State and Unknown Input Estimations for a Special Class of Nonlinear Uncertain Systems

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Abstract—In this paper, we develop a robust high gain observer for state and unknown input estimations for a special class of single-output nonlinear systems. The unknown input is characterized by a scalar disturbance that is distributed by a known nonlinear vector. Ensuring the observability of the disturbance with respect to the output, the disturbance can be estimated from the sliding surface. Under a Lipschitz condition for the nonlinear part, the high gain observers are designed under some regularity assumptions. In the sliding mode, the convergence of the estimation error dynamics is proven similar to the analysis of high-gain observers. The proposed observer can be applied for fault detection and isolation problems.

I. INTRODUCTION

Unknown inputs\(^1\) are often present in the systems due to unmodeled dynamics and/or component failures. In many practical applications (e.g. avionics, robotics, biochemical processes), the unknown dynamics/uncertainties of the system may even dominate the known dynamics of the nominal model. So development of robust estimation is very important for state monitoring and efficient control of the uncertain system.

High-gain observers (HGO) \([1],[2]\) have been proposed for a general class of single output systems that is uniformly observable. The works in \([1]–[3]\) presented some fundamental results on state estimation of systems via state transformation and nonlinear observer. The approach was generalized to a more general class of nonlinear systems in \([4],[5]\). However, most of the HGO designs rely on the nonlinear transformation to obtain the form viable for the design of high-gain observer. The gain design relies on inverse of the jacobian of the state transformation. In practical applications, computation of Lie derivatives for systems of high-order complicates the observer design and singularities may exist in the inverse of the jacobian matrix. To overcome these aspects, in \([5]\) a constant gain observer is proposed for a special class of nonlinear systems that does not require the nonlinear transformation. Khalil and co-workers \([6]–[9]\) further explored the use of high gain observer (HGO) for feedback control, numerical differentiation and sampled-data control. All the above development for observer design are for nominal systems without any disturbances or unknown inputs. The performance of the HGO degrades in the presence of uncertainty/disturbances.

On the other hand, sliding mode control has been an effective approach in handling disturbances and modeling uncertainties through the concepts of sliding surface design and equivalent control \([10]\). Based on the same concept, sliding mode observers (SMO) have been developed to robustly estimate the system states \([10]–[20]\). The Lyapunov based approach of Walcott and Zak \([11]\) considered the problems of state observation in the presence of bounded uncertainties/unknown inputs based on a matching condition. The approach in \([14],[21]\) extended the design of \([10]\) to linear systems such that the states affected by the unknown inputs are dealt with by the switching terms. The reconstruction of unknown inputs/faults from equivalent control was also discussed in the same work.

High-gain observers with sliding mode control have been used in the design of output feedback controllers due to their ability to robustly estimate the unmeasured states and to asymptotically attenuate the disturbances \([6],[8]\). The works in \([6]\) also proved a nonlinear separation principle for the stabilization of nonlinear systems employing the high-gain observer. Recently, a robust HGO with sliding mode was developed for a class of uncertain systems to guarantee the exponential convergence in the presence of uncertainties that are characterized by an unknown input \([20]\). The class of systems given in \([20]\) generally requires a nonlinear transformation as discussed earlier.

In this paper, we consider a special class of nonlinear systems \([5]\) together with an unknown/disturbance input. The design methodology follows the design of high gain observer and the sliding mode observer \([20]\). The considered class of systems does not require any state transformation for the design of the observer. Proposed SMO is integrated into the nonlinear high-gain observer design so that in the sliding mode, the disturbance under an equivalent control becomes an increment of Lipschitzian function, and the convergence of the error dynamics of the state estimation is proven. The proposed method does not require the matching condition on the unknown input distribution vector. A proper design of the estimation feedback gain ensures asymptotic convergence of the estimation residual.

The rest of the paper is organized as follows: Section 2 presents the system description and background results. Section 3 presents the design of robust nonlinear observer that incorporates a sliding mode observer. In Section 4, the convergence analysis and the design of sliding mode gain are discussed. Section 5 presents the estimation of the unknown input from the sliding mode. Section 6 concludes the paper. Throughout this paper, \(\lambda_{max}(A)\) denotes the

\(^1\)The term unknown input in general refers to all types of uncertainties/disturbances/unmodeled dynamics present in the system.
maximum eigenvalue of a matrix \(A\), \(\|A\|\) denotes the 2-norm \(\sqrt{\lambda_{\max}(A^T A)}\) of a matrix \(A\), and \(\sigma(A)\) denotes the condition number \(\frac{\sqrt{\lambda_{\max}(A)}}{\sqrt{\lambda_{\min}(A)}}\) of the matrix.

II. SYSTEM DESCRIPTION & BACKGROUND RESULTS

A. System Description

In this paper, the following class of uncertain nonlinear systems are considered for design of the robust sliding mode observer:

\[
\begin{align*}
\dot{x} &= \alpha(s,y)x + \gamma(x,u,s) + p(x)d(x,t) \\
y &= Cx
\end{align*}
\]

(1)

where \(x = [x_1 \ x_2 \ \cdots \ x_n]^T \in \mathbb{R}^n\), \(u \in \mathbb{R}^m\), \(y \in \mathbb{R}\), \(d(x,t) \in \mathbb{R}\), and \(s\) is the signal known, may represent an output injection or part of the inputs that are differentiable; the unknown input or disturbance \(d(x,t)\) is upper bounded with \(d\) and distributed with the nonlinear vector \(p(x)\). The system satisfies the following assumptions:

Assumption 1: The system (1) satisfies:

\[
\gamma(u,s,x) = \left[ \begin{array}{c}
\gamma_1(u,s,x) \\
\gamma_2(u,s,x_1,x_2) \\
\vdots \\
\gamma_n(u,s,x_1,x_2,\cdots,x_n)
\end{array} \right]^T
\]

(2)

Assumption 2: There exists a class \(U\) of bounded admissible controls, a compact set \(K \subset \mathbb{R}^n\) and two positive constants \(\beta_1, \beta_2\) such that for every \(u \in U\) and every output \(y\) associated to \(u\) and to an initial state \(x(0)\) \(\in K\), we have: \(0 < \beta_1 \leq |\alpha_i(s,y)| \leq \beta_2; i = 1, \cdots, n-1\).

Assumption 3: \(s(t)\) and its time derivative \(ds/dt\) are bounded.

Assumption 4: The functions \(\alpha_i(s,y), i = 1, \cdots, n\) belongs to class \(C^r\) \(r \geq 1\), w.r.t. their arguments.

Assumption 5: The functions \(\gamma_i(u,s,x_1,x_2,\cdots,x_i), p_i(x_1,x_2,\cdots,x_i)\); \(i = 1, \cdots, n\), are Lipschitz functions w.r.t. \(x\) uniformly in \(u\) and \(s\).

Assumption 6: The distribution vector \(p(x)\) with functions \(p_i(x_1,x_2,\cdots,x_i)\); \(i = 1, \cdots, n\) are bounded with respect to their arguments.

Remark 1: If all the \(\alpha_i(s,y), i = 1, \cdots, n-1\) are identically equal to 1, the observer design will be similar to that given in [20].

Remark 2: The structure in (3) can be obtained by normalizing \(p(x)\) with \(p_1(x)\), its first component. If \(p_1(x) = 0\), then the unknown input is not detectable from the output measurement, and the estimation will be inaccurate due to the effect of large disturbance on the system states. In general, the matching condition [10] is applicable for a constant distribution matrix. For systems with state-dependent nonlinear entries \(p(x)\), the design of a constant matrix to satisfy the matching condition is not feasible.

Assumptions 1 and 5 can be conservative, but they characterize a system that is uniformly observable for any bounded input. It has been proven in [1] that Assumption 1 is a sufficient condition, but not necessary, to ensure uniform observability for any input. The triangular structure in Assumption 1 is necessary for the equivalent control based sliding mode observers [13], [15] to facilitate successive evaluation of higher-order derivative terms from the measurable estimation error. Assumption 6 is necessary for the development of robust sliding mode observer to deal with the unknown input.

B. Background Results

Consider the nonlinear system (1) but without the disturbance, i.e. \(d(x,t) = 0\):

\[
\begin{align*}
\dot{x} &= \alpha(s,y)x + \gamma(x,u,s) \\
y &= Cx
\end{align*}
\]

(4)

For the system in the form of (4) satisfying Assumptions 1-5, the results in [5] proved the estimation convergence of the estimator

\[
\dot{x} = \alpha(s,y)x + \gamma(u,s,x) + L(y - Cx)
\]

(5)

where \(\dot{x}\) is the estimate of \(x\) and \(L\) is a properly chosen estimation gain.

For a gain design that is based on high-gain observer [1], [2], [5] \(L\) is

\[
L = \Gamma^{-1}(s,y)S_\theta^{-1}C^T
\]

(6)

where \(S_\theta\) is the unique solution of the Lyapunov equation

\[
\theta S_\theta + A^T S_\theta + S_\theta A - C^T C = 0
\]

(7)

\[
A \triangleq \begin{bmatrix}
0 & I_{(n-1)\times(n-1)} \\
0_{1\times n} & 0
\end{bmatrix}, \quad \theta \text{ is a positive parameter}
\]

which can be chosen to overcome system constants and bounds. \(\Gamma(s,y)\) is the diagonal matrix given by

\[
\Gamma(s,y) = \begin{bmatrix}
C & C\alpha(s,y) & \cdots & C\alpha^{n-1}(s,y)
\end{bmatrix}^T
\]

\[
= \begin{bmatrix}
1 & \alpha_1(s,y) & \cdots & \alpha_{n-1}(s,y) \\
0 & \alpha_1(s,y)\alpha_2(s,y) & \cdots & \alpha_{n-1}(s,y)
\end{bmatrix}
\]

The explicit solution of (7) can be obtained as

\[
S_\theta(i,j) = \frac{(-1)^{i+j}C_{i+j-2}^1}{g_{i+j-1}}, \quad 1 < i, j < n
\]

(8)

where \(C_n^r = \frac{n!}{(n-r)!r!}\)
Furthermore, $S_\theta$ is symmetric positive definite (SPD) for every $\theta > 0$ (see [1]).

In the following section, the observer design for systems with uncertainties/disturbances that are not necessarily random/structural is dealt.

III. HIGH-GAIN OBSERVER WITH SLIDING MODE

For system (1) satisfying Assumptions 1-6, the robust nonlinear estimator with a sliding mode term is of the form

$$\dot{x} = (\alpha(s, y)\dot{x} + \gamma(\dot{x}, u, s) + L(y - Cx) + p(\dot{x})u_r(t)$$  \tag{9}

In the above equation,

$$L = \begin{bmatrix} l_1 & l_2 & \cdots & l_n \end{bmatrix}^T = \Gamma^{-1}(s, y)S_\theta^{-1}CT$$  \tag{10}

is a properly chosen constant feedback estimation gain based on high-gain observer, and the design of the scalar-valued robust term $u_r$ is based on sliding mode theory, and is given as

$$u_r(t) = -\rho \text{sign}(e_1) = \rho \text{sign}(y - Cx)$$  \tag{11}

The sliding mode estimation gain $\rho$ will be discussed in Lemma 2.

It is clear that the proposed robust observer copies the form of the nonlinear system with appended feedback correction term and a switching term (robust term). The feedback gain ensures the exponential convergence where as the robust term deals with the unknown input/disturbances. The switching term serves as a “tracking element” for the unknown input which can be reconstructed from the sliding mode. The convergence analysis and the design of feedback gain and sliding mode gain will be discussed later.

IV. CONVERGENCE ANALYSIS

The convergence of the proposed estimator is discussed in this section. For ease of analysis and organization, the analysis is divided into three separate subsections. The boundedness of the estimation error is first proved in Lemma 1. With the established boundedness of error, the design of sliding mode gain $\rho$ and the error convergence to the sliding surface are then addressed in Lemma 2. With the sliding mode established, the asymptotic stability of the estimation error in the sliding mode is then proved in Theorem 1.

A. Boundedness of Error Dynamics

For the error dynamics

$$e = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}^T = \dot{x} - x$$

it can be obtained from (1) and (9) that

$$\dot{e} = (\alpha(s, y) - \Gamma^{-1}(s, y)S_\theta^{-1}CT) e + \gamma(\dot{x}, u, s) - \gamma(x, u, s) + p(\dot{x})u_r - p(x)d(x, t)$$  \tag{12}

For ease of analysis, define $\Delta_\theta$ as a diagonal matrix and deduce the following equalities:

$$\Delta_\theta = \text{diag} \left(1, \frac{1}{\theta}, \cdots, \frac{1}{\theta^{n-1}}\right), \quad S_\theta = \frac{1}{\theta} \Delta_\theta S_1 \Delta_\theta$$

$$\Delta_\theta A \Delta_\theta^{-1} = \theta A, \quad C \Delta_\theta = C \Delta_\theta^{-1} = C$$

$$\alpha(s, y) = \Gamma^{-1}(s, y)A \Gamma(s, y)$$

$$CT(s, y) = C = CT^{-1}(s, y)$$  \tag{13}

where $S_1$ is the solution of (7) for $\theta = 1$.

Lemma 1: Consider system (1) satisfying Assumptions 1-6. For the estimator (9), there exists $\theta_0 > 0$ such that $\forall \theta > \theta_0, \forall u \in U, \forall x(0) \in K, x$ and $e$ remain bounded.

Proof: Set $\xi = \Gamma(s, y)\Delta_\theta e$, then

$$\dot{\xi} = \Gamma(s, y)\Delta_\theta \left(\alpha(s, y) - \Gamma^{-1}(s, y)S_\theta^{-1}CT\right)\xi$$

$$-\Gamma^{-1}(s, y)S_\theta^{-1}CT\xi + \Gamma(s, y)\Delta_\theta \left[\gamma(\dot{x}, u, s) - \gamma(x, u, s)\right]$$

$$+ \Gamma(s, y)\Delta_\theta \left[p(\dot{x})u_r - p(x)d(x, t)\right]$$

$$+ \Gamma(s, y)\Gamma^{-1}(s, y)\xi$$  \tag{14}

Since $\Delta_\theta$ and $\Gamma(s, y)$ are diagonal matrices, they commute with each other and their inverses. Using the equalities in (13), it can be obtained that

$$\dot{\xi} = \theta (A - S_1^{-1}CTC) \xi$$

$$+ \Gamma(s, y)\Delta_\theta \left[\gamma(\dot{x}, u, s) - \gamma(x, u, s)\right]$$

$$+ \Gamma(s, y)\Delta_\theta \left[p(\dot{x})u_r - p(x)d(x, t)\right]$$

$$+ \Gamma(s, y)\Gamma^{-1}(s, y)\xi$$  \tag{15}

Using the above results and with the Lyapunov function $V = \xi^T S_1 \xi$, differentiating w.r.t. time, it can evaluated as

$$\dot{V} = 2\xi^T S_1 \dot{\xi}$$

$$= 2\xi^T S_1 \left(\theta (A - S_1^{-1}CTC) \xi + \Gamma(s, y)\Delta_\theta \left[\gamma(\dot{x}, u, s) - \gamma(x, u, s)\right] + \Gamma(s, y)\Delta_\theta \left[p(\dot{x})u_r - p(x)d(x, t)\right] + \Gamma(s, y)\Gamma^{-1}(s, y)\xi\right)$$

Further, with (7), it can be deduced that $2\xi^T S_1 A \xi = -\xi^T S_1 \xi + \xi^T C^T C \xi$. Hence

$$\dot{V} = -\theta V + 2\xi^T S_1 \Gamma(s, y)\Delta_\theta \left[\gamma(\dot{x}, u, s) - \gamma(x, u, s)\right] + 2\xi^T S_1 \Gamma(s, y)\Delta_\theta \left[p(\dot{x})u_r - p(x)d(x, t)\right] + 2\xi^T S_1 \Gamma(s, y)\Gamma^{-1}(s, y)\xi$$  \tag{16}

Together with the Lipschitz assumption in Assumption 5 and triangular structure in Assumption 1, and assuming
\( \theta \geq 1, \) it can be evaluated that
\[
\left\| \Delta_\theta \left[ \gamma(\dot{x}, u, s) - \gamma(x, u, s) \right] \right\| \\
\leq \sum_{i=1}^{n} \frac{1}{\theta_i-1} \left| \gamma_i(\dot{x}, u, s) - \gamma_i(x, u, s) \right| \\
\leq \sum_{i=1}^{n} l_{\gamma_i} \left| \bar{e}_i \right| \leq nl_{\gamma} \|\Delta_\theta e\| \tag{18}
\]
where \( \bar{e}_i = (e_1, \ldots, e_i), \) and \( l_{\gamma} = \sup_{p(u, x)} |l_{\gamma_i}| \) are the largest Lipschitz constant of \( \gamma. \) Consequently,
\[
\left\| \Delta_\theta \left[ \gamma(x, u, s) - \gamma(x, u, s) \right] \right\| \\
\leq nl_{\gamma} \|\Gamma^{-1}(s, y)\| \|\Gamma(s, y)\Delta_\theta e\| \\
\leq n l_{\gamma} \|\delta\| \|\xi\| \tag{19}
\]
where \( \delta = \sup_{t \geq 0, x \in \mathbb{R}^{n}} \{ \| \Gamma^{-1}(s, t), Cx \| \} \). Under Assumptions 5 and 6, \( p(x) \) is a Lipschitz function and bounded for some upper bound \( b_p \). Similar to (19), due to the triangular structure of \( p(x) \) in Assumption 1, it can be obtained as
\[
\|\Delta_\theta (p(\dot{x}) - p(x))\| \leq nl_p \|\xi\| \tag{20}
\]
\[
\|p(x)u_r - p(x)d(x, t)\| \leq b_p (\bar{d} + \rho) \tag{21}
\]
where \( l_p \) is the largest Lipschitz constant of \( p, |d(x, t)| \leq \bar{d} \) and \( |u_r| = \rho. \)

Using the above inequalities, the derivative of Lyapunov function can be evaluated as
\[
\dot{V} \leq \theta V + 2n\delta l_1 |D_1| \|S_1 \xi\| \|\xi\| \\
+ 2n\delta l_1 |D_1| \|S_1 \xi\| \|\xi\| + 2\delta l_1 |D_1| |b_p| (\bar{d} + \rho) \\
+ 2\delta \|S_1 \xi\| \|\xi\| \tag{22}
\]
where \( \delta_1 = \sup_{t \geq 0, x \in \mathbb{R}^{n}} \{ \| \Gamma(t, S_1), Cx \| \} \) and \( \delta_2 = \sup_{t \geq 0, x \in \mathbb{R}^{n}} \{ \| \Gamma(T), Cx \| \} \).

\[
\dot{V} \leq -[\theta - 2n\delta l_1 \delta_1 |S_1| - 2n\rho l_1 \delta_1 |S_1| - 2\delta_2 |S_1|] V \\
+ 2\delta l_1 \lambda_{\max}(S_1) |b_p| (\bar{d} + \rho) \|\xi\| \\
= -c_1 \|\xi\|^2 + c_2 \|\xi\| \tag{23}
\]
where \( c_1 \triangleq \theta - 2n\delta l_1 \delta_1 |S_1| - 2n\rho l_1 \delta_1 |S_1| - 2\delta_2 |S_1|, \) \( c_2 \triangleq 2\delta l_1 \lambda_{\max}(P) |b_p| (\bar{d} + \rho). \) With selection of \( \theta > -2n\delta l_1 \delta_1 |S_1| + 2n\rho l_1 \delta_1 |S_1| + 2\delta_2 |S_1|, \) it can be shown that \( c_1 > 0. \) Hence \( \bar{e} \) is bounded such that \( \|\bar{e}\| \leq c_2/c_1. \) Also, the bound of \( \|\bar{e}\| \) can be made small by selecting a large \( \theta. \)

\subsection{B. Sliding Mode Gain Design}

For modeling uncertainties from the scalar disturbance \( d(x, t) \) that is not necessarily Lipschitzian, the normal estimator can only achieve a bounded \( e \) instead of \( e \rightarrow 0. \)

In the proposed design, SMO with the term \( p(x)u_r \) is employed in (9) to improve the estimation accuracy. The rationale of this solution is two-fold:

(a) With the sliding surface
\[
e_1 = 0 \tag{25}
\]
the aim is to design the sliding mode estimation as (11) to reach and maintain in the sliding mode.

(b) To ensure that the term \( |u_r - d(x, t)| \) of the estimation error dynamics (12) in the sliding mode \( e_1 = 0, \) i.e., a zero dynamics, can be substituted by an increment of Lipschitzian function through an equivalent control signal, so that the asymptotic convergence of \( e \) can be proved.

The following Lemma 2 and Theorem 1 are devoted to the above mentioned points (a) and (b), respectively.

**Lemma 2:** For the system (1) satisfying Assumptions 1 - 6 and the estimator (9), the sliding mode estimation (11) ensures that the sliding surface \( e_1 = 0 \) can be reached and maintained provided there exists \( \theta_1 > 0 \) such that \( \theta > \theta_1 \) and the sliding mode gain satisfies
\[
\rho > \beta_2 e_{2max} + \bar{d} \tag{26}
\]
where \( \|e_2\| \leq e_{2max} \) and \( |\alpha_1(s, y)| \leq \beta_2. \)

**Proof:** The first dynamics \( e_1 \) from (12) can be obtained as follows:
\[
\dot{e}_1 = \alpha_1(s, y) e_2 - l_1 e_1 \\
+ [\gamma_1(\hat{x}_1, u, s) - \gamma_1(x_1, u, s)] + u_r - d(x, t) \tag{27}
\]
For the Lyapunov function \( V_1 \) is \( \frac{1}{2} e_1^2, \) using the above and the sliding mode estimation (11) it can be evaluated as
\[
\dot{V}_1 = e_1 \dot{e}_1 \\
= e_1 (e_1^2 + e_1 [\gamma_1(\hat{x}_1, u, s) - \gamma_1(x_1, u, s)] \\
+ [\alpha_1(s, y)e_2(t) - d(x, t)] e_1 - \rho |e_1| \]
Under the condition in Lemma 1 that ensures boundedness of \( e_2, \) and together with the boundedness of \( \alpha_1(s, y) \) in Assumption 2, there exists a finite constant gain satisfying (26) such that, if \( e_1 \neq 0, \) one has
\[
\dot{V}_1 < -l_1 e_1^2 + e_1 [\gamma_1(\hat{x}_1, u, s) - \gamma_1(x_1, u, s)] \\
+ [\alpha_1(s, y)e_2(t) - d(x, t)] e_1 - \rho |e_1| \]
Under the Lipschitzian condition in Assumption 5 and the boundedness of the input, it can be obtained as
\[
\dot{V}_1 < -l_1 e_1^2 + l_1 e_1^2 \\
= -l_1 e_1^2 \tag{28}
\]
where \( l_1 \) is the Lipschitz constant of \( \gamma_1(\cdot). \) From the design of high gain, \( l_1 \) of \( L \) from (6) can be evaluated as \( l_1 = n \theta \) where \( n \) is the order of the system. By selecting \( \theta \) such that
\[
\theta > \frac{l_1}{n} = \theta_1 \]
ensures \( l_1 > l_1. \) Hence, it can be shown
\[
\dot{V}_1 < 0 \text{ if } e_1 \neq 0
\]
Hence, the robust term (11) using the gain (26) ensures that the sliding surface \( e_1 = 0 \) can be reached in a finite time and maintained thereafter.

**Remark 3:** The boundedness of \( e \) is established in Lemma 1 and is dependent on \( \theta. \) As \( \theta \) is an independent
parameter, one can choose arbitrarily to reduce the bound of $e$. This bound can be used for the calculation of sliding mode gain $\rho$.

C. Error Dynamics in the Sliding Mode

Since the estimator design (9) using the robust term (11) ensures the sliding mode, it is only required to examine the convergence of the dynamics of $e$ during the sliding mode. In the sliding mode when $e_1 = 0$ and $\dot{e}_1 = 0$, $\dot{x}_1 = x_1$, the equivalent control of $u$, can be obtained from (27) as in [10]:

$$u_{eq} = d(x, t) - \alpha_1(s, y)e_{2,d} \quad (28)$$

where the subscript $d$ denotes the estimated $x$-related variables in the sliding mode, i.e., $e_d = [e_{1,d} \ e_{2,d} \ \cdots \ e_{n,d}]^T \triangleq \dot{x}_d - x$. Substituting the above equivalent control (28) into (12), the estimation error dynamics in the sliding mode of $e_1 = 0$ can be obtained as

$$\dot{e}_d = (\alpha(s, y) - \Gamma^{-1}(s, y)S_y^{-1}C^T C) e_d + \gamma(\dot{x}_d, u, s) - \gamma(x, u, s) + (p(\dot{x}_d) - p(x)) u_{eq} + p(x)[u_{eq} - d(x, t)]$$

$$= (\alpha(s, y) - \Gamma^{-1}(s, y)S_y^{-1}C^T C) e_d + \gamma(\dot{x}_d, u, s) - \gamma(x, u, s) + (p(\dot{x}_d) - p(x)) u_{eq} - p(x)\alpha_1(s, y)e_{2,d} \quad (29)$$

The equivalent control in the sliding mode clearly cancels the disturbance effect in the state estimation. The following theorem proves the asymptotic stability of the estimation error.

**Theorem 1**: Assume that system (1) satisfies Assumptions 1 - 6. For the estimator (9) with the robust term (11) and the sliding mode gain (26), there exists $\theta_2 > 0$ such that $\forall \theta > \theta_2$, the estimation error is asymptotically stable in the sliding mode of $e_1 = 0$.

**Proof**: The proof follows the similar pattern as in Lemma 1. By selecting $\xi_d = \Gamma(s, y)\Delta_\theta e_d$ and Lyapunov function $V_2 = \xi_d^T S_1 \xi_d$, it can be evaluated with (29) similar to (17):

$$\dot{V}_2 \leq -\theta V_2 + 2\xi_d^T S_1 \Gamma(s, y)\Delta_\theta [\gamma(\dot{x}_d, u, s) - \gamma(x, u, s)] + 2\xi_d^T S_1 \Gamma(s, y)\Delta_\theta [p(\dot{x}_d) - p(x_d)] u_{eq} - 2\xi_d^T S_1 \Gamma(s, y)\Delta_\theta p(x)\alpha_1(s, y)e_{2,d} + 2\xi_d^T S_1 \Gamma(s, y)\Delta_\theta \xi_d \quad (30)$$

Since $e_2$, $\alpha_1(s, y)$ and $d(x, t)$ are bounded, then according to (28), $u_{eq} \leq u_{eq}$ for some upper bound $u_{eq}$. With (21), $|p(x)| \leq b_p$ and similar to (23), can be deduced to the form (30):

$$\dot{V}_2 \leq -\theta V_2 + 2\xi_d^T S_1 \Gamma(s, y)\Delta_\theta [\gamma(\dot{x}_d, u, s) - \gamma(x, u, s)] + 2\xi_d^T S_1 \Gamma(s, y)\Delta_\theta [p(\dot{x}_d) - p(x_d)] u_{eq} - 2\xi_d^T S_1 \Gamma(s, y)\Delta_\theta p(x)\alpha_1(s, y)e_{2,d} + 2\xi_d^T S_1 \Gamma(s, y)\Delta_\theta \xi_d \quad (30)$$

By selecting $\theta > 2\theta_1\theta_2$, (21) is satisfied, then $\dot{V}_2 < 0$, therefore $e_1 \rightarrow 0$. Hence, the sliding mode gain design based on HGO guarantees asymptotic stability of the estimation error.

Obviously, the equilibrium point of the error dynamics (29) is $e_1 = 0$, i.e., $\dot{x}_d = x$. The robust term in the sliding mode can be viewed as an estimate of the disturbance $d(x, t)$ during the sliding mode. Therefore, the estimation error error dynamics in the sliding mode is discussed in the next section.

V. UNKNOWN INPUT ESTIMATION FROM SLIDING SURFACE

Once the trajectory reaches the sliding mode, all the states converge to the true states, i.e., $\dot{x}_d \rightarrow x$. Therefore

$$e_{2,d} \approx 0$$

The equivalent control $u_{eq}$ information can then be used to reconstruct the unknown input. From (28), the equivalent control can be approximated as

$$u_{eq} \approx d(x, t) \quad (31)$$

The use of a low-pass filter for recovering the equivalent control signal was given by [10]. Continuous approximation of equivalent injection signal by using a small positive scalar $\delta_e$ was also implemented in [14]. Similar to the analysis of [14], in the proposed approach, the unknown input can be estimated from equivalent control as follows:

$$\hat{d}(x, t) \approx \rho(\text{sign}(e_1))_{eq} \approx \rho \frac{e_1}{(|e_1| + \delta_e)} \quad (32)$$

The accuracy of the disturbance estimation will depend on $\delta_e$. The estimation only depends on measurement error $e_1$ and hence can be performed online with state estimation.

VI. CONCLUSIONS

A robust high-gain observer design is developed for a special class of single-output uncertain nonlinear systems by incorporating a sliding mode term into the nonlinear observer to improve estimation accuracy. The sliding surface uses only the measurable output estimation error, and the unknown input can be estimated from the measurable sliding surface. The gain design is based on HGO that guarantees exponential convergence of the estimation error.
REFERENCES


