Some inequalities for the $q$-beta and the $q$-gamma functions via some $q$-integral inequalities

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**Abstract**

Some new inequalities for the $q$-gamma, the $q$-beta and the $q$-analogue of the Psi functions are established via some $q$-integral inequalities.

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1. Introduction

In classical analysis, integral inequalities have been well-developed and leading to a wide variety of applications in mathematics and physics (see [11–13] and references therein). In a survey paper [4], Dragomir et al. used certain clever integral inequalities to provide some interesting inequalities for the Euler’s beta and gamma functions. Interested by this type of inequalities, Agarwal et al. gave in [1] some improvements and generalizations of some of the Dragomir’s ones.

In quantum-calculus, in spite of the natural difficulties, coming from the definition of the $q$-Jackson integral, the interest in the $q$-integral inequalities has grown in the last few years (see [2,6,14]). It is within this framework that this paper presents itself. The main object is provide some new $q$-integral inequalities and, as applications, we establish some inequalities for the $q$-beta and the $q$-gamma functions.

This paper is organized as follows: in Section 2, we present some standard conventional notations and notions which will be used in the sequel. In Section 3, we state $q$-analogues of the Cebyšev's integral inequalities for synchronous (asynchronous) mappings and as a direct consequence, we give some inequalities involving the $q$-beta and the $q$-gamma functions. In Section 4, we establish some inequalities for these functions via $q$-Hölder's integral inequality. Section 5 is devoted to some applications of the $q$-Grüss’ integral inequality. Finally, Section 6 shows a $q$-analogue of a Cebyšev’s type inequality and gives some related applications for the $q$-beta and the $q$-gamma functions.

2. Notations and preliminaries

For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper. All of these results can be found in [5,8] or [9]. Throughout this paper, we will fix $q \in ]0, 1]$. For $a \in \mathbb{C}$, we write

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad (a; q)_n = \prod_{k=0}^{n-1}(1 - aq^k), \quad n = 1, 2, \ldots, \infty,$$

$$[n]_q! = [1]_q [2]_q \cdots [n]_q, \quad n \in \mathbb{N}.$$
The \( q \)-derivative \( D_q f \) of a function \( f \) is given by
\[
(D_q f)(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad \text{if } x \neq 0,
\]
and \( (D_q f)(0) = f'(0) \) provided \( f'(0) \) exists.

The \( q \)-Jackson integrals from 0 to \( b \) and from 0 to \( \infty \) are defined by (see [7])
\[
\int_0^b f(x) d_q x = (1 - q) b \sum_{n=0}^{\infty} f(b q^n) q^n
\]
and
\[
\int_0^\infty f(x) d_q x = (1 - q) \sum_{n=-\infty}^{\infty} f(q^n) q^n,
\]
provided the sums converge absolutely.

The \( q \)-Jackson integral in a generic interval \([a, b]\) is given by (see [7])
\[
\int_a^b f(x) d_q x = \int_a^b f(x) d_q x - \int_0^a f(x) d_q x.
\]

We denote by \( I \) one of the following sets:
\[
\mathbb{R}_q = \{ q^n : n \in \mathbb{Z} \}, \tag{5}
\]
\[
[0, b)_q = \{ b q^n : n \in \mathbb{N} \}, \quad b > 0, \tag{6}
\]
\[
[a, b)_q = \{ b q^n : 0 \leq k \leq n \}, \quad b > 0, \quad a = b q^k, \quad n \in \mathbb{N}, \tag{7}
\]
and we note \( \int_I f(x) d_q x \) the \( q \)-integral of \( f \) on the correspondent \( I \).

We recall the two \( q \)-analogues of the exponential function (see [5,15]) given by
\[
E_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = (-1 + q)^{-x}, \quad x \neq 0, -1, -2, \ldots, \tag{8}
\]
and
\[
e^x_q = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{(1 - q) \cdot q^x}. \tag{9}
\]
These \( q \)-exponential functions satisfy the following relations: \( D_q e_q^x = e_q^x, \)
\( D_q E_q^x = E_q^x \), and \( E_q^{-x} e_q^x = e_q^x E_q^{-z} = 1. \)

The \( q \)-gamma function is defined by [7]
\[
\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}, \quad x \neq 0, -1, -2, \ldots, \tag{10}
\]
it satisfies the following functional equation:
\[
\Gamma_q(x + 1) = [x]_q \Gamma_q(x), \quad \Gamma_q(1) = 1, \tag{11}
\]
and having the following \( q \)-integral representation (see [9])
\[
\Gamma_q(x) = \int_0^{[x]_q} t^{x-1} E_q^{-t} d_q t, \quad x > 0. \tag{12}
\]
The previous \( q \)-integral representation, give that \( \Gamma_q \) is an infinitely differentiable function on \([0, +\infty]\) and
\[
\Gamma_q^{(k)}(x) = \int_0^{[x]_q} t^{x-1} (\ln t)^k E_q^{-t} d_q t, \quad x > 0, \quad k \in \mathbb{N}. \tag{13}
\]
The \( q \)-beta function is defined by (see [9])
\[
B_q(t,s) = \int_0^1 x^{t-1} (q; q)_\infty (xq^s; q)_\infty d_q x, \quad s > 0, \quad t > 0. \tag{14}
\]
It satisfies
\[
B_q(t,s) = \frac{\Gamma_q(t) \Gamma_q(s)}{\Gamma_q(t+s)}, \quad s > 0, \quad t > 0. \tag{15}
\]
3. \(q\)-Čebyšev's integral inequality and applications

This section is devoted to state a \(q\)-analogue of the classical Čebyšev's integral inequality for synchronous (asynchronous) mappings and to give some related applications for the \(q\)-beta and the \(q\)-gamma functions.

**Definition 1.** Let \(f\) and \(g\) be two functions defined on \(I\). The functions \(f\) and \(g\) are said \(q\)-synchronous (\(q\)-asynchronous) on \(I\) if

\[
(f(x) - f(y))(g(x) - g(y)) \geq (\leq) 0 \quad \forall x, y \in I.
\]

Note that if \(f\) and \(g\) are both \(q\)-increasing or \(q\)-decreasing on \(I\) then they are \(q\)-synchronous on \(I\). We begin by state the \(q\)-analogue of the Čebyšev's integral inequality.

**Proposition 1.** Let \(f\), \(g\) and \(h\) be three functions defined on \(I\) such that:

1. \(h(x) \geq 0, \quad x \in I\),
2. \(f\) and \(g\) are \(q\)-synchronous (\(q\)-asynchronous) on \(I\).

Then

\[
\int_I h(x)dx \int_I h(x)f(x)g(x)dx \geq (\leq) \int_I h(x)f(x)dx \int_I h(x)g(x)dx.
\]

**Proof.** We have

\[
\int_I h(x)dx \int_I h(x)f(x)g(x)dx - \int_I h(x)f(x)dx \int_I h(x)g(x)dx = 1/2 \int_I \int_I h(x)h(y)|f(x) - f(y)||g(x) - g(y)|dx dy
\]

So, the result follows from the conditions (1) and (2). \(\Box\)

The following theorem is a direct consequence of the previous proposition.

**Theorem 1.** Let \(m, n, p\) and \(p'\) be some positive reals such that

\((p - m)(p' - n) \leq (\geq) 0\).

Then

\[
B_q(p, p')B_q(m, n) \geq (\leq) B_q(p, n)B_q(m, p')
\]

and

\[
\Gamma_q(p + m)\Gamma_q(p' + m) \geq (\leq) \Gamma_q(p + p')\Gamma_q(m + n).
\]

**Proof.** Fix \(m, n, p\) and \(p'\) in \([0, +\infty)\), satisfying the condition of the theorem and the functions \(f\) and \(h\) on \([0, 1]_q\) by

\[
f(x) = x^{m-1}, \quad g(x) = \frac{(xq^n, q)_\infty}{(xq^p; q)_\infty} \quad \text{and} \quad h(x) = x^{m-1} \frac{(xq, q)_\infty}{(xq^p, q)_\infty}.
\]

From the conditions

\[
D_qf(x) = [p - m]_q x^{m-1}
\]

and

\[
D_qg(x) = q^p [n - p]_q \frac{(xq^{n+1}, q)_\infty}{(xq^p; q)_\infty},
\]

one can see that \(f\) and \(g\) are \(q\)-synchronous (\(q\)-asynchronous) on \([0, 1]_q\).

So, by using Proposition 1, we obtain

\[
\int_0^1 h(x)dx \int_0^1 h(x)f(x)g(x)dx \geq (\leq) \int_0^1 h(x)f(x)dx \int_0^1 h(x)g(x)dx.
\]

Thus,

\[
\int_0^1 x^{m-1} \frac{(xq, q)_\infty}{(xq^p, q)_\infty} dx \int_0^1 x^{p-1} \frac{(xq^n, q)_\infty}{(xq^p; q)_\infty} dx \geq (\leq) \int_0^1 x^{m-1} \frac{(xq, q)_\infty}{(xq^p, q)_\infty} dx \int_0^1 x^{p-1} \frac{(xq^n, q)_\infty}{(xq^p; q)_\infty} dx.
\]
which implies that
\[ B_q(m, n)B_q(p, p') \geq (\leq) B_q(p, n)B_q(m, p'). \] (25)

Now, according to the relations (15) and (18), we obtain
\[ \frac{\Gamma_q(m)\Gamma_q(n)}{\Gamma_q(m + n)} \frac{\Gamma_q(p)\Gamma_q(p')}{\Gamma_q(p + p')} \geq (\leq) \frac{\Gamma_q(p)\Gamma_q(n)}{\Gamma_q(p + n)} \frac{\Gamma_q(m)\Gamma_q(p')}{\Gamma_q(m + p')} \] (26)

Therefore,
\[ \Gamma_q(p + n)\Gamma_q(p' + m) \geq (\leq) \Gamma_q(p + p')\Gamma_q(m + n). \] \( \square \) (27)

**Corollary 1.** For all \( p, m > 0 \), we have
\[ B_q(m, n) \geq (\leq) B_q(p, p)B_q(m, m)]^{1/2} \] (28)

and
\[ \Gamma_q(p + m) \leq (\geq) \Gamma_q(2p)\Gamma_q(2m)]^{1/2}. \] (29)

**Proof.** A direct application of Theorem 1, with \( p' = p \) and \( n = m \), gives the results. \( \square \)

**Corollary 2.** For all \( u, v > 0 \), we have
\[ \Gamma_q\left(\frac{u + v}{2}\right) \leq \sqrt{\Gamma_q(u)\Gamma_q(v)}. \] (30)

**Proof.** The inequality follows from (29), by taking \( p = \frac{u}{2} \) and \( m = \frac{v}{2} \). \( \square \)

**Theorem 2.** Let \( m, p \) and \( k \) be real numbers satisfying \( m, p > 0 \) and \( p > k > -m \) and let \( n \) be a nonnegative integer. If
\[ k(p - m - k) \geq (\leq) 0 \] (31)

then
\[ \Gamma_q(2m)^{n}p\Gamma_q(2n) \geq (\leq) \Gamma_q(2n).p\Gamma_q(2n) \Gamma_q(2n) \] (32)

**Proof.** Let \( f, g \) and \( h \) be the functions defined on \( I = [0, \frac{1}{l-1}] \) by
\[ f(x) = x^p - m - k, \quad g(x) = x^k \quad \text{and} \quad h(x) = x^{m-1}E_q^{-m}(\ln x)^2. \]

We have
\[ D_qf(x) = |p - m - k|_q x^{p-m-k-1} \quad \text{and} \quad D_qg(x) = |k|_q x^{k-1}. \]

If the condition (31) holds, one can show that the functions \( f \) and \( g \) are \( q \)-synchronous (\( q \)-asynchronous) on \( I \) and Proposition 1 gives
\[ \int_0^1 x^{p-m-k}x^{m-1}E_q^{-m}(\ln x)^2dx \geq (\leq) \int_0^1 x^{p-m-k}x^{m-1}E_q^{-m}(\ln x)^2dx \]

which is equivalent to
\[ \int_0^1 x^{p-m-k}E_q^{-m}(\ln x)^2dx \geq (\leq) \int_0^1 x^{p-m-k}E_q^{-m}(\ln x)^2dx \]

Hence, the relation (13) completes the proof. \( \square \)

**Taking \( n = 0 \) in the previous theorem, we obtain the following result.**

**Corollary 3.** Let \( m, p \) and \( k \) be some real numbers under the conditions of Theorem 2, we have
\[ \Gamma_q(p)\Gamma_q(m) \geq (\leq) \Gamma_q(p - k)\Gamma_q(m + k). \] (33)
and
\[ B_q(p, m) \geq (\leq) B_q(p - k, m + k). \] (34)

**Corollary 4.** Let \( n > 0 \) be a nonnegative integer, \( p > 0 \) and \( p' \in \mathbb{R} \) such that \( \lfloor p' \rfloor < p \). Then
\[ |\Gamma_q(2n)(p)|^2 \leq \Gamma_q(2n)(p - p')\Gamma_q(2n)(p + p'). \]
(35)

**Proof.** By choosing \( m = p \) and \( k = p' \), we obtain
\[ k(p - m - k) = -\langle p' \rangle^2 \leq 0 \]
and the result turns out from Theorem 2. □

Taking in the previous result \( p = \frac{a}{1+q} \) and \( p' = \frac{a}{1+q} \), we obtain the following result:

**Corollary 5.** Let \( u, v \) be two positive real numbers and \( n \) be a nonnegative integer. Then
\[ \Gamma_q(2n)\left(1 + \frac{u + v}{2}\right) \leq \sqrt{\Gamma_q(2n)(u)\Gamma_q(2n)(v)}. \] (36)

**Corollary 6.** Let \( p > 0 \) and \( p' \in \mathbb{R} \) such that \( \lfloor p' \rfloor < p \). Then
\[ \Gamma_q^2(p) \leq \Gamma_q(p - p')\Gamma_q(p + p') \] (37)
and
\[ B_q(p, p) \leq B_q(p - p', p + p'). \] (38)

**Proof.** For \( n = 0 \), the inequality (35) becomes
\[ \Gamma_q^2(p) \leq \Gamma_q(p - p')\Gamma_q(p + p'). \]
The inequality (38) follows from (15). □

Now, let us recall the definition (see [4]).

**Definition 2.** The positive real numbers \( a \) and \( b \) may be called similarly (oppositely) unitary if
\[ (a - 1)(b - 1) \geq (\leq) 0. \]

Now, we shall prove the following result:

**Theorem 3.** If \( a, b > 0 \) be similarly (oppositely) unitary and \( n \) a nonnegative integer. Then
\[ \Gamma_q(2n)(a + b) \geq (\leq) \Gamma_q(2n)(a + 1)\Gamma_q(2n)(b + 1). \] (39)

**Proof.** In Theorem 2, set \( m = 2, p = a + b \) and \( k = b - 1 \). The condition (31) becomes
\[ k(p - m - k) = (a - 1)(b - 1) \geq (\leq) 0. \] (40)
So,
\[ \Gamma_q(2n)(2)\Gamma_q(2n)(a + b) \geq (\leq) \Gamma_q(2n)(a + 1)\Gamma_q(2n)(b + 1). \] (41)

**Corollary 7.** If \( a, b > 0 \) and be similarly (oppositely) unitary. Then
\[ \Gamma_q(a + b) \geq (\leq) |a|_q |b|_q \Gamma_q(a) \Gamma_q(b) \] (42)
and
\[ B_q(a, b) \leq (\geq) \frac{1}{|a|_q |b|_q}. \] (43)

**Proof.** The inequality (42) follows from the previous theorem by taking \( n = 0 \) and using the facts that \( \Gamma_q(2) = 1 \), \( \Gamma_q(a + 1) = |a|_q \Gamma_q(a) \) and \( \Gamma_q(b + 1) = |b|_q \Gamma_q(b) \).

Eq. (15) together with Eq. (42) gives Eq. (43). □
Corollary 8. The function $\ln \Gamma_q$ is superadditive for $x > 1$, in the sense that

$$\ln \Gamma_q(a + b) \geq \ln \Gamma_q(a) + \ln \Gamma_q(b).$$

Proof. For all $a, b \geq 1$, we have

$$\ln \Gamma_q(a + b) \geq \ln|a|^q + \ln|b|^q + \ln \Gamma_q(a) + \ln \Gamma_q(b) \geq \ln \Gamma_q(a) + \ln \Gamma_q(b),$$

which completes the proof. \(\square\)

Corollary 9. For $a \geq 1$ and $n = 1, 2, \ldots$, we have

$$\Gamma_q(n) > \left(\frac{n-1}{q} \right)^{q^n} [\Gamma_q(a)]^n.$$

Proof. We proceed by induction on $n$.

It is clear that the inequality is true for $n = 1$.

Suppose that Eq. (44) holds for an integer $n \geq 1$ and let us prove it for $n + 1$.

By Eq. (42), we have

$$\Gamma_q((n + 1)a) = \Gamma_q(na + a) \geq [na]^q [\Gamma_q(a)]^n.$$

and by hypothesis, we have

$$\Gamma_q(na) > [n-1]_q^q [\Gamma_q(a)]^{n-1}.$$  \(\square\)

Corollary 10. The mapping $\Gamma_{q,m,n}$ is supermultiplicative on $[0, \infty)$, in the sense

$$\Gamma_{q,m,n}(x + y) \geq \Gamma_{q,m,n}(x) \Gamma_{q,m,n}(y).$$

Proof. Fix $x, y$ in $[0, \infty)$ and put $p = x + y + m$ and $k = y$. We have

$$y(x + y + m - y) = xy \geq 0.$$

So, the previous theorem leads to

$$\Gamma_{q,m,n}(x + y + m) \geq \Gamma_{q,m,n}(x + m) \Gamma_{q,m,n}(y),$$

which is equivalent to

$$\Gamma_{q,m,n}(x + y) \geq \Gamma_{q,m,n}(x) \Gamma_{q,m,n}(y).$$

This achieves the proof. \(\square\)

4. Inequalities via the $q$-Hölder’s one

We begin this section by recalling the $q$-analogue of the $q$-Hölder’s integral inequality proved in [2].

Lemma 1. Let $p$ and $p'$ be two positive reals satisfying $\frac{1}{p} + \frac{1}{p'} = 1$, and $f$ and $g$ be two functions defined on $I$. Then

$$\left| \int_I f(x)g(x)dx \right| \leq \left( \int_I |f(x)|^p dx \right)^{\frac{1}{p'}} \left( \int_I |g(x)|^{p'} dx \right)^{\frac{1}{p}}.$$

Owing this lemma, one can establish some new inequalities involving the $q$-gamma and $q$-beta functions.
**Theorem 4.** Let \( n \) be a nonnegative integer, \( x, y \) be two positive real numbers and \( a, b \) be two nonnegative real numbers such that \( a + b = 1 \). Then
\[
\Gamma_q^{(2n)}(ax + by) \leq \left[ \Gamma_q^{(2n)}(x) \right]^a \left[ \Gamma_q^{(2n)}(y) \right]^b,
\] (50)
that is, the mapping \( \Gamma_q^{(2n)} \) is logarithmically convex on \((0, \infty)\).

**Proof.** Consider the following functions defined on \( I = [0, \frac{1}{2}] \):
\[
f(t) = t^{a(x-1)}(E_q^{-q}(\ln t)^{2n})^a \quad \text{and} \quad g(t) = t^{b(y-1)}(E_q^{-q}(\ln t)^{2n})^b.
\]
By application of the q-Hölder’s integral inequality, with \( p = \frac{1}{n} \), we get
\[
\int_0^{\frac{1}{n}} t^{a(x-1)}(E_q^{-q}(\ln t)^{2n})^a dt \leq \left[ \int_0^{\frac{1}{n}} t^{a(x-1)/(1/a)} E_q^{-q}(\ln t)^{2n} dt \right]^a \times \left[ \int_0^{\frac{1}{n}} t^{b(y-1)/(1/b)} E_q^{-q}(\ln t)^{2n} dt \right]^b,
\]
which is equivalent to
\[
\int_0^{\frac{1}{n}} t^{a(x-1)}(E_q^{-q}(\ln t)^{2n})^a dt \leq \left[ \int_0^{\frac{1}{n}} t^{a(x-1)}(E_q^{-q}(\ln t)^{2n})^a dt \right]^a \times \left[ \int_0^{\frac{1}{n}} t^{b(y-1)}(E_q^{-q}(\ln t)^{2n})^b dt \right]^b.
\]
Then, Eq. (50) is a direct consequence of Eq. (13). \( \square \)

**Corollary 11.** Let \((p, p'), (m, m') \in (0, \infty)^2 \) such that \( p + p' = m + m' \) and \( a, b \geq 0 \) with \( a + b = 1 \). Then, we have
\[
B_q(a(p, p') + b(m, m')) \leq \left[ B_q(p, p') \right]^a \left[ B_q(m, m') \right]^b.
\] (51)

**Proof.** On the one hand, we have
\[
B_q(a(p, p') + b(m, m')) = B_q(a(p + b(\ln m + m')) = \frac{\Gamma_q(ap + bm)\Gamma_q(ap' + bm')}{\Gamma_q(ap' + bm' + bm')},
\]
and
\[
\Gamma_q(ap' + bm') \leq \left[ \Gamma_q(p') \right]^a \left[ \Gamma_q(m') \right]^b.
\] (54)

Thus
\[
\Gamma_q(ap + bm) \Gamma_q(ap' + bm') \leq \left[ \Gamma_q(p) \right]^a \left[ \Gamma_q(m) \right]^b \left[ \Gamma_q(p') \right]^a \left[ \Gamma_q(m') \right]^b.
\] (55)

From Eq. (52), we deduce that
\[
\frac{\Gamma_q(ap + bm)\Gamma_q(ap' + bm')}{\Gamma_q(ap + b(m + m'))} \leq \frac{\Gamma_q(p)\Gamma_q(p')}{\Gamma_q(p + p')} \left[ \frac{\Gamma_q(m)\Gamma_q(m')}{\Gamma_q(m + m')} \right]^b,
\] (56)
which completes the proof. \( \square \)

Now, we recall that the logarithmic derivative of the q-gamma function is defined on \((0, \infty)\), by
\[
\varphi_q(x) = \frac{\Gamma'_q(x)}{\Gamma_q(x)}.
\]
The following result gives some properties of the function \( \varphi_q \).

**Theorem 5.** \( \varphi_q \) is monotonic non-decreasing and concave on \((0, \infty)\).

**Proof.** By taking \( n = 0 \) in **Theorem 4**, we obtain
\[
\Gamma_q(ax + by) \leq \left[ \Gamma_q(x) \right]^a \left[ \Gamma_q(y) \right]^b,
\]
for \( x, y > 0 \) and \( a, b \geq 0 \) such that \( a + b = 1 \).
So the function \( \ln \Gamma_q(x) \) is convex. Then the monotonicity of \( \Psi_q \) follows from the relation

\[
\frac{d}{dx} \left[ \ln \Gamma_q(x) \right] = \frac{\Gamma_q'(x)}{\Gamma_q(x)} = \Psi_q(x), \quad x > 0.
\]

On the other hand, since

\[
\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x},
\]

we obtain, for \( x > 0 \),

\[
\Psi_q(x) = \frac{d}{dx} \left[ \ln \Gamma_q(x) \right] = -\ln(1 - q) + \ln q \sum_{k=0}^\infty \frac{q^{x+k}}{1-q^{x+k}} = -\ln(1 - q) + \ln q \sum_{k=0}^\infty q^{x+k} \sum_{n=0}^{\infty} q^{x+k} n
\]

\[
\quad = -\ln(1 - q) + \ln q \sum_{n=0}^\infty \frac{q^{(n+1)x}}{1-q^{n+1}} = -\ln(1 - q) + \ln q \int_0^q \frac{t^{x-1}}{1-t} dt.
\]

Now, let \( x, y > 0 \) and \( a, b \geq 0 \) such that \( a + b = 1 \). Then

\[
\Psi_q(ax + by) + \ln(1 - q) = \ln q \int_0^q \frac{t^{ax+by-1}}{1-t} dt = \ln q \int_0^q \frac{t^{x-1}}{1-t} dt.
\]

(58)

Since the mapping \( x \to t^x \) is convex on \( \mathbb{R} \) for \( t \in (0, 1) \), we have

\[
t^{a(x-1)+b(y-1)} \leq at^{x-1} + bt^{y-1}, \quad \text{for } t \in [0, q]_q, x, y > 0.
\]

Thus,

\[
\frac{\ln q}{(1-q)} \int_0^q \frac{t^{ax+by-1}}{1-t} dt \geq a \left( \frac{\ln q}{(1-q)} \int_0^q \frac{t^{x-1}}{1-t} dt \right) + b \left( \frac{\ln q}{(1-q)} \int_0^q \frac{t^{y-1}}{1-t} dt \right).
\]

(59)

According to the relations (58) and (59), we have

\[
\Psi_q(ax + by) + \ln(1 - q) \geq a\Psi_q(x) + \ln(1 - q)) + b(\Psi_q(y) + \ln(1 - q)) \geq a\Psi_q(x) + b\Psi_q(y) + \ln(1 - q).
\]

This proves the concavity of the function \( \Psi_q \). \( \square \)

5. Inequalities via the \( q \)-Grüss’s one

In [6] Gauchman gave a \( q \)-analogue of the Grüss’ integral inequality namely

\textbf{Lemma 2.} Assume that \( m \leq f(x) \leq M, \varphi \leq g(x) \leq \Phi \), for each \( x \in [a, b] \), where \( m, M, \varphi, \Phi \) are given real constants. Then

\[
\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \right| \leq \frac{1}{4} (M-m)(\Phi-\varphi).
\]

(60)

As application of the previous inequality we state the following result

\textbf{Theorem 6.} Let \( m, n > 0 \), we have

\[
|G_q(m + n + 2) - G_q(m + 2)G_q(n + 2)| \leq \frac{1}{4}|m + 1|_q |n + 1|_q G_q(m + n + 2).
\]

(62)

\textbf{Proof.} Consider the functions

\[
f(x) = x^n, \quad g(x) = \frac{(xq; q)_\infty}{(xq^{n+1}; q)_\infty}, \quad x \in [0, 1], m, n > 0.
\]

We have

\[
0 \leq f(x) \leq 1 \quad \text{and} \quad 0 \leq g(x) \leq 1 \quad \forall x \in [0, 1].
\]
Then, using the $q$-Grüss' integral inequality, we obtain

$$\left| \int_0^1 x^n \frac{(xq;q)_\infty}{(xq^n;q)_\infty} \, dq \right| \leq \frac{1}{4}.$$  

The inequality (61) follows from the definition of the $q$-beta function and the following facts: $\int_0^1 x^n \, dq = B_n$ and $\int_0^1 \frac{(xq;q)_\infty}{(xq^n;q)_\infty} \, dq = B_n(1, n + 1) = \frac{1}{n + 1}$. \qed

6. $q$-Cebyshev's type inequalities and $q$-beta and $q$-gamma functions

We begin this section by recalling the following Cebyshev's type inequality:

$$\left| \int_a^b h(x) \, dx \int_a^b f(x)g(x) \, dx - \int_a^b h(x)f(x) \, dx \int_a^b g(x) \, dx \right| \leq \|f\|_{\infty} \|g\|_{\infty} \left[ \int_a^b h(x)^2 \, dx \int_a^b g(x)^2 \, dx - \left( \int_a^b h(x)g(x) \, dx \right)^2 \right] \leq \sup_{x \in I} |f(x)| |g(x)| \leq \|f\|_{\infty} \|g\|_{\infty} \leq \sup_{x \in I} |f(x)| |g(x)| .$$

provided that $h$ is positive and $f, g$ are differentiable with bounded first derivatives on $(a, b)$.

A $q$-analogue of this inequality is given in the following lemma.

**Lemma 3.** Let $f, g$ and $h$ be three functions defined on $I$ such that

1. $h(x) > 0$, for all $x \in I$,
2. $D_q(f)$ and $D_q(g)$ are bounded on $I$.

Then, provided the $q$-integrals converge, we have

$$\left| \int_a^b h(x) \, dx \int_a^b h(x)f(x)g(x) \, dx - \int_a^b h(x)f(x) \, dx \int_a^b g(x) \, dx \right| \leq \|D_q f\|_{\infty} \|D_q g\|_{\infty} \left[ \int_a^b h(x)^2 \, dx \int_a^b g(x)^2 \, dx - \left( \int_a^b h(x)g(x) \, dx \right)^2 \right],$$

where $\|D_q f\|_{\infty} = \sup_{x \in I} |D_q f(x)|$ and $\|D_q g\|_{\infty} = \sup_{x \in I} |D_q g(x)|$.

**Proof.** From the definitions of the $q$-Jackson's integrals and the $q$-derivative, we have for all $x, y \in I$ such that $y < x$,

$$f(x) - f(y) = \int_y^x D_q f(t) \, dq \quad \text{and} \quad g(x) - g(y) = \int_y^x D_q g(t) \, dq .$$

So, for all $x, y \in I$,

$$|f(x) - f(y)| \leq \|D_q f\|_{\infty} |x - y| \quad \text{and} \quad |g(x) - g(y)| \leq \|D_q g\|_{\infty} |x - y| .$$

Then,

$$\left| \int_a^b h(x) \, dx \int_a^b h(x)f(x)g(x) \, dx - \int_a^b h(x)f(x) \, dx \int_a^b g(x) \, dx \right| = \frac{1}{2} \left| \int_a^b h(x)(h(y)[f(x) - f(y)]g(x) - g(y)) \, dq \right| \leq \frac{1}{2} \int_a^b h(x)(h(y)[f(x) - f(y)]g(x) - g(y)) \, dq \leq \frac{1}{2} \|D_q f\|_{\infty} \|D_q g\|_{\infty} \int_a^b h(x) \, dq .$$

Finally, the identity

$$1/2 \int_a^b h(x)[h(y)(x - y)^2 \, dq \, dx = \int_a^b h(x) \, dq \int_a^b x^2 h(x) \, dq - \left( \int_a^b h(x) \, dq \right)^2$$

completes the proof. \qed

**Remark.** Taking account of the specificity of the interval $I$, the inequality (*) holds. As a direct application, one has the following theorem.
Taking $r = s = 0$ in Theorem 7, the following result holds.

**Corollary 12.** For $m$, $n > 1$, we have

\[
\left| B_q(m + 1, n + 1) - \frac{1}{[m + 1]_q [n + 1]_q} \right| \leq \frac{q[m]_q [n]_q}{[3]_q [2]_q^2} \tag{69}
\]

and

\[
\left| \Gamma_q(m + n + 2) - \frac{1}{\Gamma_q(m + 2) \Gamma_q(n + 2)} \right| \leq \frac{q[m]_q [n]_q}{[3]_q [2]_q^2} [m + 1]_q [n + 1]_q \Gamma_q(m + n + 2). \tag{70}
\]

**References**


