Unconditionally Stable Numerical Method for a Nonlinear Partial Integro-Differential Equation

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Abstract

The paper presents an unconditionally stable numerical scheme to solve a nonlinear integro-differential equation which arises in mathematical modelling of the penetration of a magnetic field into a substance, if the temperature is kept constant throughout the material. Numerical scheme comprises of the Galerkin finite element method [18] for the spatial discretization followed by an implicit finite difference scheme for the time stepping. We extended the results for stability estimates to a nonhomogeneous problem and derived optimal order error estimates for the semidiscretized and fully discretized equations using $H^1_0$ projection. Further, to show the efficiency, the proposed numerical method is demonstrated via numerical example.

Keywords: Nonlinear Integro-Differential equations; Galerkin finite element method; Implicit method.

1 Introduction.

The process of penetration of a magnetic field into a substance generates a variable electric field causing the currents to appear that leads to the heating which further influences the resistance of the material. The process of propagation of a magnetic field into a substance and change in temperature due to Joule heating into a medium whose electric conductivity substantially depends on temperature are mathematically modeled by the following (Maxwell’s) system of partial differential equations [1]:

\[ \frac{\partial H}{\partial t} = -rot(\nu_m rotH), \quad (1.1) \]

\[ c_v \frac{\partial \theta}{\partial t} = \nu_m (rotH)^2, \quad (1.2) \]

where $H$ represents the magnetic field intensity vector $(H_1, H_2, H_3)$, $\theta$ is the temperature, $c_v$ and $\nu_m$ characterize the heat capacity and the electric conductivity of the material. In case, $c_v$ and $\nu_m$ are the functions of temperature $\theta$, then on integrating (1.2) with respect to time $t$ and substituting the resulting equation in (1.1), the system obtained has the form

\[ \frac{\partial H}{\partial t} = -rot \left[ a \left( \int_0^t |rotH|^2 \, d\tau \right) rotH \right], \quad (1.3) \]

where $a=a(s)$ is defined for $s \in (0, \infty)$. For a planar magnetic field $H=(0,0,u)$, where $u=u(x,t)$. Then, $rotH = (0, -\frac{\partial u}{\partial x}, 0)$ and the system acquires the following form

\[ \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[ a \left( \int_0^t \left( \frac{\partial u}{\partial x} \right)^2 \, d\tau \right) \frac{\partial u}{\partial x} \right]. \quad (1.4) \]
For a system (1.3), if the temperature is kept constant throughout the material, the same process of penetration of a magnetic field into a substance can be rewritten in one dimensional analogue as (transformed by Laptev [2])

\[
\frac{\partial u}{\partial t} = a\left( \int_0^t \int_0^1 \left( \frac{\partial u}{\partial x} \right)^2 dx d\tau \right) \frac{\partial^2 u}{\partial x^2},
\]

(1.5)

Equations (1.3) and (1.4) originated in the article [3] where the existence of a weak solution to the first boundary value problem for the one dimensional spatial version for the case \(a(s) = (1+s)\) and uniqueness results for some general cases were proved. The existence and uniqueness of (1.5) has been proved in [4]. Lions [5] presented Galerkin methods and the methods based on compactness, monotonicity, regularization and successive approximations which were used to yield the existence theorems. Existence and uniqueness theorems based on the compactness methods [5] were also proved in the articles [3], [6].

In context of the theory of nonlinear integro-differential equations and systems (1.4) and 1.5, results concerning the large time behaviour, asymptotic behaviour of the solutions for the different cases of \(a(s)\) with different boundary conditions for the first boundary value problem and the initial boundary value problem were briefly discussed in [7], [8], [9], [10], [11], [12], [13], [14] and the references cited therein.

For the special case \(a(s) = (1 + s)\), study of the asymptotic behaviour of solutions (as \(t \to \infty\)) and the implementation of semidiscrete and the finite difference schemes for the model (1.5), with homogeneous Dirichlet boundary conditions was carried out in [7].

During the last decades, developing the finite difference and the finite element schemes for the parabolic integro differential equations (1.3), (1.4) and (1.5) have been the focus of intensive research. Construction and investigation of these discretization schemes were discussed by the authors (see [7], [15], [16], [17], [18], [19]) and the references therein. Recently in 2013, T.Jangveladze [19] discussed the properties of existence, uniqueness and asymptotic behaviour of the solutions of nonlinear integro-differential system (1.4).

Numerical methods for partial integro-differential equations constitute an indivisible part of modern engineering and science. Starting in 1962, Douglas and Jones [20] studied the numerical solution of parabolic integro-differential equations. The authors formulated backward difference and Crank Nicholson type methods for nonlinear parabolic integro-differential equations in one space variable subject to homogeneous Dirichlet’s boundary conditions. A somewhat similar hyperbolic integro-differential equations were also treated. Several authors examined various properties of both linear and nonlinear parabolic partial integro-differential equations in \(n\) space variables in papers presented at the conference “Integro-differential Evolution Equations and Applications, Vol.10, 1985 ” held in Trento, Italy, in 1984 and published in the Journal of Integral Equations. Later in 1988, Yanik and Fairweather [21] presented a fully discrete Galerkin finite element approximation to the solutions of a certain parabolic and hyperbolic partial integro-differential equations. Study of different types of partial integro-differential equations seem to have grown exponentially in the last few decades; so that a tremendous variety of models have now been formulated, mathematically analyzed and applied to respective areas. The breadth of the numerical analysis of various parabolic integro-differential equations is shown in the reviews of the literature([21], [22], [23], [24], [25], [26]) and the references cited therein.

In 2011, Jangveladze et.al [18] developed a conditionally stable numerical solution of (1.5) (for the special case \(a(s)=1+s\)) using the Galerkin finite element method and an explicit finite difference scheme. Recently in 2013 the work is extended to a system of integro-differential equations of the form (1.4) [19]. The authors analyzed the spatial discretization and derived error estimates for semidiscretization in the energy norm.
The goal of the present work is to obtain an unconditionally stable numerical scheme and to analyze the semidiscretize and fully discretize equations. Projection plays an important role in the derivation of optimal error estimates for the finite element approximations. The concept of projection will be seen to unify much of the analysis and the estimates are derived in an easy way.

A brief outline of this paper is as follows. In section 2, we recall the basic definitions which are used in our analysis. In section 3, we obtain the stability estimates for the non homogeneous problem, we also recall the weak formulation of the initial boundary value problem. In the subsections, spatial discretization of the concerned equation with the Galerkin finite element scheme is studied to derive an integro-differential equation system. \( H_0^1 \) projection of \( u \) in \( S^h \) is introduced and studied. We also analyze the semidiscrete error bounds. Implicit finite difference method is applied for a time discretization. Analogous stability estimates and the error estimates are also obtained for the fully discrete scheme. In section 4, the discussion continues with the construction of a numerical scheme to obtain the solution of an integro-differential equation system. In the last section, we present numerical results to illustrate the efficiency and the unconditional stability of the proposed method.

2 Preliminaries

Let \( L^2([0,1]) \) be the standard space of square integrable functions on \([0,1]\) with inner product \( <\cdot,\cdot> \) defined by \( <u,v>=\int_0^1 uvdx \) and norm \( ||u||=<u,u>^{1/2} \).

For \( m \) a nonnegative integer, \( H^m([0,1]) \) denotes the Sobolev space on \([0,1]\) with the norm \( ||v||_m = \left( \sum_{0\leq\alpha\leq m} ||\partial_x^\alpha v||^2 \right)^{1/2} \).

Further for \( v: [0,T] \to H^m([0,1]) \), define the norm \( ||\cdot||_{L^2(0,T,H^m([0,1]))} \) and \( ||\cdot||_{L^\infty(0,T,L^2([0,1]))} \) by

\[
||v||_{L^2(0,T,H^m([0,1])))} = \left( \int_0^T ||v(t)||^2_m dt \right)^{1/2},
\]

and \( ||v||_{L^\infty(0,T,L^2([0,1])))} = \sup_{0\leq t\leq T} ||v(t)|| \), respectively. We shall also consider the space of functions \( H_0^1([0,1]) = \{ u; u(0) = u(1) = 0, ||u||_1 < \infty \} \).

The following results will be used frequently in the paper

- for \( a, b > 0 \)
  \[
  ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}, \quad \epsilon > 0
  \]
  and \( a^2 + b^2 \leq (a + b)^2 \).

- **Gronwall’s Lemma in continuous form**: Let \( \alpha, \beta \) and \( \zeta \) be real valued functions on \( I = [a, \infty) \). Assume that \( \beta \) and \( u \) are continuous functions. If \( \beta \) is nonnegative, \( \alpha \) is non decreasing and if \( \zeta \) satisfies the integral inequality
  \[
  \zeta(t) \leq \alpha(t) + \int_a^t \beta(s)\zeta(s)ds, \quad \forall \ t \in I,
  \]
  then
  \[
  \zeta(t) \leq \alpha(t)exp \left( \int_a^t \beta(s)ds \right).
  \]
Gronwall’s lemma in the discrete form: If \( \zeta_n \geq 0, \alpha_n \geq \alpha_{n-1}, \beta_j \geq 0 \) and

\[
\zeta_n \leq \alpha_n + \sum_{j=0}^{n-1} \beta_j \zeta_j, \quad n \geq 0,
\]

then

\[
\zeta_n \leq \alpha_n \exp \left( \sum_{j=0}^{n-1} \beta_j \right).
\]

C will be used to denote various positive constants or finite combination of positive constants that are independent of \( h \) and \( \Delta t \) unless otherwise stated.

3 Abstract Variational Problem

In this paper, we consider a time dependent parabolic integro differential equation of the form (1.5) for the special case \( a(s) = (1 + s) \) on a bounded domain \( \Omega \)

\[
\begin{align*}
  u_t(x, t) &= (1 + s(t)) \Delta u(x, t) + f(x, t), \quad (x, t) \in \Omega = [0, 1] \times [0, T], \\
  u(0, t) &= u(1, t) = 0, \quad t \geq 0, \\
  u(x, 0) &= u_0(x), \quad x \in [0, 1],
\end{align*}
\]

where \( s(t) = f^t \int_0^1 (\frac{\partial u}{\partial x})^2 dx d\tau = \int_0^t ||\nabla u||^2 d\tau \) and \( u(x, t) \) is a real valued function in \( \Omega \).

The exact solution \( u \) satisfies the regularity (as cited in [7])

\[
\begin{align*}
  &u_t, u_x, u_{xx}, u_{xt} \in C^0([0, T], L^2([0, 1])), \\
  &u_{tt} \in L^2(0, T, L^2([0, 1])).
\end{align*}
\]

In 2009, T. Zhangveladze et.al [7] have proved the stability estimates of the problem (3.1)-(3.3) for the homogeneous case \((f = 0)\) where the authors derived the exponential stabilization of the solution of the problem in the norm of the space \( H^1([0, 1]), C^1([0, 1]) \) under appropriate assumptions on the initial function. We borrow the ideas from [7] to extend the results for non-homogeneous case when \( f \neq 0 \).

**Lemma 3.1** Let \( u \) be the solution of (3.1), if \( u_0 \in H^1_0([0, 1]), f, f_x \in L^2(0, T, L^2([0, 1])) \), then the following stability estimate holds in \( H^1([0, 1]) \) norm

\[
||u(t)|| + ||u_x(t)|| \leq A(t), \quad 0 \leq t \leq T,
\]

where \( A(t) = \exp^{-t/2} \left( c_1 \int_0^t \exp^{\tau} ||f||^2 d\tau + c_2 \right)^{1/2} \), where \( c_1 \) and \( c_2 \) are arbitrary constants.

**Proof.** Taking inner product of (3.1) with \( u \) and \( u_{xx} \), using the fact \((1 + s(t)) \geq 1\) and Cauchy Schwarz inequality for the right hand side, we obtain

\[
\begin{align*}
  \frac{1}{2} \frac{d}{dt} ||u||^2 + ||\nabla u||^2 &\leq ||f|| ||u||, \\
  \frac{1}{2} \frac{d}{dt} ||u_x||^2 + ||u_{xx}||^2 &\leq ||f_x|| ||u_x||,
\end{align*}
\]

and
respectively. Combining the inequalities (3.5), (3.6), using the inequality $ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}$, $\epsilon > 0$ and the Poincare’s inequality, we obtain

$$\exp^t \frac{d}{dt} \left(\|u\|^2 + \|u_x\|^2\right) + \exp^t \left(\|u\|^2 + \|u_x\|^2\right) \leq C \exp^t \left(\|f\|^2 + \|f_x\|^2\right), \ C > 0$$

(3.7)

which proves (3.4). □

**Lemma 3.2** Let $u$ be the solution of (3.1), if $u_0 \in H^4([0,1]) \cap H^1_0([0,1])$, $f_t \in L^2(0,t,L^2([0,1]))$, then $u_t$ is bounded in $L^2$ norm as

$$\|u_t\| \leq B(t),$$

where $B(t) = \exp^{-t/2} \left(c_1 \left(\int_0^t \exp^\tau (A(\tau))^6 d\tau + \int_0^t \exp^\tau \|f_t\|^2 d\tau\right) \right)$, $A(t)$ is as defined in Lemma 3.1 and $c_1$ and $c_2$ are arbitrary constants.

**Proof.** Differentiating (3.1) with respect to $t$ and then taking inner product with respect to $u_t$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u_t\|^2 + (1 + s(t)) \|u_{xt}\|^2 = -\|u_x\|^2 < u_x, u_{xt} > + < f_t, u_t >,$$

c.f. Lemma 2.2(as in [7]) and using Cauchy Schwarz inequality and Poincare’s inequality, we obtain

$$\frac{d}{dt} \|u_t\|^2 + \|u_{xt}\|^2 \leq (1 + s(t))^{-1} (s'(t))^2 \|u_x\|^2 + c \|f_t\|^2, \ c > 0$$

and

$$\frac{d}{dt} \|u_t\|^2 + \|u_t\|^2 \leq (1 + s(t))^{-1} (s'(t))^2 \|u_x\|^2 + c \|f_t\|^2, \ c > 0$$

which implies

$$\frac{d}{dt} (\exp^t \|u_t\|^2) \leq \exp^t (s'(t))^2 \|u_x\|^2 + \exp^t c \|f_t\|^2.$$

Using the estimates (3.4) c.f. Lemma 3.1., we obtain (3.8) which completes the proof. □

Using Lemma 3.2 the following estimate for $u_x$ in $C^1([0,1])$ can be easily obtained.

**Lemma 3.3** Let $u$ be the solution of (3.1), if $u_0 \in H^4([0,1]) \cap H^1_0([0,1])$, $f, f_t \in L^2(0,t,L^2([0,1]))$, then $u_x$ is bounded as

$$\|u_x\| \leq C(B(t) + \|f\|),$$

(3.9)

where $B(t)$ is defined as in Lemma 3.2.

**Proof.** As in [7](c.f. Theorem 2.2)

$$\|u_{xx}\| \leq C(B(t) + \|f\|)$$

which implies $\|u_x\| \leq C(B(t) + \|f\|), \ C > 0.$ □

**Remark 3.4.** If we denote $g(u_x) = u_x^2(x,t)$, then $g$ satisfies the Lipschitz’s condition since $g_{u_x} = 2u_x$ which is bounded by Lemma 3.3. Thus we assume that there exists a constant $K > 0$ for all $(x,t) \in [0,1] \times [0,T]$ such that

$$\left|\frac{\partial g}{\partial u_x}\right| \leq K.$$  

(3.10)
We will make use of the inequality (3.10) in proving the main results of the paper.

**Weak Formulation (c.f. T.Jangveladze [18]):** Forming $L^2$ inner product between (3.1), (3.3) and $v \in H^1_0([0, 1])$, and using Gauss divergence theorem, we have the weak formulation: For $f \in L^2(0, T, L^2([0, 1]))$, seek a function $u \in L^2(0, T, H^2([0, 1]) \cap H^1_0([0, 1]))$ such that for all $v \in L^2(0, T, H^2([0, 1]) \cap H^1_0([0, 1]))$ it satisfies

\[
< v, u_t > + < (1 + s(t)) \nabla u, \nabla v > = < f, v > \tag{3.11}
\]

and the initial condition

\[
< u(x, 0), v > = < u_0(x), v > . \tag{3.12}
\]

### 3.1 Finite Element Space Semidiscretization.

Let $\{S^h\}$ be a family of finite dimensional subspaces of $H^1_0([0, 1])$ with a discretization parameter $h$ and let $\varphi_1(x), \varphi_2(x), \ldots, \varphi_N(x)$ be a basis for the trial space $S^h$, then the continuous time Galerkin formulation (c.f. T.Jangveladze [18]) is to seek an approximate solution $u^h(x, t) = \sum_{i=1}^{N} \varphi_i(x) u_i(t) \in S^h$ such that it satisfies

\[
< v^h, u^h_t > + < (1 + s^h(t)) \nabla u^h, \nabla v^h > = < f, v^h > \quad \forall v^h \in S^h \tag{3.13}
\]

and

\[
u^h(0) = u^h_0, \tag{3.14}
\]

where $u^h_0$ is an approximation of $u_0$ in $S^h$ and $s^h(t) = \int_0^t ||\nabla u^h||^2 dt$.

T.Jangveladze et.al. [18] have derived error estimates in the energy norm for the semidiscrete solution (3.13)-(3.14) and we recall the main result [18]

**Theorem** Let $u, u^h$ satisfies (3.1)-(3.3) and (3.13)-(3.14), respectively, then

\[
|||u - u^h|||_1 \leq h^{j-1} C \left\{ \frac{1}{2} h^2 ||u||^2 + Ch^2 ||u||^2 + C \left[ 1 + h^{2(j-1)} ||u||^2 \right] \right\} \frac{1}{2}, \tag{3.15}
\]

where the notations $[\cdot], ||\cdot||$, $|||\cdot|||$ are as defined in the reference [18].

Our first purpose is to provide an alternative approach for deriving the error estimates for the semidiscretization with the aid of projection. The discussion and use of $H^1_0$ projection will rely on the work on Wheeler [22]. We will see that the projection will simplify the analysis. Let $v$ be a smooth function on $[0, 1]$ which vanishes at the boundary. For its nodal interpolant $I^h v \in S^h$, the following error estimates are well known, namely, for $v \in H^2 \cap H^1_0$,

\[
|||\nabla(I^h v - v)||| \leq Ch ||v||_2 \tag{3.16}
\]

and

\[
||I^h v - v|| \leq Ch^2 ||v||_2. \tag{3.17}
\]

Motivating from [22], we define a projection $T^h u(t) : H^1_0 \rightarrow S^h$ defined by

\[
< (1 + s(t)) \nabla u, \nabla \chi > = < (1 + s(t)) \nabla T^h u, \nabla \chi >, \quad \forall \chi \in S^h, t \geq 0. \tag{3.18}
\]

Using the intermediate projection (3.18), we shall write the error as $e = u^h - u = \theta + \rho$, where we define $\theta := u^h - T^h u$ and $\rho := T^h u - u$.

We shall first derive various estimates for the projection error $\rho$ of our problem which contains nonlinearity in the form of an integral term $s(t)$. We shall follow standard arguments for proving
Lemma 3.1.1. Assume that $u \in H^2 \cap H^1_0$ and $u$ be appropriately smooth. Then
\begin{align*}
||\rho|| &\leq C h^2 ||u||_2 \quad (3.19) \\
\text{and} \quad ||\nabla \rho|| &\leq C h ||u||_2, \quad (3.20)
\end{align*}
where $C$ is a positive constant depending on $u$ and its partial derivative with respect to $t$.

**Proof.** For $\chi \in S^h$, since $s(t) \geq 0$, using the coercivity condition and (3.18), it follows that
\begin{align*}
||\nabla \rho||^2 &\leq < (1 + s(t))\nabla \rho, \nabla \rho > \\
&= < (1 + s(t))\nabla \rho, \nabla (\chi - u) >.
\end{align*}

Since, $u \in L^2(0, T, H^1_0([0, 1]))$, we have $1 + s(t) \leq M$, where $M$ is a positive constant. For the other estimate, we solve the problem by duality. Consider
\begin{equation}
(1 + s(t))\Delta \psi = \phi \text{ in } (0, 1) \text{ and } \psi(0, t) = \psi(1, t) = 0. \quad (3.21)
\end{equation}
Taking inner product with $\psi$, using the boundedness of $s(t)$ and Poincare’s inequality we obtain
\begin{equation*}
||\nabla \psi|| \leq C ||\phi||.
\end{equation*}

Now,
\begin{equation*}
||\psi||_2 \leq c ||\Delta \psi|| \leq c ||(1 + s(t))\Delta \psi|| \leq c ||\phi||.
\end{equation*}
Taking inner product of (3.21) with $\rho$ and using (3.18), we have
\begin{equation*}
< \phi, \rho > = < (1 + s(t))\nabla \rho, \nabla (\psi - \chi) >, \quad \forall \chi \in S^h.
\end{equation*}
With $\chi = I^h \psi$ and using the estimates (3.16) and (3.20) we obtain (3.19) which completes the proof. \hfill \Box

Lemma 3.1.2. Assume that $u(t), u_t(t) \in H^2([0, 1]) \cap H^1_0([0, 1])$, then
\begin{align*}
||\rho_t(t)|| &\leq C h^2 \quad (3.22) \\
\text{and} \quad ||\nabla \rho_t|| &\leq C h, \text{ for } t \in [0, T], \quad (3.23)
\end{align*}
where $C$ depends on $u$ and $u_t$.

**Proof.** Differentiating (3.18) with respect to $t$, we obtain
\begin{equation}
< s'(t)\nabla \rho, \nabla \chi > + < (1 + s(t))\nabla \rho_t, \nabla \chi > = 0, \quad \forall \chi \in S^h. \quad (3.24)
\end{equation}
By definition
\begin{equation}
s'(t) = \int_0^1 u_x^2(t) dx - \int_0^1 u_x^2(0) dx, \quad (3.25)
\end{equation}
which is bounded. Using (3.18) and (3.24), we have for $\mu > 0$
\begin{align*}
\mu ||\nabla \rho_t||^2 &\leq < (1 + s(t))\nabla \rho_t, \nabla \rho_t > \\
&= < (1 + s(t))\nabla \rho_t, \nabla (\chi - u_t) > + < s'(t)\nabla \rho, \nabla (\chi - T^h u_t) > \\
&\leq C ||\nabla \rho_t|| ||\nabla (\chi - u_t)|| + ||\nabla \rho|| (||\nabla (\chi - u_t)|| + ||\nabla (u_t - T^h u_t)||),
\end{align*}
where $C$ is a positive constant. With $\chi = I^h u_t$, the estimate (3.23) is proved from the estimates (3.16) and (3.20), respectively.

As before, again by the duality argument, taking inner product of (3.21) with $\rho_t$ and using (3.18) and (3.24), we have

\[
<\rho_t, \phi> = <(1 + s(t))\nabla \rho_t, \nabla \phi>
\]
\[
= <(1 + s(t))\nabla \rho_t, \nabla (\psi - \chi) + > + s'(t)\nabla \rho, \nabla (\psi - \chi) - <s'(t)\nabla \rho, \nabla \psi>
\]
\[
= <(1 + s(t))\nabla \rho_t, \nabla (\psi - \chi) + > + s'(t)\nabla \rho, \nabla (\psi - \chi) + + s'(t)\rho, \Delta \psi>.
\]

Set $\chi = I^h \psi$. Using the boundedness of $s(t)$ and the bounds (3.16), (3.19), (3.20), (3.23), (3.25) and the equation (3.21), we obtain (3.22) which concludes the proof of Lemma.

**Lemma 3.1.3.** For $T^h u$ satisfying (3.18), there exists a positive constant $C = C(u)$ such that

\[
||\nabla T^h u|| \leq C(u).
\]

**Proof.** The assertion follows easily by setting $\chi = T^h u$ in the estimate (3.18).

Next, we apply the estimates of Lemmas to derive an optimal order error estimate for the time dependent problem.

**Theorem 3.1.4.** Let $u^h$ be the solution of (3.13). Further, assume that $u$ satisfies the following regularity

\[
u \in L^\infty(0, T, H^1_0([0, 1]) \cap H^2([0, 1]),
\]
\[
u \in L^2(0, T, H^1_0([0, 1]) \cap H^2([0, 1])
\]
\[
\text{and} \quad u^h \in L^2(0, T; H^2([0, 1]),
\]

then there is a constant $C$ depending on $u$ and independent of $h$ such that

\[
||u - u^h||_{L^\infty(0, T, L^2([0, 1]))} + ||u - u^h||_{L^2(0, T, H^1_0([0, 1]))} \leq C(u)(||u^h(0) - u_0|| + h).
\]

**Proof.** Subtracting (3.18) from (3.13) yields

\[
<\theta_t, \chi> + (1 + s^h(t))\nabla \theta, \nabla \chi = <f, \chi> - <T^h u_t, \chi> - <(1 + s^h(t))\nabla T^h(u), \nabla \chi>
\]
\[
+ <(1 + s(t))\nabla T^h(u), \nabla \chi>
\]
\[
- <(1 + s(t))\nabla u, \nabla \chi>, \quad \forall \chi \in S^h.
\]

Making use of (3.11), we obtain an error equation in $\theta$ as follows

\[
<\theta_t, \chi> + (1 + s^h(t))\nabla \theta, \nabla \chi>
\]
\[
= <-\rho_t, \chi> + <(s(t) - s^h(t))\nabla T^h u, \nabla \chi>,\quad \forall \chi \in S^h.
\]

Since $\theta \in S^h$, set $\chi = \theta$ so that

\[
<\theta_t, \theta> + (1 + s^h(t))\nabla \theta, \nabla \theta>
\]
\[
= <-\rho_t, \theta> + <(s(t) - s^h(t))\nabla T^h(u), \nabla \theta>.
\]

The left hand side is bounded as

\[
<\theta_t, \theta> + (1 + s^h(t))\nabla \theta, \nabla \theta> \geq \frac{1}{2} \frac{d}{dt} ||\theta||^2 + \mu ||\nabla \theta||^2, \quad \text{for } \mu > 0.
\]
For the right hand side of (3.28), we use the Cauchy Schwarz inequality to have
\[ | < -\rho_t, \theta > + < (s(t) - s^h(t))\nabla T^h(u), \nabla \theta > | \leq c( ||\rho_t|| ||\theta|| + |s(t) - s^h(t)|| ||\nabla T^h(u)|| ||\nabla \theta|| ). \]

(3.29)

Note that by hypothesis (3.10)
\[ s(t) - s^h(t) = \int_0^t \int_0^1 g(\nabla u) - g(\nabla u^h) dx d\tau = \int_0^t \int_0^1 \int_0^1 \frac{\partial g}{\partial u_x} (\theta \nabla u + (1 - \theta) \nabla u^h) d\theta (\nabla u - \nabla u^h) dx d\tau \leq K \int_0^t \int_0^1 \nabla e dx d\tau \]
\[ \leq K \int_0^t \int_0^1 (\int_0^1 \nabla e^2 dx) \frac{1}{2} d\tau \]
\[ = K \int_0^t ||\nabla e|| d\tau. \]

(3.30)

Using the bound (3.26) cf. Lemma 3.1.3 and substituting (3.30) in (3.29), we obtain
\[ | < -\rho_t, \theta > + < (s(t) - s^h(t))\nabla T^h(u), \nabla \theta > | \leq C ||\rho_t|| ||\theta|| + C(u)K \int_0^t ||\nabla e|| d\tau ||\nabla \theta|| \]
\[ \leq \frac{\mu}{2} ||\nabla \theta||^2 + c( ||\rho_t||^2 + ||\theta||^2 + \epsilon \int_0^t ||\nabla e||^2 d\tau ), \epsilon > 0. \]

(3.31)

Thus we obtain
\[ \frac{1}{2} \frac{d}{dt} ||\theta||^2 + \frac{\mu}{2} ||\nabla \theta||^2 \leq c( ||\rho_t||^2 + ||\theta||^2 + \epsilon \int_0^t ||\nabla e||^2 d\tau ), \mu > 0. \]

Integrating from 0 to \( t' \), \( t' \in [0, T] \), since
\[ \int_0^{t'} \int_0^t ||\nabla e||^2 d\tau dt \leq t' \int_0^{t'} ||\nabla e||^2 dt, \]
\[ \leq C( \int_0^{t'} ||\nabla \rho||^2 + ||\nabla \theta||^2 ) dt, \]
we get
\[ ||\theta(t')||^2 + \int_0^{t'} ||\nabla \theta||^2 dt \leq ||\theta(0)||^2 + C \int_0^{t'} ( ||\rho_t||^2 + ||\theta||^2 + ||\nabla \rho||^2 ) dt. \]

An application of Gronwall’s inequality yields
\[ ||\theta(t')||^2 + \int_0^{t'} ||\nabla \theta||^2 dt \leq C[ ||\theta(0)||^2 + \int_0^{t'} ||\rho_t||^2 dt + \int_0^{t'} ||\nabla \rho||^2 ]. \]

(3.32)
Now,
\[ ||\theta(0)|| \leq ||u^h(0) - u_0|| + ||u_0 - T^h u(0)|| \leq ||u^h(0) - u_0|| + Ch^2||u_0||_2. \] (3.33)

Incorporating the initial error estimates (3.33) in (3.32); together with the bounds (3.20) and (3.22) of Lemmas 3.1.1 and 3.1.2, we obtain
\[ ||\theta(t')||^2 + \int_0^{t'} ||\nabla \theta||^2 dt \leq C||u^h(0) - u_0||^2 + C(u)(h^4 + h^2). \]

Using \( a^2 + b^2 \leq (a + b)^2 \) and renaming of constants, we get
\[ ||\theta(t')||^2 + \int_0^{t'} ||\nabla \theta||^2 dt \leq C(||u^h(0) - u_0|| + c(u)h)^2, \quad t' \in [0, T], \]
which implies
\[ ||\theta(t)||_{L^\infty(0,T;L^2([0,1]))} + ||\theta(t)||_{L^2(0,T;H^1_0([0,1]))} \leq C(||u^h(0) - u_0|| + h). \] (3.34)

In view of Lemma 3.1.1 and using the triangular inequality, we obtain (3.27) which concludes the proof of the theorem.

Next, for the semidiscretization, the energy error is defined by \( ||u - u^h||_{L^2(0,T;H^1([0,1]))} \). The following inequality is obtained on using Poincare’s inequality
\[ ||u - u^h||_{L^2(0,T;H^1([0,1]))} \leq C||u^h||_{L^2(0,T;H^1_0([0,1])))}, \quad C > 0. \]

In view of the bound (3.20) in Lemma 3.1.1 and the bound (3.34) in the theorem, we obtain the energy error estimate using the triangular inequality summarized as follows.

**Corollary 3.1.5.** For \( ||u - u^h||_{L^2(0,T;H^1([0,1])),} \), the following holds
\[ ||u - u^h||_{L^2(0,T;H^1([0,1]))} \leq C(u)(h + ||u^h(0) - u_0||). \] □

### 3.2 Fully Discrete Scheme.

Introduce a partition \( \{0 = t_0, < t_1 < \ldots < T = t_J\} \) of the finite time interval \([0, T] \).

Set \( \Delta t = t_j - t_{j-1} \), where \( t_n = n\Delta t \) for \( n = 0, 1, \ldots J \). \( U^n \) approximates \( u(t_n) \) and we use the shorthand \( \partial_t U^n := \frac{U^n - U^{n-1}}{\Delta t} \) for a temporal derivative.

The fully discrete scheme consists of finding a sequence of functions \( U^n \in S^h \) such that, for each \( n = 0, 1, \ldots J \), we have
\[ < \partial_t U^n, \chi > + < \sum_{j=0}^{n-1} \Delta t w_j ||\nabla U^j||^2 + \Delta t ||\nabla U^{n-1}||^2 \nabla U^n, \nabla \chi > = < f^n, \chi >, \] (3.35)

with \( U^0 = u^h_0 \),

\[ w_j = \begin{cases} \frac{1}{2} & j = 0, n - 1 \\ 1 & 1 \leq j \leq n - 2. \end{cases} \] (3.36)

where \( w_j's \) represent the quadrature weights given by
Let $X$ be a Banach space and $v \in X$, we recall the following norms in discrete version.

\[
||v||^2_{L^2(0,T;\Delta t,X)} := \Delta t \sum_{n=0}^{J} ||v^n||^2_X,
\]

and \[
||v||_{L^\infty(0,T;\Delta t,X)} := \max_{0 \leq n \leq J} ||v^n||_X.
\]

The fully discrete scheme (3.35) is obtained by approximating the integral term $s_h(t_n)$ by the trapezoidal rule on $[0, t_{n-1}]$ and by the left rectangular rule on the proceeding interval $[t_{n-1}, t_n]$ to avoid the nonlinearity in numerical method.

We first establish stability estimate for the fully discrete approximation $U^n$.

**Lemma 3.2.1.** Let $U^m$ be a solution of (3.35). If $U^0 \in L^2([0,1])$ and $f \in L^2(0,T,\Delta t, L^2([0,1]))$, then the discrete solution is stable for $m = 1, 2 \ldots J$ in the following sense

\[
||U||_{L^\infty(0,T;\Delta t, L^2([0,1])))} + ||U||_{L^2(0,T; \Delta t, L^2([0,1])))} \leq C(||U^0|| + ||f||_{L^2(0,T; \Delta t, L^2([0,1])))}),
\]

(3.37)

**Proof.** Setting $\chi = U^n$ in (3.35), we get

\[
< \partial_t U^n, U^n > + \left( 1 + \sum_{j=0}^{n-1} \Delta t w_j ||\nabla U^j||^2 + \Delta t ||\nabla U^{n-1}||^2 \right) \nabla U^n, \nabla U^n > = < f^n, U^n >,
\]

which on using the coercivity and the Cauchy Schwarz inequality can be written as

\[
\frac{1}{2} \partial_t ||U^n||^2 + \mu ||\nabla U^n||^2 + \mu \left( \sum_{j=0}^{n-1} \Delta t w_j ||\nabla U^j||^2 + \Delta t ||\nabla U^{n-1}||^2 \right) ||\nabla U^n||^2 \leq ||f^n|| ||U^n||, \quad \mu > 0.
\]

Thus

\[
\frac{1}{2} \partial_t ||U^n||^2 + \mu ||\nabla U^n||^2 \leq ||f^n|| ||U^n||.
\]

For $\epsilon > 0$, by Young’s inequality, we have

\[
\frac{1}{2} \left( \frac{||U^n||^2 - ||U^{n-1}||^2}{\Delta t} \right) + \mu ||\nabla U^n||^2 \leq \epsilon ||U^n||^2 + \frac{||f^n||^2}{4\epsilon}.
\]

Taking summation from $n = 1$ to $m$, $1 \leq m \leq J$, we choose $\epsilon > 0$ and $\Delta t < 1$ so that $1 - 2\epsilon \Delta t > 0$ to obtain

\[
(1 - 2\epsilon \Delta t)||U^m||^2 + 2\mu \Delta t \sum_{n=1}^{m} ||\nabla U^n||^2 \leq C(||U^0||^2 + \Delta t \sum_{n=1}^{m} ||f^n||^2 + \Delta t \sum_{n=1}^{m-1} ||U^n||^2).
\]

Applying the discrete Gronwall’s inequality, for $\Delta t < 1$, we arrive at

\[
||U^m||^2 + \Delta t \sum_{n=1}^{m} ||\nabla U^n||^2 \leq C(||U^0||^2 + \Delta t \sum_{n=1}^{m} ||f^n||^2),
\]

which concludes the proof of the lemma. □
Next, our aim is to derive the error estimate for the full discretization which follows in a similar fashion as in the semi discrete case. We denote
\[ u^n = u(t_n) \]
\[ T^h u^n = T^h u(t_n). \]

We write
\[ u^n - U^n = (u^n - T^h u^n) + (T^h u^n - U^n) := \rho^n + \theta^n. \]

For the sake of simplicity, we introduce the following notations
\[ \varepsilon^n(u) := s(t_n) - \sum_{j=0}^{n-1} \Delta t w_j ||\nabla u^j||^2 - \Delta t ||\nabla u^{n-1}||^2. \]
\[ \zeta^n := \bar{\partial}_t T^h u^n - u^n. \]
\[ S^n(u) := \sum_{j=0}^{n-1} \Delta t w_j ||\nabla u^j||^2 + \Delta t ||\nabla u^{n-1}||^2. \]

We consider (3.18) at \( t = t_n \) and express it in terms of numerical quadrature as
\[ <(1 + S^n(u))\nabla T^h u^n, \nabla \chi> + <\varepsilon^n(u)\nabla T^h u^n, \nabla \chi> = <(1 + s(t_n))\nabla u^n, \nabla \chi>. \quad (3.38) \]

Subtracting (3.35) from (3.38) and using (3.11) we obtain an error equation in \( \theta \) given as
\[ <\bar{\partial}_t \theta^n, \chi> + <(1 + S^n(U))\nabla \theta^n, \nabla \chi> = <(S^n(U) - S^n(u))\nabla T^h u^n, \nabla \chi> - <\varepsilon^n(u)\nabla T^h u^n, \nabla \chi> + <\zeta^n, \chi>. \quad (3.39) \]

**Theorem 3.2.2** Let \( U^n \) be the solution of a fully discrete scheme (3.35). Let \( u \) satisfies the following regularity conditions
\[ u \in L^\infty(0, T; H^2([0, 1]) \cap H^1_0([0, 1])), \]
\[ u_t \in L^2(0, T; H^2([0, 1]) \cap L^\infty(0, T; L^\infty([0, 1])) \]
and \[ u_{tt} \in L^2(0, T; L^2([0, 1])). \]

Then, there is a constant \( C \) independent of \( h \) and \( \Delta t \) such that
\[ ||u - U||_{L^\infty(0, T; L^2([0, 1]))} + ||u - U||_{L^2(0, T; L^2([0, 1])))} \leq C(u)(h + \Delta t) + ||U^0 - u_0||. \quad (3.40) \]

**Proof.** Set \( \chi = \theta^n \) in (3.39). To bound the left hand side, we use \( \bar{\partial}_t \theta^n, \theta^n \geq \frac{1}{2} \bar{\partial}_t ||\theta^n||^2 \) and the coercivity \( \mu ||\nabla \theta^n||^2 \leq <\nabla \theta^n, \nabla \theta^n> \); and for the right hand side, using the Cauchy Schwarz inequality, we obtain
\[ \frac{1}{2} \bar{\partial}_t ||\theta^n||^2 + \mu ||\nabla \theta^n||^2 \leq ||\varepsilon^n(u)|| ||\nabla T^h u^n|| ||\nabla \theta^n|| + ||\zeta^n|| ||\theta^n|| + ||S^n(U) - S^n(u)|| ||\nabla T^h u^n|| ||\nabla \theta^n||. \]

Using the same arguments as we did in Theorem 3.1.5, we obtain
\[ \frac{||\theta^n||^2 - ||\theta^{n-1}||^2}{\Delta t} + \mu ||\nabla \theta^n||^2 \leq 3\epsilon ||\nabla \theta^n||^2 + \epsilon ||\theta^n||^2 + C_c(||\varepsilon^n(u)||^2 + ||\zeta^n||^2) + \Delta t \sum_{j=0}^{n-1} ||\nabla \bar{\theta}^j||^2, \quad \epsilon > 0. \]
Taking summation from \( n = 1 \) to \( J \), we obtain
\[
||\theta^J||^2 + (\mu - 3\epsilon)\Delta t \sum_{n=1}^{J} ||\nabla \theta^n||^2 \leq ||\theta^0||^2 + \epsilon \Delta t \sum_{n=1}^{J} ||\theta^n||^2 + C_\epsilon \Delta t \sum_{n=1}^{J} ||\varepsilon^n(u)||^2 + \sum_{n=1}^{J} ||\zeta^n||^2 + \Delta t \sum_{n=1}^{J} \sum_{j=0}^{n-1} ||\nabla \theta^j||^2 + \Delta t \sum_{n=1}^{J} \sum_{j=0}^{n-1} ||\nabla \rho^j||^2.
\]
(3.41)

Choose \( \epsilon' > 0 \), so that \( \lambda = \mu - 3\epsilon_1 - C_\epsilon > 0 \). Using the inequalities
\[
\Delta^2 t \sum_{n=1}^{J} \sum_{j=0}^{n-1} ||\nabla \theta^j||^2 \leq T \Delta t \sum_{n=0}^{J-1} ||\nabla \theta^j||^2
\]
and \( \Delta^2 t \sum_{n=1}^{J} \sum_{j=0}^{n-1} ||\nabla \rho^j||^2 \leq T \Delta t \sum_{n=0}^{J-1} ||\nabla \rho^j||^2 \),
the equation (3.41) can be written as
\[
(1 - \epsilon^0 \Delta t)||\theta^J||^2 + \lambda \Delta t \sum_{n=1}^{J} ||\nabla \theta^n||^2 \leq ||\theta^0||^2 + \epsilon \Delta t \sum_{n=1}^{J-1} ||\theta^n||^2 + C_\epsilon \Delta t W_n,
\]
where
\[
W_n = \sum_{n=1}^{J} ||\varepsilon^n(u)||^2 + \sum_{n=1}^{J} ||\zeta^n||^2 + \sum_{n=0}^{J-1} ||\nabla \rho^j||^2.
\]

For \( \Delta t < 1 \), we have
\[
||\theta^J||^2 + \lambda \Delta t \sum_{n=1}^{J} ||\nabla \theta^n||^2 \leq ||\theta^0||^2 + \epsilon \Delta t \sum_{n=1}^{J-1} ||\theta^n||^2 + C_\epsilon \Delta t W_n.
\]

Using discrete Gronwall’s inequality
\[
||\theta^J||^2 + \lambda \Delta t \sum_{n=1}^{J} ||\nabla \theta^n||^2 \leq ||\theta^0||^2 + C_\epsilon \Delta t W_n,
\]
(3.42)

\( ||\varepsilon^n(u)||^2 \) combines the quadrature errors associated with the composite Trapezoidal rule on the interval \([0, t_{n-1}]\) and the left rectangular rule on the interval \([t_{n-1}, t_n]\). Thus, it is bounded in the following way
\[
||\varepsilon^n(u)|| = \int_0^{t_n} ||\nabla u||^2 dt - \sum_{j=0}^{n-1} \Delta t w_j ||\nabla u^j||^2 - \Delta t ||\nabla u^{n-1}||^2
\]
\[
= \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} (s - t_j)(s - t_{j-1})(||\nabla u||^2)_{tt}(s) ds + \int_{t_{n-1}}^{t_n} (s - t_n)(||\nabla u||^2)_t(s) ds
\]
\[
\leq a(\Delta t)^2 \int_0^{t_n} (||\nabla u||^2)_{tt}(s) ds + b\Delta t \int_{t_{n-1}}^{t_n} (||\nabla u||^2)_t(s) ds
\]
\[
\leq C(u)\Delta t,
\]
(3.43)
where $C(u)$ is a constant depending on $u$. $(\cdot)_t$ and $(\cdot)_{tt}$ denote the first and second order partial derivatives respectively, with respect to $t$.

Analyzing the term $\zeta^n$, we rewrite it as $\zeta^n = \zeta_1^n + \zeta_2^n$, where $\zeta_1^n := \tilde{\partial}_t T^h u^n - \partial_t u^n$ and $\zeta_2^n := \partial_t u^n - u^n$. The term $\zeta_1^n$ is reformulated as follows which is bounded by using (3.22).

$$||\zeta_1^n|| = ||\tilde{\partial}_t T^h u^n - \partial_t u^n|| = ||\Delta t^{-1} \int_{t_{n-1}}^{t_n} (T^h u^n - u^n)_t(s) ds|| \leq C(u)h^2.$$ 

The second term is bounded by the Taylor’s expansion as follows

$$||\zeta_2^n|| = ||\partial_t u^n - u^n|| = ||\Delta t^{-1} \int_{t_{n-1}}^{t_n} (s - t_{n-1}) u_{tt}(s) ds|| \leq C(u)\Delta t.$$ 

Thus

$$||\zeta^n|| \leq C(u)(h^2 + \Delta t). \quad (3.44)$$

Note that

$$||\theta^0|| \leq ||U^0 - u_0|| + ||u_0 - T^h u(0)|| \leq Ch^2||u||_2 + ||U^0 - u_0||, \quad (3.45)$$

using the estimates (3.43), (3.44), (3.45) and (3.20) in (3.42), we conclude that

$$||\theta J||^2 + \Delta t \sum_{n=1}^{J} ||\nabla \theta^n||^2 \leq ||U^0 - u_0||^2 + C(u) (h + \Delta t)^2]. \quad (3.46)$$

Thus,

$$||\theta||_{L^\infty(0, T, \Delta t, L^2([0, 1]))} + ||\theta||_{L^2(0, T, \Delta t, H^1_0([0, 1])))} \leq C(u)((h + \Delta t) + ||U^0 - u_0||). \quad (3.47)$$

Now, using the triangular inequality and the error estimates (3.19) - (3.20) cf. Lemma 3.1.1, we obtain (3.40) which concludes the proof of the theorem. 

For the discrete energy error $||u - U||_{L^2(0, T, \Delta t, H^1(\Omega))}$ we have the following corollary.

**Corollary 3.2.3.** The discrete energy error as defined above, for some constant $C(u)$, satisfies

$$||u - U||_{L^2(0, T, \Delta t, H^1(\Omega))} \leq ||U^0 - u_0|| + C(u)(h + \Delta t). \quad (3.48)$$

**Proof.** By the discrete version of Poincare’s inequality

$$||u - U||_{L^2(0, T, \Delta t, H^1(\Omega))} \leq C||u - U||_{L^2(0, T, \Delta t, H^1_0(\Omega))}, \quad C > 0.$$ 

Using the bounds for $||\nabla \rho||$ (cf. estimate (3.20) in Lemma 3.1.1 ) and for $||\nabla \theta||$ (cf. estimate (3.46) in Theorem 3.2.2 ), we obtain the error estimate (3.48) using the triangular inequality. 

□
4 Numerical Computation.

Following the ideas from [18], we set \( \nu^h = \phi_k \) in (3.13) - (3.14) to obtain a system of \( N \) differential equations which can be written in the matrix notation as

\[
M\ddot{u} + k(u)u = F, \quad (4.1)
\]
\[
Mu(0) = I, \quad (4.2)
\]

here \( M = [m_{jk}]_{1 \leq j,k \leq N} \) is the mass matrix with elements \( m_{jk} = <\phi_k, \phi_j> \),

\[
k(u)_{jk} = \left< (1 + s^h(t))\phi'_k, \phi'_j \right>, \quad (4.3)
\]

where the evaluation of \( s^h(t) \) is given as follows

\[
s^h(t) = \int_0^t \int_0^1 \left( \sum_{l=1}^N u_l(\tau)\phi'_l(x) \right)^2 dx d\tau
\]
\[
= \int_0^t \int_0^1 \sum_{l=1}^N \phi'_l(x)\phi'_m(x) dx u_l(\tau) u_m(\tau) d\tau
\]
\[
= \sum_{l=1}^N \sum_{m=1}^N \int_0^t u_l(\tau) u_m(\tau) \left( \int_0^1 \phi'_l(x)\phi'_m(x) dx \right) d\tau,
\]

denoting \( \int_0^1 \phi'_l(x)\phi'_m(x) dx = \tilde{K}_{lm} \), the above equation becomes

\[
s^h(t) = \sum_{l=1}^N \sum_{m=1}^N \tilde{K}_{lm} \int_0^t u_l(\tau) u_m(\tau) d\tau, \quad (4.4)
\]

\( F = [F_j]_{1 \leq j \leq N}, \quad I = [I_j]_{1 \leq j \leq N} \) are the vectors with entries \( F_j = <\phi_j, f> \) and \( I_j = <\phi_j, u_0> \) respectively and \( u(t) \) is the vector of unknowns.

All these arrays have the same dimension as that of \( V^h \) i.e \( N \).

An approximation \( U(t + \Delta t) \) of the solution at the time \( (t + \Delta t) \) is calculated using an approximation \( u \) at the time \( t \) and time step \( \Delta t \) as

\[
U(t + \Delta t) = u(t) + \Delta t \dot{U}(t + \Delta t),
\]

multiplying both the sides by \( M \), we get

\[
MU(t + \Delta t) = Mu(t) + \Delta t M\dot{U}(t + \Delta t). \quad (4.5)
\]

To evaluate \( Mu(t) \) at \( t = t + \Delta t \), we take into account the equation (4.1) as

\[
M\dot{U}(t + \Delta t) = \left( F - k(u)U \right) \bigg|_{t+\Delta t} \quad (4.6)
\]

From (4.4), we have

\[
s^h(t + \Delta t) = \sum_{l=1}^N \sum_{m=1}^N \tilde{K}_{lm} \int_0^t+\Delta t u_l(\tau) u_m(\tau) d\tau.
\]
which can be written as
\[ s^h(t + \Delta t) = \sum_{l=1}^{N} \sum_{m=1}^{N} \widetilde{K}_{lm} \left( \int_{0}^{t} u_l(\tau)u_m(\tau)d\tau + \int_{t}^{t+\Delta t} u_l(\tau)u_m(\tau)d\tau \right). \]  

(4.7)

Substitution of (4.7) in equation (4.3) yields
\[ k(uk)|_{t+\Delta t} = \left\langle \left( 1 + \sum_{l=1}^{N} \sum_{m=1}^{N} \widetilde{K}_{lm} \left( \int_{0}^{t} u_l(\tau)u_m(\tau)d\tau + \int_{t}^{t+\Delta t} u_l(\tau)u_m(\tau)d\tau \right) \phi'_k, \phi'_j \right) \right\rangle. \]

Let us write
\[ u^T(t) \tilde{K} u(t) = v(t), \]  

(4.8)

then the above equation becomes
\[ k(uk)|_{t+\Delta t} = \left\langle \left( 1 + \int_{0}^{t} v(\tau)d\tau + \int_{t}^{t+\Delta t} v(\tau)d\tau \right) \phi'_k, \phi'_j \right\rangle \]
\[ = \left( 1 + \int_{0}^{t} v(\tau)d\tau + \int_{t}^{t+\Delta t} v(\tau)d\tau \right) \tilde{K}_{jk}. \]  

(4.9)

Incorporating (4.9) for \( k(u) \) in (4.6) and the resulting equation for \( \widetilde{M}\dot{U}(t + \Delta t) \) in (4.5). For a partition \( \varphi : \{0 = t_0 < t_1 < \ldots < T = t_J\} \) of the finite time interval \([0,T] \), \( \Delta t = t_n - t_{n-1} \), where \( t_n = n\Delta t \) for \( n = 0,1,\ldots, J \).

Numerical scheme starting at the initial condition \( M\dot{u}(0) = I \) creates an approximation of the exact solution in the form of a sequence of discrete states \( u^1, u^2, \ldots, u^J \) at \( t_1, t_2, \ldots, t_J \). The approximation \( u^n \) for \( 1 \leq n \leq J \) can be computed as follows.

From (4.9), letting \( t = t_{n-1} \), we employed the trapezoidal rule to approximate the first integral and the left hand rectangular rule for the second integral to obtain an approximation \( \tilde{k}(uk)|_{t_n} \) of \( k(uk)|_{t_n} \). Therefore,
\[ \tilde{k}(uk)|_{t_n} = \left( 1 + \sum_{p=0}^{n-1} \Delta tw_pv^p + \Delta tv^{n-1} \right) \tilde{K}_{jk}. \]  

(4.10)

Here, the weights \( w_p = 1/2 \) for \( p = 0, n-1 \) and \( w_p = 1 \) for \( p = 1,2,\ldots,n-2 \). \( v^p \) approximates \( v(t_p) \). For the sake of simplicity, denote \( (1 + \sum_{p=0}^{n-1} \Delta tw_pv^p + \Delta tv^{n-1} = \psi^{n-1} \) and \( F(t_n) = F^n \), then
\[ \tilde{k}(uk)|_{t_n} = \psi^{n-1} \tilde{K}_{jk}, \]  

(4.11)

which computes the numerical solution as
\[ Mu^n = Mu^{n-1} + \Delta t \left( F^n - \psi^{n-1} \tilde{K} u^n \right), \]  

so that
\[ u^n = \left( M + \Delta t\psi^{n-1} \tilde{K} \right)^{-1} \left[ Mu^{n-1} + \Delta t F^n \right]. \]

Remark 4.1. \( \psi^n > 0 \) for \( n = 0, 1, \ldots, J \).

Proof. Since \( \Delta t \) and \( w_p \) are positive, so enough to show the positivity of \( v^p \).

Consider \( v^p = w^T \tilde{K} w^p \), for \( p = 0, 1, \ldots, n \). Here \( w^p = [u^p_1, u^p_2, \ldots, u^p_N]^T \). Then
\[ v^p = \sum_{l=1}^{N} \sum_{m=1}^{N} \tilde{K}_{lm} u^p_l u^p_m = \sum_{l=1}^{N} \sum_{m=1}^{N} (\phi'_m, \phi'_l) u^p_l u^p_m. \]
Let $\sum_{i=1}^{N} \phi_i u_i^p = z^p$. Note that $\langle \cdot, \cdot \rangle$ is symmetric and bilinear, therefore $\langle z^p, z^p \rangle \geq 0$, which immediately yields $\psi^a > 0$. \hfill \Box

**Remark 4.2.** $(M + \Delta t \psi^{n-1} \tilde{K})$ is invertible

**Proof.** The proof follows from the contradiction. Let us assume that $(M + \Delta t \psi^{n-1} \tilde{K})$ is singular, then there exists a vector $Y = (y_1, y_2, \ldots, y_N)^T \in \mathbb{R}^N$, $Y \neq 0$, such that

$$(M + \Delta t \psi^{n-1} \tilde{K})Y = 0.$$ 

Then necessarily

$$Y^T(M + \Delta t \psi^{n-1} \tilde{K})Y = Y^TMY + Y^T\Delta t \psi^{n-1} \tilde{K}Y = 0,$$

which implies

$$\sum_{i=1}^{N} \sum_{j=1}^{N} m_{ij} y_i y_j + \Delta t \psi^{n-1} \sum_{i=1}^{N} \sum_{j=1}^{N} \tilde{K}_{ij} y_i y_j = \sum_{i=1}^{N} \sum_{j=1}^{N} \langle \phi_i, \phi_j \rangle y_i y_j + \Delta t \psi^{n-1} \sum_{i=1}^{N} \sum_{j=1}^{N} \langle \phi_i', \phi_j' \rangle y_i y_j = 0.$$ 

Let $\sum_{i=1}^{N} \phi_i y_i = z$ and $\sum_{i=1}^{N} \phi_i' y_i = z'$, then the above equation becomes

$$\langle z, z \rangle + \Delta t \psi^{n-1} \langle z', z \rangle = 0. \quad (4.12)$$

Since the linear independene of $\phi_i'$s guarantees $\langle z, z \rangle > 0$ and using Remark 4.1, it follows that $\langle z, z \rangle + \Delta t \psi^{n} \langle z', z \rangle > 0$ which contradicts $(4.12)$. \hfill \Box

### 5 Demonstration of the method.

In this section, we present some numerical results validating our theoretical analysis. In order to check the efficiency of the proposed method, for $h = 1/N, \Delta t = T/J$ the error of approximation

$$E(h, \Delta t) := u(nh, j\Delta t) - U(n, j), \quad 1 \leq n \leq N, 1 \leq j \leq J,$$

and the order of convergence $\gamma$ are tabulated, where $u$ is the exact solution and $U$ is the approximation at the grid point $x = nh$ and $t = j\Delta t$.

**Example** Consider (3.1) - (3.3) at $T = 0.5$ with the right hand side function

$$f(x, t) = x(1 - x)\cos(x + t) - \left(1 + \frac{11}{60} t - \frac{1}{8} \cos(t) \sin(t) - \frac{1}{8} \cos(1 + t) \sin(1 + t) + \frac{1}{8} \cos(1) \sin(1)\right) * \left(-2\sin(x + t) + 2(1 - x)\cos(x + t) - 2\cos(x + t) - x(1 - x)\sin(x + t)\right).$$

The exact solution of the above problem is given by

$$u(x, t) = x(1 - x)\sin(x + t).$$

Tables 1 and 2 tabulate the error of approximation and the order of convergence in $L^2(0, T, H^1([0, 1]))$ norm. In Table 1, we fix $\Delta t = 0.001$ and a moderate sequence of $h = 1/N$ is chosen. The order of convergence is given by

$$\gamma_1(h_i, h_{i+1}) = \log_2(E(h_i)/E(h_{i+1})), \quad 1 \leq i \leq 4,$$
Table 1: Error of Approximation and order of convergence.

<table>
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<tr>
<th>N</th>
<th>E(h)</th>
<th>γ</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>0.024484</td>
<td>0.999656</td>
</tr>
<tr>
<td>50</td>
<td>0.012245</td>
<td>0.999913</td>
</tr>
<tr>
<td>100</td>
<td>0.006122</td>
<td>0.999976</td>
</tr>
<tr>
<td>200</td>
<td>0.0030615</td>
<td>0.999987</td>
</tr>
<tr>
<td>400</td>
<td>0.0015307</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 2: Error of Approximation and order of convergence.

<table>
<thead>
<tr>
<th>J</th>
<th>E(Δt)</th>
<th>γ</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>0.13751e-003</td>
<td>0.94670</td>
</tr>
<tr>
<td>50</td>
<td>0.07134e-003</td>
<td>0.97338</td>
</tr>
<tr>
<td>100</td>
<td>0.03633e-003</td>
<td>0.98625</td>
</tr>
<tr>
<td>200</td>
<td>0.01834e-003</td>
<td>0.99212</td>
</tr>
<tr>
<td>400</td>
<td>0.00922e-003</td>
<td>–</td>
</tr>
</tbody>
</table>

where $E(h)$ is calculated in $L^2(0, T, H^1([0, 1]))$.

In Table 2 we fix $h = 0.001$ and a moderate sequence of $Δt$ is chosen. Here, the order of convergence is computed by

$$\gamma(Δt_i, Δt_{i+1}) = \log_2(E(Δt_i)/E(Δt_{i+1})), 1 \leq i \leq 4.$$ 

Tables 1 and 2 confirm the first order convergence of the proposed method in $L^2(0, T, Δt, H^1([0, 1]))$ norm in both space and time directions respectively.

6 Conclusion.

We study the numerical solution of a nonlinear parabolic integro-differential equation in bounded domain. Galerkin finite element method [7] and an implicit finite difference scheme are used to solve the problem. The method is analyzed for stability and convergence via the concept of projection which shows that the method is unconditionally stable with first order convergence in both space and time directions. We demonstrate the efficiency of the proposed method by displaying the numerical results which are in accordance with the theoretical analysis.

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References


