$k$-Additivity and $C$-decomposability
of bi-capacities and its integral

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Abstract

$k$-Additivity is a convenient way to have less complex (bi-)capacities. This paper gives a new characterization of $k$-additivity, introduced by Grabisch and Labreuche, of bi-capacities and contrasts between the existing characterization and the new one, that differs from the one of Saminger and Mesiar. Besides, in the same way for capacities, a concept of $C$-decomposability, distinct from the proposal of Saminger and Mesiar, but closely-linked to $k$-additivity, is introduced for bi-capacities. Moreover, the concept of $C$-decomposability applies to the Choquet integral with respect to bi-capacities.

Key words: Bi-capacity, the bipolar Möbius transform, $k$-additivity, $C$-decomposability

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1 Introduction

Let $N = \{1, \ldots, n\}$ be a set of criteria describing the preference of a decision maker over a set $U$ of objects in a multicriteria decision problem, or a set of players in a cooperative game, etc. Capacities [5], set functions which vanish at the empty set and are monotone with respect to set inclusion, have become a fundamental tool in decision making, especially decision under uncertainty and under multiple criteria. Characteristic functions [22], in cooperative game theory, are set functions vanishing at the empty set. These concepts have been defined on the set of all subsets of $N$ (i.e., on $2^N := \{A \mid A \subseteq N\}$). Recently, there have been some attempts to define more general concepts both in game theory and multicriteria decision theory. In game theory, Aubin [1] has proposed the concept of generalized coalition as a function $c : N \to [-1, 1]$ which associates each player $i \in N$ with his/her level of participation $c(i) \in [-1, 1]$. Later, Bilbao et al. [3] have proposed what they call bi-cooperative games, which generalize the idea of ternary voting games proposed by Felsenthal and Machover [7] on the set of all signed coalitions $T : N \to \{-1, 0, 1\}$ (recall that the power set $2^N$ is isomorphic to the set of all functions $c : N \to \{0, 1\}$). Then, we have a family $\{T : N \to \{-1, 0, 1\}\}$, which is isomorphic to $3^N := \{(S_1, S_2) \mid S_1, S_2 \subseteq N, S_1 \cap S_2 = \emptyset\}$, as a natural extension of $\{c : N \to \{0, 1\}\}$. In multicriteria decision theory, Grabisch and Labreuche [11–13] have proposed what they call bi-capacities on the lattice $(3^N, \sqsubseteq)$, which vanish at $(\emptyset, \emptyset)$ and are monotone with respect to the order $\sqsubseteq$ on $3^N$, where $(S_1, S_2) \sqsubseteq (T_1, T_2)$ if and only if $S_1 \subseteq T_1$ and $S_2 \supseteq T_2$ (recall that the power set $2^N$ can be regarded as the Boolean lattice $(2^N, \subseteq)$ equipped with the order $\subseteq$, and ordinary capacities as monotone functions
with respect to $\subseteq$ on the lattice $(2^N, \subseteq)$. However, bi-capacities generally require $3^n - 1$ parameters. In order to reduce the number of these parameters, (i.e., to reduce complexity of bi-capacities), Grabisch and Labreuche [11,12] have proposed the notion of $k$-additivity of bi-capacities by using the Möbius transform via the order $\subseteq$ on $2^N$ (note that Saminger and Mesiar [25] also have generalized and characterized the notion of $k$-additivity by taking a different approach). On the other hand, as another approach to the problem, Saminger and Mesiar [25] have proposed the notion of decomposable bi-capacities (see, Note 2 in section 3.2). In this paper, we give a new characterization of $k$-additivity of bi-capacities and propose another concept to reduce complexity of bi-capacities, $C$-decomposability, which is closely-linked to $k$-additivity but not to decomposable bi-capacities. Moreover, the concept of $C$-decomposability applies to the Choquet integral with respect to bi-capacities which have been proposed axiomatically by Grabisch and Labreuche [16] (note that Greco, Matarazzo, and Slowinski [14,15] also have introduced and discussed the Choquet integral with respect to bipolar measures).

In the remainder of this introduction, we shall indicate the contents of four sections as follows: Section 2 introduces the concept of a bi-capacity and its equivalent representations, the Möbius transform and the bipolar Möbius transform. Section 3 discusses the notions to avoid/reduce complexity of bi-capacities, $k$-additivity and $C$-decomposability of bi-capacities. Section 4 discusses $C$-decomposition of the Choquet integral with respect to bi-capacities. Section 5 concludes and presents ideas for future works.

Throughout the paper, we denote $\mathbb{R}$ as the set of all real numbers and $\mathbb{R}^+$ the set of all non-negative real numbers; for a real number $x \in \mathbb{R}$, we denote $x^+ := \max\{x, 0\}$ and $x^- := (-x)^+$; for a set of non-negative real numbers $\{x_i\} \subseteq \mathbb{R}^+$,
we adopt the convention that $\bigwedge_{i} x_i = \infty$ and $0 \cdot \bigwedge_{i} x_i = 0$. In order to avoid heavy notation, we will often omit braces for singletons, e.g., by writing $v(i)$, $U \setminus i$ instead of $v\{i\}$, $U \setminus \{i\}$, respectively. Similarly, we will often write $ij$, $ijk$ instead of $\{i, j\}$, $\{i, j, k\}$, respectively.

2 Bi-capacities

We denote $\mathcal{P}(N) := 2^N = \{S \subseteq N\}$ and $\mathcal{Q}(N) := 3^N = \{(A_1, A_2) \in \mathcal{P}(N) \times \mathcal{P}(N) \mid A_1 \cap A_2 = \emptyset\}$. When equipped with the following order $\sqsubseteq$: for arbitrary $(A_1, A_2), (B_1, B_2) \in \mathcal{Q}(N)$

$$(A_1, A_2) \sqsubseteq (B_1, B_2) \text{ iff } A_1 \subseteq B_1 \text{ and } A_2 \supseteq B_2,$$

$(\mathcal{Q}(N), \sqsubseteq)$ becomes a lattice. Sup and inf are denoted by $\sqcup, \sqcap$, and are given by

$$(A_1, A_2) \sqcup (B_1, B_2) = (A_1 \cup B_1, A_2 \cap B_2),$$

$$(A_1, A_2) \sqcap (B_1, B_2) = (A_1 \cap B_1, A_2 \cup B_2),$$

respectively; the top and bottom are respectively $(N, \emptyset)$ and $(\emptyset, N)$.

**Definition 1 (irreducible elements (e.g., [6]))** Let $(L, \leq, \vee, \wedge, \top, \bot)$ be a lattice, where $\vee, \wedge, \top, \bot$ denote sup, inf, the top and bottom element, respectively. An element $x \in L$ is $\vee$-irreducible if $x \neq \bot$ and $x = a \vee b$ implies $x = a$ or $x = b$ for any $a, b \in L$.

**Proposition 2** [11] The $\sqcup$-irreducible elements of $\mathcal{Q}(N)$ are $(\emptyset, i^c)$ and $(i, i^c)$ for all $i \in N$. Moreover, for any $(A_1, A_2) \in \mathcal{Q}(N)$,

$$\bigcup_{i \in A_1} (i, i^c) \sqcup \bigcup_{j \in N \setminus (A_1 \cup A_2)} (\emptyset, j^c). \tag{1}$$

Eq. (1) is called the minimal decomposition of $(A_1, A_2)$.  

Note 1 [11] ⊔-irreducible elements permit to define layers in \( \mathcal{Q}(N) \) as follows: \((\emptyset, N)\) is the bottom layer (layer 0: the black square on Fig.1), the set of all ⊔-irreducible elements forms layer 1, black circles on Fig.1, and layer \( k \), for \( k = 2, \ldots n \), consists of all elements whose minimal decomposition contains exactly \( k \) ⊔-irreducible elements. In other words, layer \( k \) consists of all elements \((A_1, A_2) \in \mathcal{Q}(N)\) such that \( |A_2^c| = k \), for \( k = 2, \ldots, n \). On the other hand, let us consider the Boolean lattice \((\mathcal{P}(N), \subseteq, \cup, \cap, N, \emptyset)\). Then, the empty set is the bottom layer; all singletons are ⊔-irreducible elements, (i.e., in layer 1); the set of all \( A \in \mathcal{P}(N)\) whose cardinality is \( k \) forms layer \( k \), for \( k = 2, \ldots, n \).

Fig. 1. The lattice \( \mathcal{Q}(123) \): the element in layer 0 is indicated by the black square and elements in layer 1 black circles.

Definition 3 (capacity (e.g., [4])) A function \( \mu : \mathcal{P}(N) \rightarrow \mathbb{R} \) is a capacity on \( \mathcal{P}(N) \) if it satisfies:

(i) \( \mu(\emptyset) = 0 \),

(ii) \( A, B \in \mathcal{P}(N), \ A \subseteq B \) implies \( \mu(A) \leq \mu(B) \).
Definition 4 (Möbius transform of a capacity (e.g., [23])) To any capacity \( \mu \) on \( \mathcal{P}(N) \), another function \( m_\mu : \mathcal{P}(N) \to \mathbb{R} \) can be associated by

\[
\mu(A) = \sum_{B \subseteq A} m_\mu(B) \quad \forall A \in \mathcal{P}(N).
\]

In combinatorics [23], the function \( m_\mu \) is called the Möbius transform of \( \mu \) and is given by

\[
m_\mu(A) := \sum_{B \subseteq A} (-1)^{|A|\backslash |B|} \mu(B) \quad \text{for} \ A \in \mathcal{P}(N).
\]

Definition 5 (bi-capacity [11]) A function \( v : \mathcal{Q}(N) \to \mathbb{R} \) is a bi-capacity on \( \mathcal{Q}(N) \) if it satisfies:

(i) \( v(\emptyset, \emptyset) = 0 \),

(ii) \( (A_1, A_2), (B_1, B_2) \in \mathcal{Q}(N), \ (A_1, A_2) \sqsubseteq (B_1, B_2) \) implies \( v(A_1, A_2) \leq v(B_1, B_2) \).

In addition, \( v \) is called additive if it satisfies

\[
v(A_1, A_2) = \sum_{i \in A_1} v(i, \emptyset) + \sum_{j \in A_2} v(\emptyset, j) \quad \forall (A_1, A_2) \in \mathcal{Q}(N);
\]

A function \( v : \mathcal{Q}(N) \to \mathbb{R} \) is said to be a bipolar set function on \( \mathcal{Q}(N) \) if it satisfies the condition (i) above.

Definition 6 (Möbius transform of a bi-capacity [11]) To any bi-capacity \( v \) on \( \mathcal{Q}(N) \), another function \( m : \mathcal{Q}(N) \to \mathbb{R} \) can be associated by

\[
v(A_1, A_2) = \sum_{(B_1, B_2) \sqsubseteq (A_1, A_2)} m(B_1, B_2) \quad \forall (A_1, A_2) \in \mathcal{Q}(N).
\]

This correspondence proves to be one-to-one, since conversely

\[
m(A_1, A_2) := \sum_{B_1 \subseteq A_1, A_2 \subseteq B_2 \subseteq A_1^c} (-1)^{|A_1\backslash B_1| + |B_2\backslash A_2|} v(B_1, B_2) \quad \text{for} \ (A_1, A_2) \in \mathcal{Q}(N).
\]
The validity of Eq. (2) is proved by Grabisch and Labreuche [11] who call $m$ the Möbius transform of a bi-capacity $v$. Indeed, it is the Möbius transform in the sense of Rota [23] for a function on the partially ordered set $(3^N, \sqsubseteq)$.

Another equivalent representation of bi-capacities has been proposed, to facilitate the expression of the Choquet integral with respect to bi-capacities (see Proposition 26), by Fujimoto [8] as follows:

**Definition 7 (bipolar Möbius transform of a bi-capacity [8])** To any bi-capacity $v$ on $Q(N)$, another function $b : Q(N) \rightarrow \mathbb{R}$ can be associated by

$$b(A_1, A_2) := \sum_{B_1 \subseteq A_1 \atop B_2 \subseteq A_2} (-1)^{|A_1 \setminus B_1| + |A_2 \setminus B_2|} v(B_1, B_2)$$

$$= \sum_{(\emptyset, A_2) \subseteq (B_1, B_2) \subseteq (A_1, \emptyset)} (-1)^{|A_1 \setminus B_1| + |A_2 \setminus B_2|} v(B_1, B_2)$$

for $(A_1, A_2) \in Q(N)$. Then, the function defined by Eq. (3) is called the bipolar Möbius transform of $v$. Conversely, it holds that

$$v(A_1, A_2) = \sum_{B_1 \subseteq A_1 \atop B_2 \subseteq A_2} b(B_1, B_2) \quad \forall (A_1, A_2) \in Q(N).$$

The next proposition shows a correspondence between the Möbius transform and the bipolar Möbius transform.

**Proposition 8** [9] Let $v$ be a bi-capacity on $Q(N)$, $m$ the Möbius transform of $v$, and $b$ the bipolar Möbius transform of $v$. Then, it holds, for any $(A_1, A_2) \in Q(N)$, that

$$m(A_1, A_2) = (-1)^{|A_1 \setminus A_2|} \sum_{A_1 \subseteq C_2 \subseteq A_1 \setminus A_2} b(A_1, C_2)$$

and

$$b(A_1, A_2) = (-1)^{|A_2|} \sum_{C_2 \subseteq A_1 \setminus A_2} m(A_1, C_2).$$
Proposition 9 [9] Let \( v \) be a bi-capacity on \( Q(N) \), \( m \) the Möbius transform of \( v \), and \( b \) the bipolar Möbius transform of \( v \). If \( v \) is additive, then for any \((A_1, A_2) \in Q(N)\),

\[
v(A_1, A_2) = \sum_{i \in A_1} m(i, i^c) + \sum_{j \in A_2} m(\emptyset, j^c)
\]

\[
= \sum_{i \in A_1} b(i, \emptyset) + \sum_{j \in A_2} b(\emptyset, j).
\]

3 \( k \)-Additivity and \( C \)-decomposability of bi-capacities

3.1 \( k \)-Additivity

Bi-capacities on \( Q(N) \) generally require \( 3^n - 1 \) parameters. In order to reduce the number of these parameters, (i.e., to reduce complexity of bi-capacities), Grabisch and Labreuche [11] have proposed the notion of \( k \)-additivity of bi-capacities, similar to the way they have done for capacities (see also [10]).

Definition 10 (\( k \)-additivity of capacities (e.g., [10])) Given a positive integer \( k < n \), a capacity \( \mu \) is said to be \( k \)-additive if \( m_\mu(A) = 0 \) whenever \( |A| > k \). In addition, a \( k \)-additive capacity \( \mu \) is said to be purely \( k \)-additive if there exists some \( A \in \mathcal{P}(N) \) such that \( |A| = k \) and \( m_\mu(A) \neq 0 \).

Definition 11 (\( k \)-additivity of bi-capacities (e.g., [11])) Given a positive integer \( k < n \), a bi-capacity is said to be \( k \)-additive if its Möbius transform vanishes for all elements in layer \( l \) whenever \( l > k \). Equivalently, a bi-capacity \( v \) is \( k \)-additive if and only if \( m(A_1, A_2) = 0 \) whenever \( |A_2^c| > k \). In addition, a \( k \)-additive bi-capacity \( v \) is said to be purely \( k \)-additive if there exists some \((A_1, A_2) \in Q(N)\) such that \( |A_2^c| = k \) and \( m(A_1, A_2) \neq 0 \).
1-Additivity coincides with ordinary additivity. If \( v \) is \( k \)-additive, then \( v \) is \( k' \)-additive for any \( n > k' \geq k \).

**Lemma 12** Let \( v \) be a bi-capacity on \( \mathcal{Q}(N) \), \( m \) the Möbius transform of \( v \), \( b \) the bipolar Möbius transform of \( v \), and \( S \) a subset of \( N \). Then,

\[
m(A_1, A_2) = 0 \text{ whenever } A_2^c \supseteq S \iff b(B_1, B_2) = 0 \text{ whenever } B_1 \cup B_2 \supseteq S.
\]

**Proof.** In order to prove the necessity, assume that \( b(B_1, B_2) = 0 \) whenever \( B_1 \cup B_2 \supseteq S \). For any \( (A_1, A_2) \in \mathcal{Q}(N) \) such that \( A_2^c \supseteq S \), from Proposition 8,

\[
m(A_1, A_2) = (-1)^{|A_1^c \setminus A_2|} \sum_{A_1^c \setminus A_2 \subseteq C_2 \subseteq A_1^c} b(A_1, C_2) = 0
\]

since \( A_1^c \setminus A_2 \subseteq C_2 \) implies \( A_1 \cup C_2 \supseteq A_1 \cup (A_1^c \setminus A_2) = A_2^c \supseteq S \).

For showing the sufficiency, we assume that \( m(A_1, A_2) = 0 \) whenever \( A_2^c \supseteq S \). For any \( (B_1, B_2) \in \mathcal{Q}(N) \) such that \( B_1 \cup B_2 \supseteq S \), from Proposition 8,

\[
b(B_1, B_2) = (-1)^{|B_2|} \sum_{C_2 \subseteq B_1^c \setminus B_2} m(B_1, C_2) = 0
\]

since \( C_2 \subseteq B_1^c \setminus B_2 \) implies \( C_2^c \supseteq (B_1^c \setminus B_2)^c = B_1 \cup B_2 \supseteq S \). \( \square \)

**Proposition 13** Given a positive integer \( k < n \), a bi-capacity is \( k \)-additive if and only if \( b(A_1, A_2) = 0 \) whenever \( |A_1 \cup A_2| > k \).

**Proof.** We denote by \( \binom{N}{k+1} \) the set of all subsets of \( N \) whose cardinalities are \( k + 1 \), i.e., \( \binom{N}{k+1} := \{ S \subseteq N \mid |S| = k + 1 \} \). Then, \( v \) is \( k \)-additive

\[
\iff m(B_1, B_2) = 0 \text{ whenever } |B_2^c| > k
\]

\[
\iff m(B_1, B_2) = 0 \text{ whenever } B_2^c \supseteq S \text{ for some } S \in \binom{N}{k+1}
\]
\[ b(A_1, A_2) = 0 \text{ whenever } A_1 \cup A_2 \supseteq S \text{ for some } S \in \binom{N}{k+1} \]

\[ b(A_1, A_2) = 0 \text{ whenever } |A_1 \cup A_2| > k; \]

the third equivalence follows from Lemma 12. \[ \square \]

Now we provide an intuitive illustration of the above two representations (Definition 11 and Proposition 13) of \( k \)-additivity of bi-capacities as follows.

Let us introduce the following two coordinate-like systems for the graphic as displayed in Fig. 1.

**Coordinate-like system I**: It has the origin \((\emptyset, 123)\) and three coordinate axes:

- Axis 1) through \((\emptyset, 123)\) and \((1, 23)\),
- Axis 2) through \((\emptyset, 123)\) and \((2, 13)\),
- Axis 3) through \((\emptyset, 123)\) and \((3, 12)\).

**Coordinate-like system II**: It has the origin \((\emptyset, \emptyset)\) and three coordinate axes:

- Axis 1) through \((\emptyset, 1)\) and \((1, \emptyset)\),
- Axis 2) through \((\emptyset, 2)\) and \((2, \emptyset)\),
- Axis 3) through \((\emptyset, 3)\) and \((3, \emptyset)\).

Then, if \( v \) is 1-additive, \( m(A_1, A_2) = 0 \) (resp. \( b(A_1, A_2) = 0 \)) for all points \((A_1, A_2) \in Q(123)\) which are not on any axis in coordinate-like system I (resp. II); if \( v \) is 2-additive, \( m(A_1, A_2) = 0 \) (resp. \( b(A_1, A_2) = 0 \)) for all points \((A_1, A_2) \in Q(123)\) which are not on any plane spanned by two among three axes in coordinate-like system I (resp. II).
3.2 \( C \)-decomposability

Decomposition of bi-capacities into subdomains is another approach to reduce complexity of defining its values. In this subsection, the notion of \( C \)-decomposability of bi-capacities is proposed and discussed.

**Definition 14** (e.g., [24]) A family \( C \subseteq \mathcal{P}(N) \) is called a *covering* of \( N \) if \( \bigcup C = N \). A family \( A \subseteq \mathcal{P}(N) \) is called an *antichain* if

\[
A \subseteq A', \{A, A'\} \subseteq A \Rightarrow A = A'.
\]

A family \( \mathcal{H} \subseteq \mathcal{P}(N) \) is called *hereditary* (or a *downset* in the poset \((\mathcal{P}(N), \subseteq)\)) if

\[
H' \subseteq H \in \mathcal{H} \Rightarrow H' \in \mathcal{H}.
\]

For a family \( C \subseteq \mathcal{P}(N) \), we denote by \( \mathcal{A}(C) \) the *antichain consisting of all the maximal elements of \( C \) with respect to set inclusion \( \subseteq \), and by \( \mathcal{H}(C) \) the *hereditary family generated by \( C \), i.e.,

\[
\mathcal{A}(C) := C \setminus \bigcup_{C \in C} \{A \in \mathcal{P}(N) \mid A \subsetneq C\}
\]

and

\[
\mathcal{H}(C) := \bigcup_{C \in C} \{H \in \mathcal{P}(N) \mid H \subseteq C\}.
\]

Note that \( \mathcal{A}(C) \subseteq C \subseteq \mathcal{H}(C) \) and \( \mathcal{H}(\mathcal{A}(C)) = \mathcal{H}(C) = \mathcal{H}(\mathcal{H}(C)) \).

**Definition 15** Let \( C \) be a family of subsets of \( N \) and \( v \) a bi-capacity on \( \mathcal{Q}(N) \).

A family \( \{v_C\}_{C \in C} \) is called a *\( C \)-decomposition* of \( v \) if each \( v_C \) is a bipolar set function on \( \mathcal{Q}(C) \) and the following holds:

\[
v(A_1, A_2) = \sum_{C \in C} v_C(A_1 \cap C, A_2 \cap C) \quad \forall (A_1, A_2) \in \mathcal{Q}(N). \quad (8)
\]

Then, \( v \) is said to be *\( C \)-decomposable*.

Table 1 shows the values of a \( C \)-decomposable bi-capacity \( v \), where \( N = \{1, 2, 3\} \) and \( C = \{\{1, 2\}, \{2, 3\}\} \). The values of a \( C \)-decomposition \( \{v_{12}, v_{23}\} \)
of $v$ are displayed in Table 2.

**Table 1.** $\{\{1,2\},\{2,3\}\}$-decomposable bi-capacity $v$.

<table>
<thead>
<tr>
<th>$v(A_1,A_2)$</th>
<th>$\emptyset$</th>
<th>${1}$</th>
<th>${2}$</th>
<th>${3}$</th>
<th>${1,2}$</th>
<th>${2,3}$</th>
<th>${1,3}$</th>
<th>$N$</th>
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<tbody>
<tr>
<td>$\emptyset$</td>
<td>0</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
<td>-6</td>
<td>-6</td>
<td>-4</td>
<td>-10</td>
</tr>
<tr>
<td>${1}$</td>
<td>2</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-</td>
<td>-</td>
<td>-4</td>
<td>-</td>
</tr>
<tr>
<td>${2}$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>-4</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>${3}$</td>
<td>2</td>
<td>6</td>
<td>3</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>${1,2}$</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>${2,3}$</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>${1,3}$</td>
<td>10</td>
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**Table 2.** $\{\{1,2\},\{2,3\}\}$-decomposition of $v$ defined in Table 1.

<table>
<thead>
<tr>
<th>$v_{12}(A_1,A_2)$</th>
<th>$\emptyset$</th>
<th>${1}$</th>
<th>${2}$</th>
<th>${1,2}$</th>
<th>$\emptyset$</th>
<th>${2}$</th>
<th>${3}$</th>
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<td>-1</td>
<td>-5</td>
<td>0</td>
<td>-1</td>
<td>-2</td>
<td>-5</td>
</tr>
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<td>${1}$</td>
<td>2</td>
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<td>1</td>
<td>-</td>
<td>1</td>
<td>-2</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>${2}$</td>
<td>1</td>
<td>-2</td>
<td>-</td>
<td>-</td>
<td>2</td>
<td>1</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>${1,2}$</td>
<td>5</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>5</td>
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</table>

<table>
<thead>
<tr>
<th>$v_{23}(A_1,A_2)$</th>
<th>$\emptyset$</th>
<th>${2}$</th>
<th>${3}$</th>
<th>${2,3}$</th>
<th>$\emptyset$</th>
<th>${2}$</th>
<th>${3}$</th>
<th>${2,3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>0</td>
<td>-1</td>
<td>-2</td>
<td>-5</td>
<td>0</td>
<td>-1</td>
<td>-2</td>
<td>-5</td>
</tr>
<tr>
<td>${2}$</td>
<td>1</td>
<td>-</td>
<td>-2</td>
<td>-</td>
<td>1</td>
<td>-2</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>${3}$</td>
<td>2</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>2</td>
<td>1</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>${2,3}$</td>
<td>5</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>5</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

**Note 2** We recall another approach to decomposition ($U$-decomposition) of bi-capacities, introduced by Saminger and Mesiar [25] and show the differences with respect to our approach.

**Definition 16 (U-decomposable bi-capacites) [25]** For an arbitrary interval $I \supseteq [-1,1]$, consider some uninorm $U : I^2 \to I$ with natural element 0, i.e., $U$ is an associative and commutative mapping with $U(x,0) = U(0,x) = x$. A (normalized) bi-capacity $v : Q(N) \to [-1,1]$ is called decomposable ($U$-decomposable) if for all $(A_1, A_2), (B_1, B_2) \in Q(N)$,

$$U(v(A_1, A_2), v(B_1, B_2)) = U(v((A_1, A_2) \sqcup (B_1, B_2)), v((A_1, A_2) \sqcap (B_1, B_2))).$$
Therefore, for any \((A_1, A_2) \in \mathcal{Q}(N)\),

\[
v(A_1, A_2) = U(U_{i \in A_1} v(\{i\}, \emptyset), U_{j \in A_2} v(\emptyset, \{j\}). \tag{9}
\]

Eq. (9) means that the value of a bi-capacity \(v(A_1, A_2)\) can be decomposed into the values \(v(\{i\}, \emptyset)\) and \(v(\emptyset, \{j\})\) for \(i \in A_1, j \in A_2\). That is, decomposability in the sense of Saminger and Mesiar [25] means that a \((U)\)-decomposable bi-capacity can be constructed, through a uninorm \(U\), by fixing the values of \(v(\{i\}, \emptyset)\) and \(v(\emptyset, \{j\})\) for all \(i, j \in N\). On the other hand, decomposability in our sense means that a \((C)\)-decomposable bi-capacity can be constructed by a sum of bi-capacities on subdomains \(\{\mathcal{Q}(C)\}_{C \in \mathcal{C}}\).

**Proposition 17** Let \(v\) be a bi-capacity on \(\mathcal{Q}(N)\), \(m\) the Möbius transform of \(v\), \(b\) the bipolar Möbius transform of \(v\), and \(\mathcal{C}\) a family of subsets of \(N\). Then, the following three conditions are equivalent to each other.

(i) \(v\) has a \(\mathcal{C}\)-decomposition.

(ii) \(m(A_1, A_2) = 0\) if \(A_2^c \notin \mathcal{H}(\mathcal{C})\).

(iii) \(b(A_1, A_2) = 0\) if \((A_1 \cup A_2) \notin \mathcal{H}(\mathcal{C})\).

**Proof.** \((i) \Rightarrow (iii)\). Let \(v\) be a bi-capacity on \(\mathcal{Q}(N)\) which has a \(\mathcal{C}\)-decomposition \(\{v_C\}_{C \in \mathcal{C}}\), i.e., each \(v_C\) is a bipolar set function on \(\mathcal{Q}(C)\) and the following holds:

\[
v(A_1, A_2) = \sum_{C \in \mathcal{C}} v_C(A_1 \cap C, A_2 \cap C) \quad \forall (A_1, A_2) \in \mathcal{Q}(N).
\]
For any \((S_1, S_2) \in \mathcal{Q}(N)\) such that \(S_1 \cup S_2 \not\in \mathcal{H}(C)\),

\[
b(S_1, S_2) = \sum_{\substack{T_1 \subseteq S_1 \\ T_2 \subseteq S_2}} (-1)^{|S_1 \setminus T_1|+|S_2 \setminus T_2|} v(T_1, T_2)
\]

\[
= (-1)^{|S_1 \cup S_2|} \sum_{\substack{T_1 \subseteq S_1 \\ T_2 \subseteq S_2}} (-1)^{|T_1 \cup T_2|} v(T_1, T_2)
\]

\[
= (-1)^{|S_1 \cup S_2|} \sum_{C \in \mathcal{C}} \sum_{\substack{T_1 \subseteq S_1 \\ T_2 \subseteq S_2}} (-1)^{|T_1 \cup T_2|} \sum_{C \subseteq C} v_C(T_1 \cap C, T_2 \cap C)
\]

\[
= (-1)^{|S_1 \cup S_2|} \sum_{C \in \mathcal{C}} \sum_{\substack{T_1 \subseteq S_1 \\ T_2 \subseteq S_2 \subseteq C \cap (S_1 \cup S_2) \cap C}} \sum_{C \subseteq C} (-1)^{|U_1 \cup U_2|+|V_1 \cup V_2|} v_C(U_1, U_2)
\]

\[
= (-1)^{|S_1 \cup S_2|} \sum_{C \in \mathcal{C}} \sum_{\substack{T_1 \subseteq S_1 \\ T_2 \subseteq S_2 \subseteq C \cap (S_1 \cup S_2) \cap C}} (-1)^{|U_1 \cup U_2|} v_C(U_1, U_2) \sum_{V \subseteq (S_1 \cup S_2) \cap C} (-1)^{|V|}
\]

\[
= 0;
\]

the last equality follows from the assumption \(S_1 \cup S_2 \not\in \mathcal{H}(C)\), i.e., \((S_1 \cup S_2) \setminus C \neq \emptyset\) for all \(C \in \mathcal{C}\).

(iii) \(\Rightarrow\) (i). Put, for \((A_1, A_2) \in \mathcal{Q}(N),\)

\[
\mathcal{C}(A_1, A_2) := \{C \in \mathcal{C} \mid C \supseteq (A_1 \cup A_2)\}.
\]

For each \(C \in \mathcal{C}\), we define \(b_C\) and \(v_C\) on \(\mathcal{Q}(C)\) as follows:

\[
b_C(S_1, S_2) := \frac{1}{|\mathcal{C}(S_1, S_2)|} b(S_1, S_2) \quad \text{for} \ (S_1, S_2) \in \mathcal{Q}(C),
\]

\[
v_C(S_1, S_2) := \sum_{\substack{T_1 \subseteq S_1 \\ T_2 \subseteq S_2}} b_C(T_1, T_2) \quad \text{for} \ (S_1, S_2) \in \mathcal{Q}(C).
\]
Then, for any \((S_1, S_2) \in \mathcal{Q}(N)\),

\[
v(S_1, S_2) = \sum_{T_1 \subseteq S_1, T_2 \subseteq S_2} b(T_1, T_2)
\]

\[
= \sum_{T_1 \subseteq S_1, T_2 \subseteq S_2} b(T_1, T_2)
\]

\[
= \sum_{C \in \mathcal{C}} \sum_{T_1 \subseteq S_1 \cap C, T_2 \subseteq S_2 \cap C} \frac{1}{\mathcal{C}(T_1, T_2)} b(T_1, T_2)
\]

\[
= \sum_{C \in \mathcal{C}} \sum_{T_1 \subseteq S_1 \cap C, T_2 \subseteq S_2 \cap C} b_C(T_1, T_2)
\]

\[
= \sum_{C \in \mathcal{C}} v_C(S_1 \cap C, S_2 \cap C);
\]

the second equality follows from the assumption, \(b(T_1, T_2) = 0\) if \((T_1 \cup T_2) \not\in \mathcal{H}(C)\), i.e., \(b(T_1, T_2) = 0\) if \((T_1 \cup T_2) \not\in C\) for all \(C \in \mathcal{C}\).

(ii) ⇔ (iii).

\[m(A_1, A_2) = 0\] if \(A_2^c \not\in \mathcal{H}(C)\)

\[\iff m(A_1, A_2) = 0\] if \(A_2^c \supseteq H\) for some \(H \not\in \mathcal{H}(C)\)

\[\iff b(B_1, B_2) = 0\] if \((B_1 \cup B_2) \supseteq H\) for some \(H \not\in \mathcal{H}(C)\)

\[\iff b(B_1, B_2) = 0\] if \((B_1 \cup B_2) \not\in \mathcal{H}(C)\);

the second equivalence follows from Lemma 12. \(\square\)

The following corollary is immediately obtained from Proposition 17 and the fact that \(\mathcal{H}(\mathcal{C}) = \mathcal{H}(\mathcal{A}(\mathcal{C})) = \mathcal{H}(\mathcal{H}(\mathcal{C}))\).

**Corollary 18** Let \(\mathcal{C}\) be a family of subsets of \(N\) and \(v\) a bi-capacity on \(\mathcal{Q}(N)\).

Then, the following three conditions are equivalent to each other.

(i) \(v\) has a \(\mathcal{C}\)-decomposition.

(ii) \(v\) has an \(\mathcal{A}(\mathcal{C})\)-decomposition.

(iii) \(v\) has an \(\mathcal{H}(\mathcal{C})\)-decomposition.
The following proposition shows that $\mathcal{C}$-decomposability and $k$-additivity are closely-linked to each other.

**Proposition 19** Let $v$ be a bi-capacity on $\mathcal{Q}(N)$. Then, for any positive integer $k$, the following three conditions are equivalent to each other.

(i) $v$ is $k$-additive.

(ii) $v$ has an $\binom{N}{k}$-decomposition, where $\binom{N}{k} = \{S \subseteq N \mid |S| = k\}$.

(iii) $v$ has a $\mathcal{C}$-decomposition for some $\mathcal{C} \subseteq \mathcal{P}(N)$ such that $k \geq \max_{C \in \mathcal{C}} |C|$.

**Proof.** (i) $\Rightarrow$ (ii). Assume that $v$ is $k$-additive, that is, $m(A_1, A_2) = 0$ whenever $|A_2^c| > k$. Then, $m(A_1, A_2) = 0$ whenever $A_2^c \notin \mathcal{H}(\binom{N}{k}) = \{A \subseteq N \mid |A| \leq k\}$. Therefore, from (ii) $\Rightarrow$ (i) in Proposition 17, $v$ has an $\binom{N}{k}$-decomposition.

(ii) $\Rightarrow$ (iii). It is trivial, indeed, $\binom{N}{k}$ is a covering of $N$, i.e., $\binom{N}{k} \subseteq \mathcal{P}(N)$, such that $k \geq \max_{C \in \binom{N}{k}} |C|$.

(iii) $\Rightarrow$ (i). Assume that $v$ has a $\mathcal{C}$-decomposition for some $\mathcal{C} \subseteq \mathcal{P}(N)$ such that $k \geq \max_{C \in \mathcal{C}} |C|$. Then, $B \notin \mathcal{H}(\mathcal{C})$ if $|B| > k$. Therefore, from (i) $\Rightarrow$ (ii) in Proposition 17, $m(A_1, A_2) = 0$ whenever $|A_2^c| > k$. Hence, $v$ is $k$-additive. □

**Proposition 20** Let $\mathcal{C}$ and $\mathcal{D}$ be two families of subsets of $N$ and $v$ a bi-capacity on $\mathcal{Q}(N)$. If $\mathcal{C} \subseteq \mathcal{H}(\mathcal{D})$ (i.e., for any $C \in \mathcal{C}$ there exists $D \in \mathcal{D}$ such that $C \subseteq D$) and if $v$ has a $\mathcal{C}$-decomposition, then $v$ has a $\mathcal{D}$-decomposition.

**Proof.** Assume that $\mathcal{C} \subseteq \mathcal{H}(\mathcal{D})$, then $\mathcal{H}(\mathcal{C}) \subseteq \mathcal{H}(\mathcal{H}(\mathcal{D})) = \mathcal{H}(\mathcal{D})$. If $v$ has a $\mathcal{C}$-decomposition, then, from (i) $\Rightarrow$ (ii) in Proposition 17, $m(A_1, A_2) = 0$ whenever $A_2^c \notin \mathcal{H}(\mathcal{C})$. Therefore, $m(A_1, A_2) = 0$ whenever $A_2^c \notin \mathcal{H}(\mathcal{D})$. Indeed, $A_2^c \notin \mathcal{H}(\mathcal{D})$ implies $A_2^c \notin \mathcal{H}(\mathcal{C})$. Hence, from (ii) $\Rightarrow$ (i) in Proposition 17, $v$ has a $\mathcal{D}$-decomposition. □
By Corollary 18, it is sufficient to consider only antichain as families of subsets of \( N \). The next corollary is obtained immediately from the proofs of Proposition 17 and Corollary 18.

**Corollary 21** Let \( v \) be a bi-capacity on \( Q(N) \), \( b \) the bipolar Möbius transform of \( v \), and \( \mathcal{F}(v) := \{ A_1 \cup A_2 \mid b(A_1, A_2) \neq 0 \} \). Then for each \( F \in \mathcal{A}(\mathcal{F}(v)) \) there exists a bipolar set function \( v_F : Q(F) \to \mathbb{R} \) such that

\[
v(A_1, A_2) = \sum_{F \in \mathcal{A}(\mathcal{F}(v))} v_F(A_1 \cap F, A_2 \cap F) \quad \forall (A_1, A_2) \in Q(N). \tag{10}
\]

Table 3 shows the values of the bipolar Möbius transform of the \( \{\{1, 2\}, \{2, 3\}\} \)-decomposable bi-capacity \( v \) defined in Table 1. Indeed, \( b(A_1, A_2) = 0 \) whenever \( A_1 \cup A_2 \not\in \mathcal{H}((\{1, 2\}, \{2, 3\})) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\} \) and/or \( \mathcal{A}(\mathcal{F}(v)) = \mathcal{A}((\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}) = \{\{1, 2\}, \{2, 3\}\} \).

**Table 3.** Bipolar Möbius transform of \( v \) defined in Table 1.

<table>
<thead>
<tr>
<th>( b(A_1, A_2) )</th>
<th>( \emptyset )</th>
<th>( {1} )</th>
<th>( {2} )</th>
<th>( {3} )</th>
<th>( {1, 2} )</th>
<th>( {2, 3} )</th>
<th>( {1, 3} )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \emptyset )</td>
<td>0</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( {1} )</td>
<td>2</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( {2} )</td>
<td>2</td>
<td>-1</td>
<td>-</td>
<td>-1</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>( {3} )</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( {1, 2} )</td>
<td>2</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( {2, 3} )</td>
<td>2</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( {1, 3} )</td>
<td>0</td>
<td>-</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( N )</td>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

**Proposition 22** Let \( v \) be a bi-capacity on \( Q(N) \) and \( \mathcal{C} \) a family of subsets of \( N \). Then, the following two conditions are equivalent to each other.

(i) \( v \) has a \( \mathcal{C} \)-decomposition.

(ii) \[ v(A_1, A_2) = \sum_{D \subseteq \mathcal{C}, D \neq \emptyset} (-1)^{|D|+1} v \left( \bigcap_{D \in \mathcal{D}} D \cap A_1, \bigcap_{D \in \mathcal{D}} D \cap A_2 \right) \] for any \( (A_1, A_2) \in Q(N) \).
Proof. (i) ⇒ (ii). Assume that \( v \) has a \( \mathcal{C} \)-decomposition \( \{ v_C \}_{C \in \mathcal{C}} \).

Then, for any \( (A_1, A_2) \in \mathcal{Q}(N) \),

\[
\sum_{\substack{D \subseteq \mathcal{C} \\ D \neq \emptyset}} (-1)^{|D|+1} v \left( \bigcap_{D \in D} D \cap A_1, \bigcap_{D \in D} D \cap A_2 \right) \\
= \sum_{\substack{D \subseteq \mathcal{C} \\ D \neq \emptyset}} (-1)^{|D|+1} \sum_{C \in \mathcal{C}} v_C \left( \bigcap_{D \in D} D \cap A_1 \cap C, \bigcap_{D \in D} D \cap A_2 \cap C \right) \\
= \sum_{C \in \mathcal{C}} \left( \sum_{\substack{D \subseteq \mathcal{C} \\ D \not
\}} (-1)^{|D|+1} v_C \left( \bigcap_{D \in D} D \cap A_1 \cap C, \bigcap_{D \in D} D \cap A_2 \cap C \right) \\
+ \sum_{\substack{D \subseteq \mathcal{C} \\ D \not
\}} (-1)^{|D|+1} v_C \left( \bigcap_{D \in D} D \cap A_1 \cap C, \bigcap_{D \in D} D \cap A_2 \cap C \right) \right) \\
= \sum_{C \in \mathcal{C}} \left( \sum_{\substack{D \subseteq \mathcal{C} \\ D \not

}} (-1)^{|D|+2} v_C \left( \bigcap_{D \in D} D \cap A_1 \cap C, \bigcap_{D \in D} D \cap A_2 \cap C \right) \\
+ \sum_{\substack{D \subseteq \mathcal{C} \\ D \not

}} (-1)^{|D|+1} v_C \left( \bigcap_{D \in D} D \cap A_1 \cap C, \bigcap_{D \in D} D \cap A_2 \cap C \right) \right) \\
= \sum_{C \in \mathcal{C}} v_C(A_1 \cap C, A_2 \cap C) \\
= v(A_1, A_2);
\]

the first and last equalities above follow from the assumption.

(ii) ⇒ (i). Suppose that

\[
v(A_1, A_2) = \sum_{\substack{D \subseteq \mathcal{C} \\ D \neq \emptyset}} (-1)^{|D|+1} v \left( \bigcap_{D \in D} D \cap A_1, \bigcap_{D \in D} D \cap A_2 \right) \quad \forall (A_1, A_2) \in \mathcal{Q}(N).
\]

Now we put, for each \( C \in \mathcal{C} \),

\[
v_C(A_1 \cap C, A_2 \cap C) := \sum_{\substack{D \subseteq \mathcal{C} \\ D \subseteq C}} (-1)^{|D|+1} \frac{v \left( \bigcap_{D \in D} D \cap A_1, \bigcap_{D \in D} D \cap A_2 \right)}{|D|}
\]
for all \((A_1, A_2) \in \mathcal{Q}(N)\). Then, for any \((A_1, A_2) \in \mathcal{Q}(N)\),

\[
\sum_{C \in \mathcal{C}} v_C(A_1 \cap C, A_2 \cap C)
= \sum_{C \in \mathcal{C}} \sum_{\substack{\mathcal{D} \subseteq \mathcal{C} \\ \mathcal{D} \neq \emptyset}} (-1)^{|\mathcal{D}|+1} \frac{v \left( \bigcap_{D \in \mathcal{D}} D \cap A_1, \bigcap_{D \in \mathcal{D}} D \cap A_2 \right)}{|\mathcal{D}|}
= \sum_{\mathcal{D} \subseteq \mathcal{C} \atop \mathcal{D} \neq \emptyset} (-1)^{|\mathcal{D}|+1} v \left( \bigcap_{D \in \mathcal{D}} D \cap A_1, \bigcap_{D \in \mathcal{D}} D \cap A_2 \right)
= v(A_1, A_2).
\]

Hence, \(v\) has a \(\mathcal{C}\)-decomposition. \(\square\)

4 \(\mathcal{C}\)-decomposability of the Choquet integral with respect to bi-capacities

The Choquet integral with respect to capacities has been well-known as an aggregation operator in decision problems under uncertainty (see e.g., [2]). The Choquet integral with respect to bi-capacities has been introduced axiomatically by Labreuche and Grabisch [16]. In this section, we discuss \(\mathcal{C}\)-decomposability of the Choquet integral with respect to bi-capacities. In order to simplify the notations we denote, for a function \(f\) over \(N\), by \(f_i\) the value \(f(i)\) for each \(i \in N\).

**Definition 23 (Choquet integral w.r.t. capacities (e.g., [5]))** Let \(\mu\) be a capacity on \(\mathcal{Q}(N)\) and \(f : N \to \mathbb{R}^+\) a non-negative function on \(N\). The **Choquet integral** \(\mathcal{I}^\mu_N(f)\) of \(f\) over \(N\) with respect to \(\mu\) is defined by

\[
\mathcal{I}^\mu_N(f) := \sum_{i=1}^{n} f_{\sigma(i)} \left( \mu(A_{\sigma(i)}) - \mu(A_{\sigma(i+1)}) \right),
\]
where $\sigma$ is a permutation on $N$ such that $f_{\sigma(1)} \leq \cdots \leq f_{\sigma(n)}$, and $A_{\sigma(i)} := \{\sigma(i), \ldots, \sigma(n)\}$ and $A_{\sigma(n+1)} := \emptyset$.

An important remark is that $\mu$ needs not to be a capacity (i.e., monotone) in order to define the Choquet integral properly; any set function vanishing at the empty set could do [21].

**Definition 24 (Choquet integral w.r.t. bi-capacities [16])** Let $v$ be a bi-capacity on $Q(N)$ and $f : N \to \mathbb{R}$ a real-valued function on $N$. The Choquet integral $\mathcal{I}_N^v(f)$ of $f$ over $N$ with respect to $v$ is given by

$$\mathcal{I}_N^v(f) := \mathcal{I}_N^{\mu_f^N}(|f|),$$

where $\mu_f^N$ is a real-valued set function on $\mathcal{P}(N)$ defined by

$$\mu_f^N(S) = v(S \cap N_f^+, S \cap N_f^-), \quad (11)$$

and $N_f^+ := \{i \in N \mid f_i \geq 0\}$, $N_f^- := N \setminus N_f^+$.

It should be noticed that $\mu_f^N$ is not a capacity in general, since it may be not monotone, and even take negative values. Therefore, $v$ also needs not to be a bi-capacity (i.e., monotone) in order to define the Choquet integral properly; any bipolar set function could do. Observe that we have $\mathcal{I}_N^v(1_{(A_1, A_2)}) = v(A_1, A_2)$ for any $(A_1, A_2) \in Q(N)$, where $1_{(A_1, A_2)}$ denotes the characteristic function of $(A_1, A_2)$ which is the function on $N$ whose value for $i \in N$ is 1 if $i \in A_1$, $-1$ if $i \in A_2$, and 0 otherwise. Hence, the Choquet integral is indeed an extension of $v$.

The next proposition gives a necessary and sufficient condition for the Choquet integral with respect to bi-capacities to be decomposable into a sum of Choquet integrals as shown in Fig.2.
Fig. 2. Decomposition of the Choquet integral via $\mathcal{C} = \{C_1, C_2, C_3\}$, where $N = \{1, 2, 3, 4, 5, 6\}$, $C_1 = \{1, 2, 3\}$, $C_2 = \{3, 4, 5\}$, and $C_3 = \{4, 5, 6\}$.

**Proposition 25** Let $\mathcal{C}$ be a covering of $N$ and $v$ a bi-capacity on $\mathcal{Q}(N)$. Then, for each $C \in \mathcal{C}$ there exists a bipolar set function $v_C$ on $\mathcal{Q}(C)$ such that

$$\mathcal{I}_N^v(f) = \sum_{C \in \mathcal{C}} \mathcal{I}_C^v(f|_C)$$

for all functions $f : N \to \mathbb{R}$, where $f|_C$ denotes the restriction of $f$ to $C$, if and only if $v$ has a $\mathcal{C}$-decomposition.

**Proof.** It is easily verified from the definition of the Choquet integral with respect to bi-capacities that the Choquet integral is additive with respect to bi-capacities, that is,

$$\mathcal{I}_N^{\sum_{i=1}^n v_i}(f) = \sum_{i=1}^n \mathcal{I}_N^{v_i}(f)$$

for any $f : N \to \mathbb{R}$. Suppose that $v$ has a $\mathcal{C}$-decomposition. Then for each $C \in \mathcal{C}$ there exists a bipolar set function $v_C$ on $\mathcal{Q}(C)$ such that

$$v(S_1, S_2) = \sum_{C \in \mathcal{C}} v_C(S_1 \cap C, S_2 \cap C)$$

for any $(S_1, S_2) \in \mathcal{Q}(N)$. For each $C \in \mathcal{C}$, we define a bipolar set function $v'_C$
on $\mathcal{Q}(N)$ such that
\[
v'(C(S_1, S_2)) := v(C(S_1 \cap C, S_2 \cap C))
\]
for $(S_1, S_2) \in \mathcal{Q}(N)$. Then we have that $v(S_1, S_2) = \sum_{C \in \mathcal{C}} v'_C(S_1, S_2)$ for any $(S_1, S_2) \in \mathcal{Q}(N)$. Therefore,
\[
\mathcal{I}_N^v(f) = \sum_{C \in \mathcal{C}} \mathcal{I}_N^{v_C}(f) = \sum_{C \in \mathcal{C}} \mathcal{I}_C^{v_C}(f|_C)
\]
for any $f : N \to \mathbb{R}$ (for details on the last equality, see Appendix A1).

In order to prove the sufficiency, we show that Eq.(12) implies that $v$ has a $\mathcal{C}$-decomposition. For any $(A_1, A_2) \in \mathcal{Q}(N)$, by considering the Choquet integral of $1_{(A_1, A_2)}$ with respect to $v$, we obtain that
\[
v(A_1, A_2) = \mathcal{I}_N^v(1_{(A_1, A_2)}) = \sum_{C \in \mathcal{C}} \mathcal{I}_C^{v_C}(1_{(A_1 \cap C, A_2 \cap C)}) = \sum_{C \in \mathcal{C}} v_C(A_1 \cap C, A_2 \cap C).
\]

The Choquet integral has an equivalent representation by means of the bipolar Möbius transform as shown in the following proposition. By using this proposition, we can give another proof of Proposition 25 above (see Appendix A2).

**Proposition 26** [9] Let $v$ be a bi-capacity on $\mathcal{Q}(N)$. The Choquet integral $\mathcal{I}_N^v(f)$ of a function $f$ over $N$ with respect to $v$ can be represented as
\[
\mathcal{I}_N^v(f) = \sum_{A \in \mathcal{P}(N)} b(A \cap N_f^+, A \cap N_f^-) \bigwedge_{i \in A} |f_i| \quad (13)
\]
\[
= \sum_{(S_1, S_2) \in \mathcal{Q}(N)} b(S_1, S_2) \left( \bigwedge_{i \in S_1} f_i^+ \land \bigwedge_{j \in S_2} f_j^- \right) \quad (14)
\]

Through the discussion in sections 3 and 4, we obtain the following corollary.

**Corollary 27** Let $\mathcal{C}$ be a family of subsets of $N$ and $v$ a bi-capacity on $\mathcal{Q}(N)$. 

\[22\]
Then, the following four conditions are each equivalent to the $\mathcal{C}$-decomposability of $v$.

(i) For each $C \in \mathcal{C}$ there exists a bipolar set function $v_C$ on $\mathcal{Q}(C)$ such that $\mathcal{S}^v_N(f) = \sum_{C \in \mathcal{C}} \mathcal{S}^v_C(f|_C)$ for any function $f : N \rightarrow \mathbb{R}$.

(ii) $m(A_1, A_2) = 0$ if $A_2^c \notin \mathcal{H}(C)$.

(iii) $b(A_1, A_2) = 0$ if $(A_1 \cup A_2) \notin \mathcal{H}(C)$.

(iv) $v(A_1, A_2) = \sum_{D \subseteq C \setminus \emptyset}(-1)^{|D|+1}v(\bigcap_{D \in D} D \cap A_1, \bigcap_{D \in D} D \cap A_2)$ for any $(A_1, A_2) \in \mathcal{Q}(N)$.

5 Concluding remarks and future works

This paper discusses notions to reduce complexity of bi-capacities, $k$-additivity and $\mathcal{C}$-decomposability of that. These notions are closely-linked to each other via the Möbius and bipolar Möbius transforms. $\mathcal{C}$-decomposability of bi-capacities is naturally extended to that of the Choquet integral with respect to bi-capacities. It should be noticed that monotonicity of bi-capacities do not play an essential role in this paper, indeed, almost all notions and results for bi-capacities given in the paper hold for merely bipolar set functions. However, monotonicity of bi-capacities and of the Choquet integral is one of the most important properties in decision problems. Therefore, to investigate monotone $\mathcal{C}$-decomposition (i.e., every $v_C$ in $\mathcal{C}$-decomposition $\{v_C\}_{C \in \mathcal{C}}$ is a bi-capacity) of bi-capacities is an important subject for future work.
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References


Appendix

A1 Supplement to the proof of Proposition 25

Proof of $\mathcal{N}^C(f) = \mathcal{C}^C(f|_C)$:
Since for any $S \subseteq N$, using Eq.(11),

\[
(\mu'_C)^N_f(S) = v'_C(S \cap N^+_f, S \cap N^-_f)
\]
\[
= v_C(S \cap N^+_f \cap C, S \cap N^-_f \cap C)
\]
\[
= v_C((S \cap C) \cap N^+_f, (S \cap C) \cap N^-_f)
\]
\[
= v_C((S \cap C) \cap C^+_f \cap C, (S \cap C) \cap C^-_f \cap C)
\]
\[
= (\mu_C)^C_f(S \cap C) \quad ( = (\mu'_C)^N_f(S \cap C) ),
\]

it follows that $N \setminus C$ is a null set (see e.g., [20]) with respect to $(\mu'_C)^N_f$, and hence from a well-known property (see e.g., [20]) of the Choquet integral that for any function $g : N \to \mathbb{R}$

\[
I^{(\mu'_C)^N_f}_N(g) = I^{(\mu_C)^C_f}_C(g|C).
\]

Therefore, we have

\[
I^{v'_C}_N(f) = I^{(\mu'_C)^N_f}_N(|f|) = I^{(\mu_C)^C_f}_C(|f| |C) = I^{(\mu_C)^C_f}_C(|f| |C) = I^{v_C}_C(f |C).
\]

### A2 Another proof of the “necessity” part of Proposition 25

Suppose that $v$ has a $C$-decomposition. Put

\[
\mathcal{I}(C) := \{(A_1, A_2) \in \mathcal{Q}(N) \mid (A_1 \cup A_2) \in \mathcal{H}(C)\},
\]

\[
\mathcal{O}(C) := \{(A_1, A_2) \in \mathcal{Q}(N) \mid (A_1 \cup A_2) \not\in \mathcal{H}(C)\},
\]

and for $(A_1, A_2) \in \mathcal{Q}(N)$

\[
\mathcal{C}(A_1, A_2) := \{C \in \mathcal{C} \mid C \supseteq (A_1 \cup A_2)\}.
\]
Then, clearly, $\{I(C), O(C)\}$ is a partition of $Q(N)$. For each $C \in \mathcal{C}$, we define two functions on $Q(C)$, $b_C$ and $v_C$, as follows:

$$b_C(A_1, A_2) := \frac{1}{|C(A_1, A_2)|} b(A_1, A_2) \quad \text{for} \quad (A_1, A_2) \in Q(C),$$

$$v_C(A_1, A_2) := \sum_{B_1 \subseteq A_1 \atop B_2 \subseteq A_2} b_C(B_1, B_2) \quad \text{for} \quad (A_1, A_2) \in Q(C).$$

Then $v_C(\emptyset, \emptyset) = 0$ for any $C \in \mathcal{C}$, and $b(A_1, A_2) = 0$ for any $(A_1, A_2) \in O(C)$ from (i) $\Rightarrow$ (iii) in Proposition 17. Therefore, for any $f : N \rightarrow \mathbb{R}$,

$$\mathcal{I}_N^c(f) = \sum_{(A_1, A_2) \in Q(N)} b(A_1, A_2) \left( \bigwedge_{i \in A_1} f_i^+ \land \bigwedge_{j \in A_2} f_j^- \right)$$

$$= \sum_{(A_1, A_2) \in I(C) \cup O(C)} b(A_1, A_2) \left( \bigwedge_{i \in A_1} f_i^+ \land \bigwedge_{j \in A_2} f_j^- \right)$$

$$= \sum_{(A_1, A_2) \in I(C)} b(A_1, A_2) \left( \bigwedge_{i \in A_1} f_i^+ \land \bigwedge_{j \in A_2} f_j^- \right)$$

$$= \sum_{C \in \mathcal{C}} \sum_{(A_1, A_2) \in Q(C)} \frac{b(A_1, A_2)}{|C(A_1, A_2)|} \left( \bigwedge_{i \in A_1} (f|_C)^+_i \land \bigwedge_{j \in A_2} (f|_C)^-_j \right)$$

$$= \sum_{C \in \mathcal{C}} \sum_{(A_1, A_2) \in Q(C)} b_C(A_1, A_2) \left( \bigwedge_{i \in A_1} (f|_C)^+_i \land \bigwedge_{j \in A_2} (f|_C)^-_j \right)$$

$$= \sum_{C \in \mathcal{C}} \mathcal{I}_C^c(f|_C). \quad \Box$$