A weak Moufang condition suffices

Katrin Tent

Mathematisches Institut, Universität Würzburg, Am Hubland, D-97074 Würzburg, Germany

Received 30 January 2004; received in revised form 15 June 2004; accepted 21 June 2004
Available online 7 August 2004

Abstract

If $\Gamma$ is a 2-Moufang generalized $n$-gon for $n \leq 6$, then $\Gamma$ is Moufang.

MSC: 20E42; 51E12; 05C25

1. Introduction

The classification of the Moufang polygons by Tits and Weiss [8] shows that Moufang polygons and higher rank buildings all arise from classical or algebraic groups. However, for identifying a group as one of these groups through its action on the associated geometry, often only an apparently weaker form of the Moufang condition can be verified and hence it is very useful to know that this form is already sufficient for the classification. The results of [9] and [11] imply that finite 2-Moufang polygons are Moufang. The proof given in [11] is group theoretic and uses the results of Fong and Seitz [1]. We here give a direct proof of the corresponding result for arbitrary generalized $n$-gons with $n \leq 6$. Note that for infinite generalized $n$-gons it is still not known whether the 2-Moufang condition implies a bound on the possible $n$ occurring.
2. Background and definitions

A generalized $n$-gon $\Gamma$ is a bipartite graph with valencies at least 3, diameter $n$ and girth $2n$. For $n = 3, 4, 5$ and $6$, we call $\Gamma$ a projective plane, generalized quadrangle, generalized pentagon, and generalized hexagon, respectively. The vertices of this graph are called the elements of $\Gamma$. The set of elements adjacent to some element $x \in \Gamma$ is denoted by $\Gamma_1(x)$, and more generally $\Gamma_i(x)$ denotes the set of elements of (graph theoretic) distance $i$ from $x$. We call $(x_0, \ldots, x_k)$ a simple path if the $x_i$ are pairwise distinct and $x_i$ is adjacent to $x_{i+1}$ for $i = 0, \ldots, k-1$.

If $G \leq \text{Aut}(\Gamma)$, we denote by $G_{\Gamma}^{[i]}$ the subgroup of $G$ fixing all elements of $\Gamma_i(x_0)$ (and then it automatically fixes all sets $\Gamma_j(x_0)$ pointwise, for $0 \leq j \leq i \leq n$). Further, for elements $x_0, \ldots, x_k$, we set $G_{x_0, x_1, \ldots, x_k}^{[i]} = G_{x_0}^{[i]} \cap G_{x_1}^{[i]} \cap \cdots \cap G_{x_k}^{[i]}$. For $i = 0$, we usually omit the superscript $[0]$.

**Crucial Fact 2.1.** For every simple path $(x_0, \ldots, x_{n+1})$ of length $n + 1$ and every $i$ with $0 \leq i \leq n$, we have $G_{x_0, \ldots, x_{i+1}} \cap G_{x_1, x_{i+1}}^{[i]} = 1$ (see e.g. [10] 4.4.2 (v)).

For $2 \leq k \leq n$, the generalized $n$-gon $\Gamma$ is said to be $k$-Moufang with respect to $G \leq \text{Aut}(\Gamma)$ if for each simple $k$-path $(x_0, \ldots, x_k)$ the group $G_{x_1, \ldots, x_{k-1}}^{[i]}$ acts transitively on the set of $2n$-cycles through $(x_0, \ldots, x_k)$. If $\Gamma$ is $n$-Moufang with respect to some group $G$, then we say that $\Gamma$ is a Moufang polygon. If $G_{x_0, x_1}$ acts transitively on the set of $2n$-cycles through $(x_0, x_1)$ for all paths $(x_0, x_1)$ (sometimes referred to as the 1-Moufang condition), then $G$ acts transitively on the set of ordered $2n$-cycles. This is equivalent to $G$ having a BN-pair of rank 2, which is in general too weak to allow a classification, see examples in [7,5].

An elation (or root elation) $g$ of $\Gamma$ is a member of $G_{x_1, \ldots, x_{n-1}}^{[i]}$ for some simple path $(x_1, x_2, \ldots, x_{n-1})$ of $\Gamma$, in which case $g$ is also called an $(x_1, x_2, \ldots, x_{n-1})$-elation. It acts freely on $\Gamma_i(x_0) \setminus \{x_1\}$, for every element $x_0 \in \Gamma_1(x_1) \setminus \{x_2\}$. The group generated by all root elations of a Moufang polygon is called its little projective group. By the classification [8], this group is always a (possibly twisted) Chevalley group.

It is well-known that finite or Moufang generalized $n$-gons exist only for $n = 3, 4, 6, 8$. Background on the Moufang condition for generalized $n$-gons can be found in [10] and [8].

**Notation.** Regarding commutators and conjugation, we use the notation $g^h = h^{-1}gh$. We also write $g^{-h} = (g^{-1})^h$ and $[g, h] = g^{-1}h^{-1}gh = h^{-g}h = g^{-1}g^h$. We let automorphisms act on the right, so we use exponential notation.

If $H$ is a group acting on a set $\Omega$, and $A \subseteq \Omega$, we let $H_A$ denote the pointwise stabilizer of $A$ in $H$.

The following observations are at the heart of many arguments.

**Lemma 2.2.** Let $\Gamma$ be a generalized $n$-gon, $G \leq \text{Aut}(\Gamma)$. Then the following holds for $g, h \in G$:

(i) if $g \in G_{x}^{[i]}$, $h \in G$, then $[g, h] = 1$ implies $g \in G_{x}^{[i]}$;
(ii) if $g \in G_{x}^{[i]}$, $h \in G_x$, then $[g, h] \in G_x^{[i]}$. 
Throughout this paper, for any 2-Moufang generalized \( n \)-gon we let \( G \) denote the group generated by automorphisms of \( \Gamma \) which fix \( \Gamma_1(x_0) \cup \Gamma_1(x_1) \) elementwise for some path \((x_0, x_1)\).

We will repeatedly use the following lemma (see [4] Lemma 2.3):

**Lemma 2.3.** Let \( \Gamma \) be a 2-Moufang generalized \( n \)-gon, \((x_0, x_1, \ldots, x_k)\) a path in \( \Gamma \) and suppose that \( U = G_{x_1, x_2}^{[1]} \cap G_{x_4, \ldots, x_k} \) acts freely on \( \Gamma_1(x_k) \setminus \{x_{k-1}\} \). Then \( [U, U] \leq \bigcap_{y \in \Gamma_1(x_2)\setminus\{x_3\}} G_{y}^{[1]} \cap G_{x_4, \ldots, x_k} \).

**Proof.** Let \( \alpha, \beta \in U \) and let \( g \in G_{x_k}^{[1]} \cap G_{x_4, \ldots, x_k-2} \). Notice that since \( U \) acts freely on \( \Gamma_1(x_k) \setminus \{x_{k-1}\} \), we have \([\alpha, \beta] = [\alpha, \beta^g] \in U \cap U^g \leq G_{x_1, x_2}^{[1]} \cap G_{x_4, \ldots, x_k} \).

Similarly, for \( \alpha \in U, \beta \in U^g \), we have \([\alpha, \beta] = [\alpha, \beta^{-1}] \in [U, U]^g \). This easily implies \([U, U] = [U, U^g] \leq G_{x_1, x_2}^{[1]} \cap G_{x_4, \ldots, x_k} \). By the 2-Moufang condition the claim now follows. \( \square \)

Note that in light of Fact 2.1 the previous lemma applies in particular when \( k = n \).

3. 2-Moufang quadrangles

The following result was first proved by F. Haot in her Ph.D. thesis (see [3]), but we here give a much shorter proof.

**Theorem 3.1.** Let \( \Gamma \) be a generalized quadrangle which is 2-Moufang with respect to \( G \). Then \( \Gamma \) is Moufang and \( G \) contains its little projective group.

**Proof.** Let \((x_0, \ldots, x_7, x_8 = x_0)\) be an 8-cycle. By [9], \( \Gamma \) is 3-Moufang. We directly prove the Moufang property by constructing all \((x_7, x_0, x_1, x_2, x_3)\)-elations from the elements of \( G_{x_1, x_2}^{[1]} \) given by the 3-Moufang property, i.e., we show that \( G_{x_1, x_2}^{[1]} \) acts transitively on \( \Gamma_1(x_3) \setminus \{x_2\} \).

Let \((x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 = x_0)\) be another 8-cycle with \( x_0 \neq x_4 \). Let \( a_1 \in G_{x_1, x_2}^{[1]} \cap G_{x_4}^{[1]} \setminus \{1\} \), let \( y_1 = x_5^{a_1} \) and let \((y_1, y_2, y_3, y_4)\) be the path from \( y_1 \) to \( y_4 \) and let \((x_1, y_4, y_5, y_2)\) be the path from \( x_1 \) to \( y_2 \). By the 3-Moufang assumption there is some \( g \in G_{x_0}^{[1]} \) with \( x_4^g = y_2 \). Then \( a_2 = a_1^{-g} \in G_{x_1, y_4}^{[1]} \cap G_{y_2}^{[1]} \) satisfies \( y_1^{a_2} = y_3^{a_2} = y_3 \) and \( a_1 a_2 = [a_1^{-1}, g] \in G_{x_0}^{[1]} \). Similarly, there is some \( h \in G_{x_2}^{[1]} \) with \( y_2^h = x_6^h \). Put \( a_3 = a_2^{-h} \in G_{x_0, x_7}^{[1]} \), so \( y_3^{a_3} = x_5^{a_3} \) and \( a_2 a_3 = [a_2^{-1}, h] \in G_{x_2}^{[1]} \). Then, the composition \( a_1 a_2 a_3 \in G_{x_0, x_1, x_2}^{[1]} \) is the desired element taking \( x_5 \) to \( x_5' \) and hence \( x_4 \) to \( x_4' \). \( \square \)

4. 2-Moufang pentagons

In this section, let \( \Gamma \) be a generalized pentagon which is 2-Moufang with respect to \( G \), and let \((x_0, \ldots, x_9, x_{10} = x_0)\) be a closed path.

**Lemma 4.1.** For all \( x \in \Gamma \), we have \( G_{x}^{[2]} = 1 \).
Proof. Suppose towards a contradiction that \( \alpha \in G_{x_1}^{[2]} \setminus \{1\} \). Since \( G_{x_1}^{[3]} = 1 \) by Fact 2.1, we may assume that \( \alpha \not\in G_{x_0} \). By the 2-Moufang assumption \( G_{x_0} \cap G_{x_1, x_7, x_6} \) is transitive on \( \Gamma_1(x_6) \setminus \{x_7\} \). If for \( \beta \in G_{x_0} \cap G_{x_1, x_7, x_6} \setminus \{1\} \), we have \([\alpha, \beta] = 1\), then \( \beta \) fixes \( x_6^\alpha \) by Lemma 2.2(i) and hence the unique element \( z \in \Gamma_1(x_6) \) with \( d(z, x_6^\alpha) = 3 \). But \( G_{x_0} \cap G_{x_1, x_7, x_6} \) is transitive on \( \Gamma_1(x_6) \setminus \{x_7\} \), and hence there is some \( \beta \in G_{x_0} \cap G_{x_1, x_7, x_6} \setminus \{1\} \) such that \( 1 \neq [\alpha, \beta] \in G_{x_0}^{[2]} \cap G_{x_0}^{[1]} \). Now let \( \delta \in G_{x_0}^{[2]} \cap G_{x_0}^{[1]} \). Then \([\delta, \alpha] \in G_{x_1, x_9} \) by Lemma 2.2(ii). By Lemma 2.2(i), we have \( \delta \in G_{x_0}^{[2]} \cap G_{x_0}^{[1]} = 1\), a contradiction. \( \square \)

Lemma 4.2. We have \( \bigcap_{y \in \Gamma_1(x_3) \setminus \{x_2\}} G_y^{[1]} = 1 \).

Proof. Suppose towards a contradiction that \( \alpha \in \bigcap_{y \in \Gamma_1(x_3) \setminus \{x_2\}} G_y^{[1]} \setminus \{1\} \), and let \( \beta \in \bigcap_{y \in \Gamma_1(x_3) \setminus \{x_2\}} G_y^{[1]} \setminus G_{x_3} \). Then \([\alpha, \beta] \in G_{x_3}^{[2]} = 1 \) by Lemma 2.2(ii) and Lemma 4.1. But by Lemma 2.2(ii) \([\alpha, \beta] = 1 \) implies \( \alpha \in \bigcap_{y \in \Gamma_1(x_3) \setminus \{x_2\}} G_y^{[1]} \cap \bigcap_{y \in \Gamma_1(x_3) \setminus \{x_2\}} G_y^{[1]} = 1 \), a contradiction. \( \square \)

Lemma 4.3. The group \( W = G_{x_1, x_3} \cap G_{x_1, x_2} \) is trivial.

Proof. By Lemmas 2.3 and 4.2, we have \([W, W] = 1\), and hence \( W \) is abelian. Suppose towards a contradiction that \( \alpha \in W \setminus \{1\}, \beta \in G_{x_3}^{[1]} \cap G_{x_8} \setminus \{1\} \). Then \([\alpha, \beta] \in G_{x_4, x_5, x_6}^{[1]} \) by Lemma 2.2(ii). We now claim that \([\alpha, \beta] \in G_{x_3}^{[2]} \) : let \( y \in \Gamma_1(x_3) \setminus \{x_4, x_6\} \), and let \( g \in G_{x_4} \) with \( x_6^g = x_1 \). Then \( \beta^g \in W \), and hence \([\alpha, \beta^g] = 1 \). So \([\alpha, \beta] \in G_{y}^{[1]} \) and since \( y \in \Gamma_1(x_3) \setminus \{x_4, x_6\} \) was arbitrary and \([\alpha, \beta] \in G_{x_4, x_6}^{[1]} \) by Lemma 2.2(ii), we have \([\alpha, \beta] \in G_{x_3}^{[2]} \). By Lemma 4.1 we thus have \([\alpha, \beta] = 1 \), implying by Lemma 2.2(i) that \( \alpha \in G_{x_1, x_2, x_7, x_6}^{[1]} \cap G_{x_4, x_5}^{[1]} \). But this last group is trivial by Fact 2.1 and hence \( \alpha = 1 \), a contradiction. \( \square \)

Lemma 4.4. \( G_{x_4}^{[1]} \cap G_{x_0, x_1, x_2} = G_{x_0}^{[1]} \).

Proof. Let \( \alpha \in G_{x_4}^{[1]} \cap G_{x_0, x_1, x_2} \setminus \{1\} \). Suppose that \( \alpha \not\in G_{x_4}^{[1]} \), so \( \alpha \not\in G_{y}^{[1]} \) for some \( y \in \Gamma_1(x_1) \). By the 2-Moufang assumption, let \( \beta \in G_{x_5}^{[1]} \cap G_{x_1} \) with \( x_0^\beta = y \). Then \([\alpha, \beta] \not\in G_{x_4}^{[1]} \) by Lemma 2.2(i). But on the other hand by Lemma 2.2(ii) we have \([\alpha, \beta] \in G_{x_1}^{[1]} \cap G_{x_1, x_2} = 1\), a contradiction. \( \square \)

Notice that the group \( G_{x_0}^{[1]} \cap G_{x_0, x_1, x_2} = G_{x_0}^{[1]} \) is nontrivial by the 2-Moufang assumption.

Theorem 4.5. There are no 2-Moufang pentagons.

Proof. Let \( \alpha \in G_{x_0}^{[1]} \setminus \{1\} \). Then by Lemma 4.3, we have \( \alpha \not\in G_{x_2}^{[1]} \), say \( \alpha \not\in G_{y}^{[1]} \) for some \( y \in \Gamma_1(x_2) \). By the 2-Moufang assumption, let \( \beta \in G_{x_3}^{[1]} \cap G_{x_2} \) with \( x_0^\beta = y \), and so by Lemma 2.2(i) and (ii) we have \( 1 \neq [\alpha, \beta] \in W := G_{x_4, x_5}^{[1]} \cap G_{x_2} \). By Lemma 4.3, \( W \cap G_{x_4} = 1 \) and so \( W \) has to act freely on \( \Gamma_1(x_2) \setminus \{x_3\} \).

Next we show that \( W \) acts freely on \( \Gamma_1(x_6) \setminus \{x_5, x_7\} \). Suppose that \( g \in W \cap G_{x_7} \setminus \{1\} \).

If \( g \not\in G_{y}^{[1]} \) for some \( y \in \Gamma_1(x_6) \setminus \{x_5, x_7\} \), let \( h \in G_{x_0}^{[1]} \cap G_{x_6} \) with \( x_6^h = y \). Then
5. 2-Moufang hexagons

Lemma 5.1. Let $\Gamma$ be a generalized hexagon which is 2-Moufang with respect to $G$, and let $(x_0, \ldots, x_{12} = x_0)$ be a closed path. If $G_{x_2} \cap G_{x_3, x_6} \neq 1$, then $\Gamma$ is Moufang and if $|\Gamma_1(x_0)| \geq 4$, then $G$ contains its little projective group.

Proof. Let $\alpha \in G_{x_2}^{[2]} \cap G_{x_3, x_6} \setminus \{1\}$, and let $\beta \in G_{x_2}^{[2]} \cap G_{x_7, x_8} \setminus \{1\}$. Then by Lemma 2.2, we have $[\alpha, \beta] \in G_{x_2, x_4} \setminus \{1\}$, and the claim follows from [6] Proposition 3.5 and Theorem 4.7.

Proposition 5.2. Let $\Gamma$ be a generalized hexagon which is 2-Moufang with respect to $G$, and let $(x_0, \ldots, x_{12} = x_0)$ be a closed path. If $|\Gamma_1(x_0)| = 3$, then $\Gamma$ is Moufang and $G$ contains its little projective group.

Proof. Clearly we may assume that $|\Gamma_1(x_1)| \geq 4$, as otherwise the conclusion is clear. First assume that $G_{x_2}^{[2]} \cap G_{x_3, x_6} = 1$. Let $\alpha \in G_{x_2, x_3, x_4} \cap G_{x_0, x_6} \setminus \{1\}$. If $\alpha \in G_{x_2}^{[1]}$, let $\beta \in G_{x_3, x_4, x_6} \cap G_{x_3} \setminus \{1\}$. Using Lemma 2.2(i) and (ii) it is easy to see that $[\alpha, \beta] \in G_{x_2}^{[2]} \cap G_{x_6, x_7} \setminus \{1\}$ and we are done by Lemma 5.1. If $\alpha \notin G_{x_2}^{[2]}$, let $y \in \Gamma_1(x_1)$ such that $\alpha \notin G_y$, and let $g \in G_{x_2, x_3, x_4, x_6}$ such that $x_6^g = y$. Then we have $[\alpha, g] \in G_{x_3} \cap G_{x_6, x_7} \setminus \{1\}$ and we are again done by Lemma 5.1.

Now assume that $\alpha \in G_{x_2}^{[2]} \cap G_{x_4, x_6} \setminus \{1\}$ and let $\beta \in G_{x_3}^{[2]} \cap G_{x_7, x_8} \setminus \{1\}$. Then since $|\Gamma_1(x_0)| = 3$, we have $[\alpha, \beta] \in G_{x_2, x_3, x_4} \setminus \{1\}$ and this is the unique nontrivial elation for $(x_1, \ldots, x_5)$. Notice that by the 2-Moufang assumption the group $V = G_{x_0, x_1, x_2, x_4}^{[1]}$ is active on $\Gamma_1(x_3) \setminus \{x_4\}$. We claim that $V$ consists of elations. To see this, let $g \in V$ and $\beta' \in G_{x_3, x_5, x_6} \setminus \{1\}$. Then $[g, \beta'] \in G_{x_4}^{[2]} \cap G_{x_2}^{[2]}$. If $[g, \beta'] \neq 1$, then we are done by Lemma 5.1. If $[g, \beta'] = 1$, then $g \in G_{x_4}^{[1], x_3}$ by Lemma 2.2(i). In this case, let $\alpha' \in G_{x_3, x_1, x_0} \setminus \{1\}$. Then $x_1^{\alpha'} = x_2^{\alpha'}$ and so $[g, \alpha'] \in G_{x_4}$; hence $[g, \alpha'] = 1$ by Lemma 2.2(ii) and Fact 2.1. Thus, $g \in G_{x_4}^{[1]}$ and hence $V$ consists of elations as claimed. So also in this case, $G$ contains the little projective group of $\Gamma$.

Lemma 5.3 (cp. [6] 3.1). Let $\Gamma$ be a generalized hexagon with $|\Gamma_1(x)| \geq 4$. Assume that $\Gamma$ is 2-Moufang with respect to $G$, and let $(x_0, \ldots, x_6)$ be a path. The group $V = G_{x_0, x_2}^{[1]} \cap G_{x_4, x_5}$ acts transitively on $\Gamma_1(x_3) \setminus \{x_4\}$. 

1 $\neq [g, h] \in G_{x_2, x_3, x_5}^{[2]} = 1$, a contradiction. So $g \in W \cap G_{x_6}^{[1]} = G_{x_4, x_5, x_6} \cap G_{x_2} \setminus \{1\}$. By Lemma 4.4, $g \in G_{x_2}^{[1]} \cap G_{x_4, x_5, x_6}$. But this group is trivial by Lemma 4.3, again a contradiction.

So $W$ acts freely on $\Gamma_1(x_6) \setminus \{x_5\}$. Let $\alpha \in W \setminus \{1\}$ and let $\beta \in G_{x_3, x_8}^{[1]} \setminus \{1\}$. We claim that $[\alpha, \beta] = \prod_{\gamma \in \Gamma_1(x_5) \setminus \{x_6\}} G_{y}^{[1]}$. To see this, let $y \in \Gamma_1(x_5) \setminus \{x_4, x_6\}$, and let $g \in G_{y}^{[1]}$ with $x_6^g = x_2$. Then $\beta^\gamma \in G_{x_2}^{[1]} \cap G_{x_3, x_5, x_6} = 1$. Thus $[\alpha, \beta] \in G_{x_2}^{[1]}$. Since $y \in \Gamma_1(x_5) \setminus \{x_4, x_6\}$ was arbitrary and $[\alpha, \beta] \in G_{x_2}^{[1]}$, we have proved the claim. Hence $[\alpha, \beta] = 1$ by Lemma 4.2, and so by Lemma 2.2(i) $\beta \in G_{x_4, x_5}^{[1]} = 1$. This final contradiction finishes the proof. □
Lemma 5.5. Let $x_4 \neq x'_4$. Choose two distinct 12-cycles $(x'_4, x'_6, x'_3, x'_4, x_3, x_4, x_5, x_6, \ldots, x_{10}, x_{11} = x'_7)$ and $(x'_7, x'_4, x'_4, x_3, x_4, x_5, y_6, \ldots, y_{10}, y_{11} = x'_7)$. Let $p_0$ and $p_1$ denote the respective projections of $x_9$ and $y_9$ onto $x_2$ (i.e., $p_0$ and $p_1$ are the unique elements in $\Gamma_1(x_2)$ with $d(x_9, p_0) = 4$ and $d(y_9, p_01) = 4$, respectively), and let $q \in \Gamma_1(x_2) \setminus \{x_3, p_0, p_1\}$. Then $d(x_8, x_2) = d(y_8, x_2) = d(x_{10}, x_2) = d(y_{10}, x_2) = 6$. Let $y_1 = (x_8, \ldots, q, x_2)$ and $y_2 = (x_{10}, \ldots, q, x_2)$ be paths of length 6. By the 2-Moufang assumption let $\alpha_1 \in G_q^1 \cap G_{y_1}$ with $x'^{y_1} = x_9$ (and so $x'^{\alpha_1} = p_0$); similarly, let $h_1 \in G_q^1$ with $y_1 = y_2$. Put $\alpha_2 = \alpha_1^{-1} \in G_q^1 \cap G_{y_2}$ so $x'^{\alpha_2} = x'_7$ (and hence $p_0^{\alpha_2} = x_3$). Note that $\alpha_1 \alpha_2 \in G_q^1 \setminus \{q\}$. Let $\gamma_3 = (y_8, \ldots, q, x_2)$ and $\gamma_4 = (y_8, \ldots, q, x_2)$ be paths of length 6 and let $\alpha_3 \in G_q^1 \cap G_{y_3}$ with $x'^{\gamma_3} = y_9$ and so $x'^{\alpha_3} = p_1$. Let $h_2 \in G_q^1$ with $y_2 = y_4$, and put $\alpha_4 = \alpha_3^{-1}$, so $y'^{\alpha_4} = y_7$ and $p_0^{\alpha_4} = x_3$.

Then $g = \alpha_1 \alpha_2 \alpha_3 \alpha_4 \in G_q^1 \cap x'^{x_2}$ and $x'^{x_6} = y_6$. Let $\beta \in G_{x_3}^1$ with $q^{\beta} = x_1$.

Then $h = g^{\beta} \in G_{x_3} \cap G_{x_4}$ and $x'^{x_6} = y_6$, proving the claim. □

From now on we suppose that $\Gamma$ is a generalized hexagon with $|\Gamma_1(x)| \geq 4$. We assume that $\Gamma$ is 2-Moufang with respect to $G$, and we fix a closed path $(x_0, \ldots, x_{12} = x_0)$.

Lemma 5.4. If $G_{x_3} \cap G_{x_1, x_5} \neq 1$, then $\Gamma$ is Moufang and $G$ contains its little projective group.

Proof. Let $h \in G_{x_3}^1 \cap G_{x_1, x_5} \setminus \{1\}$. By Lemma 5.1 we may assume that $h \notin G_{x_3}^1$, say, $h \notin G_{x_3}^{11}$ for some $y \in \Gamma_1(x_3)$. By Lemma 5.3 there is some $g' \in G_{y_5}^1 \cap x_9$ with $x'^{g'} = y$. Then we have $1 \neq [h, g']$ by Lemma 2.2(i) and $[h, g'] \in G_{x_3}^1 \cap G_{x_3}^{11}$ by Lemma 2.2(ii). Now we are done by Lemma 5.1. □

Lemma 5.5. If $G_{x_3}^1 \neq 1$, then $\Gamma$ is Moufang and $G$ contains its little projective group.

Proof. Let $h \in G_{x_3}^1 \setminus \{1\}$. By Lemma 5.4 (and after conjugating if necessary) we may assume that $h \in G_{x_3}^2 \setminus G_{x_3}^6$. Let $h' \in G_{x_3}^1 \cap G_{x_3, x_7, x_9} \setminus \{1\}$. Then $\delta = [h, h'] \in G_{x_3}^1 \cap G_{x_3}^{11} \setminus \{1\}$ by Lemma 2.2(i) and (ii). By Lemma 5.4 again, we may similarly assume that $\delta \in G_{x_3}^2 \cap G_{x_3}^{11} \setminus G_{x_0}$. Let $\delta' \in G_{x_3}^1 \setminus G_{x_4}$, so $[\delta, \delta'] \in G_{x_3, x_5}^2$. If $[\delta, \delta'] = 1$, then $\delta \in G_{x_3}^1 \cap G_{x_3}^{11}$ by Lemma 2.2(i), and we are done by Lemma 5.4. If $1 \neq [\delta, \delta'] \in G_{x_3, x_7}^2$, then we are done by Lemma 5.1. □

Lemma 5.6. If $W = G_{x_2, x_3}^1 \cap G_{x_{10}, x_{11}}$ is abelian, then for any $\alpha \in W$ and $\beta \in G_{x_3, x_4} \cap G_{x_6, x_7, x_9}$ we have $[\alpha, \beta] \in G_{x_2}^2$.

If $W_1 = G_{x_2, x_3}^1 \cap G_{x_1, x_5}$ is abelian, the same holds for $\alpha \in W_1$, $\beta \in G_{x_3, x_4}^1 \cap G_{x_6, x_7}$ and if $W_2 = G_{x_3, x_4, x_2, x_3}^1$ is abelian for $\alpha \in W_2$, $\beta \in G_{x_3, x_4, x_5, x_6, x_7}$.

Proof. Let $\alpha \in W$, and let $\beta \in G_{x_3, x_4} \cap G_{x_6, x_7, x_9}$. Let $y \in \Gamma_1(x_3) \setminus \{x_2, x_4\}$, and let $g \in G_{y}^1$ such that $x'^{g} = x_9$. Then $x'^{g} \in W$. So, $[\alpha, g] = 1$. Hence $[\alpha, \beta] \in G_{y}^1$. Since
Lemma 5.9.

Proof. The assumption on $\bigcap \neq \{\alpha \in G_{x_2,x_4}\}$ is arbitrary and $[\alpha, \beta] \in G_{x_2,x_4}$ by Lemma 2.2(ii), this proves the claim.

The proofs for $W_1$ and $W_2$ are similar. $\square$

Proposition 5.7. If $1 \neq W = G_{x_1,x_2}^{[1]} \cap G_{x_4,x_5,x_6}$, then $\Gamma$ is Moufang and $G$ contains its little projective group.

Proof. First assume that $W$ (and also its conjugate $G_{x_{10},x_{11},x_0} \cap G_{x_2,x_3}^{[1]}$) is abelian. Let $\alpha \in G_{x_{10},x_{11},x_0} \cap G_{x_2,x_3}^{[1]} \setminus \{1\}$, and let $\beta \in G_{x_3,x_4} \cap G_{x_6,x_7,x_8}$. By Lemma 2.2(i), $1 \neq [\alpha, \beta] \in G_{x_2,x_4}^{[1]}$ and by Lemma 5.6 we have $[\alpha, \beta] \in G_{x_3}^{[2]} \setminus \{1\}$. Now we are done by Lemma 5.5.

Now assume that $W$ is not abelian. Then by Lemma 2.3 we have $1 \neq [W, W] \leq \bigcap_{y \in G_{x_2}^{[1]} \setminus \{x_1\}} G_{x_2,x_3,x_5}^{[1]} \cap G_{x_4,x_5,x_6}$. Let $\alpha \in [W, W] \setminus \{1\}$ and let $\beta \in G_{x_8,x_9,x_{10}} \cap \bigcap_{y \in G_{x_2}^{[1]} \setminus \{x_1\}} G_{x_2,x_3,x_5}^{[1]} \setminus \{1\}$. Since $|\Gamma(x_0)| \geq 4$, choose $x_1' \in \Gamma(x_0) \setminus \{x_1, x_1'\}$ and $x_3' \in \Gamma(x_2) \setminus \{x_3, x_5\}$ and let $W_2 := G_{x_{11},x_0,x_1,x_2,x_3}^{[1]}$. Then $[\alpha, \beta] \in W_2$ by Lemma 2.2(ii).

If $[\alpha, \beta] = 1$, then by Lemma 2.2(i) we conclude that $\alpha \in G_{x_4,x_5,x_6,x_3' x_4', x_5'} \cap \bigcap_{y \in G_{x_2}^{[1]} \setminus \{x_1\}} G_{x_2,x_3,x_5}^{[1]} = 1$, contradicting our assumption on $\alpha$. So $[\alpha, \beta] \neq 1$, and hence $W_2 \neq 1$.

If $W_2$ is abelian, then for $\alpha \in W_2$ and $\beta \in G_{x_{11},x_0,x_1,x_2,x_3}^{[1]}$ we have $[\alpha, \beta] \neq 1$ by Lemma 2.2(i). By Lemma 5.6 we have $[\alpha, \beta] \in G_{x_3}^{[2]} \setminus \{1\}$, and we finish by Lemma 5.5.

If $W_2$ is not abelian, then by Lemma 2.3 we have $1 \neq [W_2, W_2] \leq G_{x_0,x_2}$ and we are done by Lemma 5.1. $\square$

Note that the previous proposition also holds if $1 \neq W = G_{x_{10},x_{11}} \cap G_{x_3,x_4,x_5}$.

Lemma 5.8. If $V = G_{x_{11}}^{[1]} \cap G_{x_4,x_5}$ does not induce a free action on $\Gamma(x_0)$, then $\Gamma$ is Moufang and $G$ contains its little projective group.

Proof. Let $g \in V \cap G_{x_{11}} \setminus G_y$ for some $y \in \Gamma(x_0)$, and let $h \in G_{x_5}^{[1]}$ with $x_1^h = y$. Then $1 \neq [h, g] \in G_{x_{11},x_2,x_3}$, and we are done by Proposition 5.7. $\square$

Lemma 5.9. The group $V = G_{x_{11},x_2}^{[1]} \cap G_{x_4,x_5}$ cannot act freely on $(\Gamma(x_0) \setminus \{x_1\}) \cup (\Gamma(x_3) \setminus \{x_4\})$.

Proof. Suppose towards a contradiction that $V$ acts freely on $(\Gamma(x_0) \setminus \{x_1\}) \cup (\Gamma(x_3) \setminus \{x_4\})$. Then we may apply Lemma 2.3 to see that $[V, V] \leq \bigcap_{y \in \Gamma(x_2) \setminus \{x_1\}} G_{x_2,x_3,x_5}^{[1]} \cap G_{x_4,x_5}$.

If $V$ is not abelian, let $\alpha \in [V, V] \setminus \{1\}$ and $\beta \in G_{x_8,x_9,x_{10}} \cap \bigcap_{y \in \Gamma(x_2) \setminus \{x_1\}} G_{x_2,x_3,x_5}^{[1]} \setminus \{1\}$. Since $|\Gamma(x_0)| \geq 4$, we have $[\alpha, \beta] \in G_{x_{11},x_0,x_1,x_2,x_3}^{[1]}$ for all $x_1' \in \Gamma(x_0) \setminus \{x_1, x_1'\}$ and all $x_3' \in \Gamma(x_2) \setminus \{x_3, x_5\}$. By the assumption on $V$ and Proposition 5.7 we thus have $[\alpha, \beta] = 1$, and hence $\beta \in G_{x_8,x_9,x_{10}} \cap G_{x_0,x_1}^{[1]}$ by Lemma 2.2(i), contradicting our assumption on $V$ and Proposition 5.7.
Thus, we may assume that $V$ is abelian. Let $\alpha \in V$, $\beta \in G^{[1]}_{x_0, x_1} \cap G_{x_0, x_9}$. By Lemma 5.6 we have $[\alpha, \beta] \in G_{x_1}^{[1]}$.

If $[\alpha, \beta] = 1$, then by Lemma 2.2 $\alpha \in G^{[1]}_{x_0, x_2} \cap G_{x_4, x_5, x_4, x_5}$. By assumption, $\alpha \not\in G_{x_1}$.

Let $g \in G_{x_3}^{[1]} \cap G_{x_5}$ with $x_1^g = x_5^g$. By Lemma 2.2(ii), we thus have $[g, \alpha] \not\in G_{x_1}$. Hence $1 \neq [\alpha, g] \in G_{x_1}^{[1]}$. If $\gamma = [\alpha, g] \in G_{x_1, x_2, x_3}$, we again contradict our assumption on $V$ and Proposition 5.7. If $\gamma \not\in G_{x_1}^{[1]}$, say $\gamma \not\in G_\gamma$ for some $\gamma \in \Gamma_1(x_1)$, let $h \in G_{x_6}^{[1]}$ with $x_0^h = y$. Then $[\gamma, h] \in G_{x_2, x_5, x_6} \setminus \{1\}$, contradicting once again our assumption on $V$ and Proposition 5.7.

So we have $[\alpha, \beta] \in G_{x_1}^{[1]} \setminus \{1\}$, and thus $\Gamma$ is Moufang and $G$ contains its little projective group by Lemma 5.5, again contradicting our assumption on $V$ and Proposition 5.7. □

**Theorem 5.10.** If $\Gamma$ is a 2-Moufang hexagon, then $\Gamma$ is Moufang and $G$ contains its little projective group.

**Proof.** The group $V = G_{x_1, x_2}^{[1]} \cap G_{x_4, x_5}$ cannot act freely on $(\Gamma_1(x_0) \setminus \{x_1\}) \cup (\Gamma_1(x_5) \setminus \{x_4\})$. Hence either $G_{x_1, x_2}^{[1]} \cap G_{x_4, x_5, x_6} \neq 1$ or $G_{x_1, x_2}^{[1]} \cap G_{x_1, x_4, x_5} \neq 1$. In the first case we are done by Proposition 5.7. In the second case, we are done by Lemma 5.8. □

For projective planes it is easily seen that any 2-Moufang projective plane is in fact Moufang (see also [10, Theorem 6.8.5]). Namely, fix any path $(x_0, x_1, x_2, x_3)$. Let $p, q \in \Gamma_1(x_0) \setminus \{x_1\}$ and let $\alpha \in G_{x_2}^{[1]} \cap G_{x_0}$ with $p^\alpha = q$. It is well-known (see e.g. [2, Theorem 4.9]) that there is some $x \in \Gamma_1(x_0)$ such that $\alpha \in G_{x_0}^{[1]}$. If $x = x_1$, then $\alpha$ is the required elation. Otherwise, let $\beta \in G_{x_2}^{[1]} \cap G_{x_5}$, such that $x_1^\beta = p$. Then we have $\gamma = [\beta, \alpha] \in G_{x_1, x_2}^{[1]}$ with $p^\gamma = q$, proving the Moufang condition.

We have thus established the following theorem stated in the abstract:

**Theorem 5.11.** Let $\Gamma$ be a generalized $n$-gon with $n \leq 6$. If $\Gamma$ is 2-Moufang with respect to $G$, then $n = 3, 4$ or $6$, $\Gamma$ is Moufang and $G$ contains its little projective group.

**Acknowledgment**

The author was supported by a Heisenberg-Stipendium.

**References**


