finite covering satisfying the following:

\[
C = D, \quad \text{whenever} \quad C \subseteq D, \ C, D \in \mathcal{C}.
\]  \( (12) \)

**Lemma 4.1** Suppose that \( I \) and \( J \) are non-empty finite sets, and \( \mathcal{I}_{I,J} \equiv \{ I \in 2^{I \times J} | \bigcup_{(i,j) \in I} \{i\} = I, \ \bigcup_{(i,j) \in I} \{j\} = J \} \), then

\[
\sum_{\mathcal{I} \subseteq \mathcal{I}_{I,J} | \mathcal{I} \neq \emptyset} (-1)^{|\mathcal{I}|+1} = (-1)^{|I|+|J|}.
\]  \( (13) \)

**proof**

Suppose that \( \mathcal{I}_{I,J}^k = \{ I \in \mathcal{I}_{I,J} | |I| = k \} \). Then

\[
\sum_{\mathcal{I} \subseteq \mathcal{I}_{I,J} | \mathcal{I} \neq \emptyset} (-1)^{|\mathcal{I}|+1}
\]

\[
= \sum_{k=1}^{|I||J|} \sum_{\mathcal{I} \subseteq \mathcal{I}_{I,J}^k | |I| = k} (-1)^{k+1}
\]
Let $C\cap D$ be two IECs. Then the following two statements hold:

(a) $C\cap D$ is an IEC.

(b) $C\cup D$ is an IEC.

In other words the set of all IECs is closed under the formation of inter-refinement and union.

**Proof**

Proof of (a):

Assume that $C = \{C_i\}_{i \in N}$ and $D = \{D_j\}_{j \in M}$. Suppose that $S \equiv \{(i, j) \mid i \in N, j \in M\}$ and $I_{ij} \equiv \{I \in 2^{I \times J} \mid I \cap \{i\} \neq \emptyset, I \cap \{j\} = \emptyset\}$. Then

$$\sum_{T \subseteq S} (-1)^{|T|+1} \mu(\bigcap_{(i,j) \in T} (C_i \cap D_j) \cap A)$$

$$= \sum_{I \times J \subseteq N \times M} \sum_{I \cap \{i\} \neq \emptyset, J \neq \emptyset} (-1)^{|I|+1} \mu(\bigcap_{(i,j) \in I \times J} (C_i \cap D_j) \cap A)$$

$$= \sum_{I \times J \subseteq N \times M} (-1)^{|I|+|J|+2} \mu(\bigcap_{(i,j) \in I \times J} (C_i \cap D_j) \cap A)$$

(by lemma 4.1)
\[ \sum_{I \subseteq N, I \neq \emptyset} \sum_{J \subseteq M, J \neq \emptyset} (-1)^{|I|+|J|+2} \mu(\bigcap_{i \in I} C_i \cap \bigcap_{j \in J} C_j \cap A) \]
\[ = \sum_{I \subseteq N, I \neq \emptyset} (-1)^{|I|+1} \sum_{J \subseteq M, J \neq \emptyset} (-1)^{|J|+1} \mu \left( \bigcap_{j \in J} C_j \cap \left( \bigcap_{i \in I} C_i \cap A \right) \right) \]
\[ = \sum_{I \subseteq N, I \neq \emptyset} (-1)^{|I|+1} \mu \left( \bigcap_{i \in I} C_i \cap A \right) \]
\[ = \mu(A). \]

Proof of (b):
It is clear from proposition 4.2. \(\square\)

**Lemma 4.2** For each finite measurable covering \(\mathcal{C}\) of \(X\), there exists a finite unique measurable irreducible covering \(\mathcal{D}\) such that \(\mathcal{C} \subseteq \mathcal{D}\) and \(\mathcal{D} \subseteq \mathcal{C}\). This \(\mathcal{D}\) is called the irreducible covering of \(\mathcal{C}\)

**proof**

We define that \(\mathcal{D} = \mathcal{C} - \{C_i \in \mathcal{C} | \exists C_j \in \mathcal{C} such that C_i \subsetneq C_j\}\). It is clear that \(\mathcal{D}\) is a finite measurable irreducible covering such that \(\mathcal{C} \subseteq \mathcal{D}\) and \(\mathcal{D} \subseteq \mathcal{C}\).

If \(\mathcal{E}\) is a finite measurable irreducible covering such that \(\mathcal{C} \subseteq \mathcal{E}\) and \(\mathcal{E} \subseteq \mathcal{C}\), we obtain from transitivity of \(\subseteq\) that \(\mathcal{E} \subseteq \mathcal{D}\) and \(\mathcal{D} \subseteq \mathcal{E}\). For each \(D_j \in \mathcal{D}\), there exists \(E_k \in \mathcal{E}\) such that \(D_j \subseteq E_k\) because of the fact that \(\mathcal{D} \subseteq \mathcal{E}\). For this \(E_k \in \mathcal{E}\), there exists \(D_l \in \mathcal{D}\) such that \(E_k \subseteq D_l\) because of the fact that \(\mathcal{E} \subseteq \mathcal{D}\). Hence it follows from the fact that \(\mathcal{D}\) is an irreducible covering that \(D_j = E_k\). So that \(\mathcal{D} \subseteq \mathcal{E}\). It is similarly proved that \(\mathcal{D} \subseteq \mathcal{E}\). \(\square\)

**Corollary 4.2** For each IEC \(\mathcal{C}\) of \(X\), there exists a unique irreducible IEC \(\mathcal{D}\) such that \(\mathcal{C} \subseteq \mathcal{D}\) and \(\mathcal{D} \subseteq \mathcal{C}\).

**Theorem 4.2** Let \(X\) be a finite set. Then there exists a unique irreducible IEC \(\mathcal{C}\) such that \(\mathcal{C} \subseteq \mathcal{D}\) for every IEC \(\mathcal{D}\).

**proof**

Suppose that \(\mathcal{C}_0\) is the inter-refinement of all IECs. It is clear that \(\mathcal{C}_0 \subseteq \mathcal{D}\) for every IEC \(\mathcal{D}\). It follows from proposition 4.1(a) that \(\mathcal{C}_0\) is an IEC. Now let \(\mathcal{C}\) be the irreducible covering of \(\mathcal{C}_0\). Then it follows from lemma 4.2 that this \(\mathcal{C}\) satisfies the statement of this proposition. \(\square\)

By the argument above, Let \(X = \{x_1, \ldots, x_4\}\). Suppose that \(\mathcal{C} = \{\{x_1, x_4\}, \{x_2, x_3\}\}\), and \(\mathcal{D} = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}\}\) are IEC's. Then \(\mathcal{C} \cap \mathcal{D} = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}\}\). Suppose that \(\mathcal{E} = \mathcal{C} \cap_\mathcal{D} \mathcal{D}, \mathcal{E} = \{\{x_1\}, \{x_2, x_3\}, \{x_4\}\}\). Hence if there are two hierarchical Choquet integral models as shown in Figure 4(a)(b), then a hierarchical Choquet integral model as shown in Figure 4(c) is obtained immediately.
Now we give a relation $\sim$, on the set of all finite measurable coverings of $X$, defined as follows:

**Definition 4.3** Let $C$ and $D$ be two finite measurable coverings of $X$. If $C \subseteq D$ and $C \supseteq D$, we write $C \sim D$.

The relation $\sim$ is clearly an equivalence relation. Moreover the following proposition is obtained immediately.

**Proposition 4.3** Let $\mathcal{MC}$ be the set of all finite measurable coverings of $X$ and $\mathcal{IRC}$ the set of all irreducible coverings of $X$. Then $\mathcal{MC}/\sim$ is isomorphic to $\mathcal{IRC}$.

**Corollary 4.3** Let $\mathcal{IEC}$ be the set of all IECs of $X$ and $\mathcal{IRIEC}$ the set of all irreducible IECs of $X$. Then $\mathcal{IEC}/\sim$ is isomorphic to $\mathcal{IRIEC}$.

### 4.2. Structures on the set of irreducible IECs

**Definition 4.4** Let $C$ and $D$ be two irreducible coverings. The irreducible interrefinement (irreducible union) of $C$ and $D$ is denoted by $C \cap_{ir} D$ ($C \cup_{ir} D$), and defined as the irreducible covering of $C \cap D$ ($C \cup D$).
The following proposition is easily verified.

**Theorem 4.3** Let $\mathcal{IRC}$ be the set of all measurable irreducible coverings. Then the ordered set $(\mathcal{IRC}, \subseteq)$ is a distributive lattice. Furthermore, $\sup\{C, D\} = C \sqcup_D D$ and $\inf\{C, D\} = C \sqcap_D D$ for every $C, D \in \mathcal{IRC}$.

**Corollary 4.4** Let $\mathcal{IRIEC}$ be the set of all measurable irreducible IEC. Then the ordered set $(\mathcal{IRIEC}, \subseteq)$ is a distributive lattice.

5. **Conclusions**

We gave the necessary and sufficient conditions for an ordinary Choquet integral model to be decomposable into an equivalent hierarchical Choquet integral model. The hierarchical decomposition based on this condition is expected to be useful as a structural analysis method for not only subjective evaluation model, but also various phenomena represented by Choquet integral model. Furthermore, it is also expected to be useful as the model lying between linear model and ordinary Choquet integral model.

However, since identified fuzzy measures generally yield errors when dealing with real problems, the hierarchical decomposition theorem in the paper cannot be expected to be satisfied exactly. Recently, hierarchical decomposition based on the results are applied.

We will show the result, in case the (non-)monotonic fuzzy measures $\nu$ of macrot-attributes are not additive (i.e., the general case), on another occasion.

**References**