Research Article

On Prime-Gamma-Near-Rings with Generalized Derivations

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Let \( N \) be a 2-torsion free prime \( \Gamma \)-near-ring with center \( Z(N) \). Let \( (f, d) \) and \( (g, h) \) be two generalized derivations on \( N \). We prove the following results:

(i) \( f(x, y, \alpha) = 0 \) or \( f(x, y, \alpha) = \pm x, y \alpha \) or \( f^2(x) \in Z(N) \) for all \( x, y \in N, \alpha \in \Gamma \), then \( N \) is a commutative \( \Gamma \)-ring.

(ii) If \( a \in N \) and \( [f(x), a] = 0 \) for all \( x \in N, \alpha \in \Gamma \), then \( d(a) \in Z(N) \). (iii) If \( (fg, dh) \) acts as a generalized derivation on \( N \), then \( f = 0 \) or \( g = 0 \).

1. Introduction

The derivations in \( \Gamma \)-near-rings have been introduced by Bell and Mason [1]. They studied basic properties of derivations in \( \Gamma \)-near-rings. Then Aşçı [2] obtained commutativity conditions for a \( \Gamma \)-near-ring with derivations. Some characterizations of \( \Gamma \)-near-rings and regularity conditions were obtained by Cho [3]. Kazaz and Alkan [4] introduced the notion of two-sided \( \Gamma \)-\( \alpha \)-derivation of a \( \Gamma \)-near-ring and investigated the commutativity of a prime and semiprime \( \Gamma \)-near-rings. Uçkun et al. [5] worked on prime \( \Gamma \)-near-rings with derivations and they found conditions for a \( \Gamma \)-near-ring to be commutative. In [6] Dey et al. studied commutativity of prime \( \Gamma \)-near-ring with generalized derivations.

In this paper, we obtain the conditions of a prime \( \Gamma \)-near-ring to be a commutative \( \Gamma \)-ring. If \( a \in N \), and \( [f(x), a] = 0 \) for all \( x \in N, \alpha \in \Gamma \), then \( d \) is central. Also we prove that if \( (fg, dh) \) is the generalized derivation on \( N \), then \( f \) and \( g \) are trivial.
2. Preliminaries

A Γ-near-ring is a triple \((N, +, \Gamma)\), where

(i) \((N, +)\) is a group (not necessarily abelian);

(ii) \(\Gamma\) is a nonempty set of binary operations on \(N\) such that for each \(\alpha \in \Gamma\), \((N, +, \alpha)\) is a left near-ring;

(iii) \(\alpha x(yz) = (\alpha xy)z\), for all \(x, y, z \in N\) and \(\alpha, \beta \in \Gamma\).

We will use the word Γ-near-ring to mean left Γ-near-ring. For a near-ring \(N\), the set \(N_0 = \{x \in N : 0ax = 0, a \in \Gamma\}\) is called the zero-symmetric part of \(N\). A Γ-near-ring \(N\) is said to be zero-symmetric if \(N = N_0\). Throughout this paper, \(N\) will denote a zero symmetric left Γ-near-ring with multiplicative centre \(Z(N)\). Recall that a Γ-near-ring \(N\) is prime if \(x\Gamma N y = 0\) implies \(x = 0\) or \(y = 0\). An additive mapping \(d : N \to N\) is said to be a derivation on \(N\) if \(d(xy) = xad(y) + d(x)ay\) for all \(x, y \in N, \alpha \in \Gamma\), or equivalently, as noted in [1], that \(d(xy) = d(x)ay + xad(y)\) for all \(x, y \in N, \alpha \in \Gamma\). Further, an element \(x \in N\) for which \(d(x) = 0\) is called a constant. For \(x, y \in N, \alpha \in \Gamma\), the symbol \([x, y]_{\alpha}\) will denote the commutator \(xy - yx\), while the symbol \((x, y)\) will denote the additive-group commutator \(x + y - x - y\). An additive mapping \(f : N \to N\) is called a generalized derivation if there exits a derivation \(d\) of \(N\) such that \(f(xy) = f(x)ay + xad(y)\) for all \(x, y \in N, \alpha \in \Gamma\). The concept of generalized derivation covers also the concept of a derivation.

3. Derivations on Γ-Near-Rings

In this section we prove that a few subsidiary results (Lemmas 3.1, 3.2, 3.4, 3.8, 3.9, 3.10 and 3.11) to use them for proving of our main results (Theorems 3.3, 3.5, 3.6, 3.12 and 3.13).

**Lemma 3.1.** Let \(d\) be an arbitrary derivation on a Γ-near-ring \(N\). Then \(N\) satisfies the following partial distributive law: \((aad(b) + d(a)ab)b\)c = \(aad(b)b\)c + \(d(a)ab\)bc and \((d(a)ab + aad(b))b\)c = \(d(a)abbc + aad(b)b\)c for all \(a, b, c \in N, \alpha, \beta \in \Gamma\).

**Proof.** For all \(a, b, c \in N, \alpha, \beta \in \Gamma\), we get \(d((aad)b)c) = aab\beta d(c) + (aad(b) + d(a)ab)b\)c and \(d(aa(b)c)) = aad(b)c + d(a)ab\)bc = \(aa(b\beta d(c) + d(b)b\)c + d(a)ab\)bc = \(aab\beta d(c) + aad(b)b\)c + d(a)ab\)bc. Equating these two relations for \(d(aab\)bc\) now yields the required partial distributive law.

**Lemma 3.2.** Let \(d\) be a derivation on a Γ-near-ring \(N\) and suppose \(u \in N\) is not a left zero divisor. If \([u, d(u)]_{\alpha} = 0, \alpha \in \Gamma\), then \((x, u)\) is a constant for every \(x \in N\).

**Proof.** From \(ua(u + x) = uau + uax\), for all \(x \in N, \alpha \in \Gamma\), we obtain \(uad(u + x) + d(u)\alpha (u + x) = uad(u) + d(u)au + uad(x) + d(u)ax\), which reduces \(uad(x) + d(u)au = d(u)au + uad(x)\), for all \(\alpha \in \Gamma\).

Since \(d(u)au = uad(u), \alpha \in \Gamma\), this equation is expressible as \(ua(d(x) + d(u) - d(x) - d(u)) = 0 = uad((x, u))\). Thus \(d((x, u)) = 0\).

**Theorem 3.3.** Let \(N\) be a Γ-near-ring having no nonzero divisors of zero. If \(N\) admits a nontrivial commuting derivation \(d\), then \((N, +)\) is abelian.
Proof. Let c be any additive commutator. Then c is a constant by Lemma 3.2. Moreover, for any \( w \in N, \ a \in \Gamma, \) \( wac \) is an additive commutator, hence also a constant. Thus, \( 0 = d(wac) = wad(c) + d(w)ac \) and \( d(w)ac = 0, \) for all \( a \in \Gamma. \) Since \( d(w) \neq 0 \) for all \( w \in N, \) we conclude that \( c = 0. \)

Lemma 3.4. Let \( N \) be a prime \( \Gamma \)-near-ring.

(i) If \( z \in Z(N) - \{0\}, \) then \( z \) is not a zero divisor in \( N. \)

(ii) If \( Z(N) - \{0\} \) contains an element \( z \) for which \( z + z \in Z(N), \) then \( (N,+) \) is abelian.

(iii) Let \( d \) be a nonzero derivation on \( N. \) Then \( xt \Gamma d(N) = \{0\} \) implies \( x = 0, \) and \( d(N) \Gamma x = \{0\} \) implies \( x = 0. \)

(iv) If \( N \) is \( 2 \)-torsion free and \( d \) is a derivation on \( N \) such that \( d^2 = 0, \) then \( d = 0. \)

Proof. (i) If \( z \in Z(N) - \{0\} \) and \( zax = 0, \ a, x \in N, \ a \in \Gamma, \) then \( zarb = 0, \ z \in \Gamma. \) Thus we get \( z \Gamma N \Gamma x = 0, \) by primeness of \( N, x = 0. \)

(ii) Let \( z \in Z(N) - \{0\} \) be an element such that \( z + z \in Z(N), \) and let \( x, y \in N, \ a, b \in \Gamma. \) Since \( z + z \) is distributive, we get \( (x + y)a(z + z) = xa(z + z) + ya(z + z) = xaz + xaz + yaz + yaz = za(x + x + y + y). \)

On the other hand, \( (x + y)x + y = y + y + y + y \) and therefore \( x + y = y + x. \) Hence \( (N,+) \) is abelian.

(iii) Let \( xt \Gamma d(N) = 0, \) and let \( r, s \) be arbitrary elements of \( N \) and \( a, b, \beta \in \Gamma. \) Then \( 0 = xad(r \beta s) = xarb \Gamma d(s) + xad(r) \beta s = xarb \Gamma d(s). \) Thus \( x \Gamma N \Gamma d(N) = \{0\}, \) and since \( d(N) \neq \{0\}, \) \( x = 0. \)

A similar argument works if \( d(N) \Gamma x = \{0\}, \) since Lemma 3.1 provides enough distributivity to carry it through.

(iv) For arbitrary \( x, y \in N, a, b \in \Gamma, \) we have \( 0 = d^2(xy) = d(xad(y) + d(x)ay) = xad^2(y) + d(x)ad(y) + d(x)ad(y) + d^2(xy) = 2d(x)ad(y). \) Since \( N \) is \( 2 \)-torsion free, \( d(xy) = 0, \) \( x, y \in N, a, b \in \Gamma. \) Thus \( d(xy) \Gamma d(N) = \{0\} \) for each \( x \in N, \) and (iii) yields \( d(a) = 0. \) Thus \( d = 0. \)

Theorem 3.5. If a prime \( \Gamma \)-near-ring \( N \) admits a nontrivial derivation \( d \) for which \( d(N) \in Z(N), \) then \( (N,+) \) is abelian. Moreover, if \( N \) is \( 2 \)-torsion free, then \( N \) is a commutative \( \Gamma \)-ring.

Proof. Let \( c \) be an arbitrary constant, and let \( x \) be anon-constant. Then \( d(xac) = xad(c) + d(x)ac = d(x)ac \in Z(N), a \in \Gamma. \) Since \( d(x) \in Z(N) - \{0\}, \) it follows easily that \( c \in Z(N). \) Since \( c + c \) is a constant for all constants \( c, \) it follows from Lemma 3.4(iii) that \( (N,+) \) is abelian, provided that there exists a nonzero constant.

Assume, then, that \( 0 = \) the only constant. Since \( d \) is obviously commuting, it follows from Lemma 3.2 that all \( u \) which are not zero divisors belong to the center of \( (N, +), \) denoted by \( Z(N). \) In particular, if \( x \neq 0, d(x) \in Z(N). \) But then for all \( y \in N, d(y) + d(x) - d(y) - d(x) = d((y, x)) = 0, \) hence \( (y, x) = 0. \)

Now we assume that \( N \) is \( 2 \)-torsion free. By Lemma 3.1, \( (aad(b) + d(a)ab)b \beta c = aad(b)b \beta c + d(a)ab \beta c \) for all \( a, b, c \in N, a, b \in \Gamma. \) and using the fact that \( d(aab) \in Z(N), a \in \Gamma, \) we get \( aad(b) + cad(a)b \beta c = aad(b)b \beta c + aad(b)b \beta c, a, b \in \Gamma. \) Since \( (N,+) \) is abelian and \( d(N) \subseteq Z(N), \) this equation can be rearranged to yield \( d(b)a[b, c]_\beta = d(a)a[b, c]_\beta \) for all \( a, b, c \in N, a, b \in \Gamma. \)

Suppose now that \( N \) is not commutative. Choosing \( b, c \in N, \) with \( [b, c]_\beta \neq 0, \beta \in \Gamma, \) and letting \( a = d(x), \) we get \( d^2(x)a[b, c]_\beta = 0, \) for all \( x \in N, a, b \in \Gamma, \) and since the central
elements $d^2(x)$ cannot be a nonzero divisor of zero, we conclude that $d^2(x) = 0$ for all $x \in N$. But by Lemma 3.4(iv), this cannot happen for nontrivial $d$.

**Theorem 3.6.** Let $N$ be a prime $\Gamma$-near-ring admitting a nontrivial derivation $d$ such that $[d(x), d(y)]_\alpha = 0$ for all $x, y \in N$, $\alpha \in \Gamma$. Then $(N, +)$ is abelian. Moreover, if $N$ is 2-torsion free, then $N$ is a commutative $\Gamma$-ring.

**Proof.** By Lemma 3.4(ii), if both $z$ and $z + z$ commute element-wise with $d(N)$, then $zad(c) = 0$, $\alpha \in \Gamma$, for all additive commutators $c$. Thus, taking $z = d(x)$, we get $d(x)ad(c) = 0$ for all $x \in N$, $\alpha \in \Gamma$, so $d(c) = 0$ by Lemma 3.4(iii). Since $wac$ is also an additive commutator for any $w \in N$, $\alpha \in \Gamma$, we have $d(wac) = 0$ $d(w)ac$, and another application of Lemma 3.4(iii) gives $c = 0$.

Now we assume that $N$ is 2-torsion free. By the partial distributive law, $d(d(x)ay)\beta d(z) = d(x)ad(y)\beta d(z) + d^2(x)ay\beta d(z)$ for all $x, y, z \in N, \alpha, \beta \in \Gamma$, hence,

\[
d^2(x)ay\beta d(z) = d(d(x)ay)\beta d(z) - d(x)ad(y)\beta d(z) = d(x)ad(y)\beta d(z) = d(x)ad(y)\beta d(z) = d(z)\alpha(d(x)\beta y) - d(x)\beta d(y)) = d(z)d^2(x)\beta y = d^2(x)ad(z)\beta y, \alpha, \beta \in \Gamma.\]

Thus $d^2(x)\alpha(\beta d(z) - d(z)\beta y) = 0$ for all $x, y, z \in N, \alpha, \beta \in \Gamma$.

Replacing $y\delta t, \delta \in \Gamma$, we obtain $d^2(x)\alpha(\beta d(z) - d(z)\beta y) = d^2(x)\alpha(\beta d(z) - d(z)\beta y)\delta t, \delta \in \Gamma$, so that $d^2(x)\alpha(\beta d(z) - d(z)\beta y)\delta t = 0$ for all $x, y, z, t \in N, \alpha, \beta, \delta \in \Gamma$. The primeness of $N$ shows that either $d^2 = 0$ or $d(N) \subseteq Z(N)$, and since the first of these conditions is impossible by Lemma 3.4(iv), the second must hold and $N$ is a commutative $\Gamma$-ring by Theorem 3.5.

**Definition 3.7.** Let $N$ be a $\Gamma$-near-ring and $d$ a derivation of $N$. An additive mapping $f : N \to N$ is said to be a right generalized derivation of $N$ associated with $d$ if

\[
f(xy) = f(xy) + xad(y) \quad \forall x, y \in R, \quad \alpha \in \Gamma, \tag{3.1}
\]

and $f$ is said to be a left generalized derivation of $N$ associated with $d$ if

\[
f(xy) = d(xy) + xaf(y) \quad \forall x, y \in R, \quad \alpha \in \Gamma. \tag{3.2}
\]

Finally, $f$ is said to be a generalized derivation of $N$ associated with $d$ if it is both a left and right generalized derivation of $N$ associated with $d$.

**Lemma 3.8.** Let $f$ be a right generalized derivation of a $\Gamma$-near ring $N$ associated with $d$. Then

(i) $f(xy) = xad(y) + f(x)ay$ for all $x, y \in N, \alpha \in \Gamma$;

(ii) $f(xy) = xaf(y) + d(xy)ay$ for all $x, y \in N, \alpha \in \Gamma$.

**Proof.** (i) For any $x, y \in N, \alpha \in \Gamma$, we get

\[
f(xa(y + y)) = f(xa(y + y) + xad(y + y) = f(x)ay + f(x)ay + xad(y) + xad(y),
\]

\[
f(xy + xay) = f(xy) + xad(y) + f(xy) + xad(y).
\]

(3.3)
Comparing these two expressions, we obtain

$$f(x)ay + xad(y) = xad(y) + f(x)ay \quad \forall x, y \in N, \ a \in \Gamma,$$

(3.4)

and so,

$$f(xay) = xad(y) + f(x)ay \quad \forall x, y \in N, \ a \in \Gamma.$$

(3.5)

(ii) In a similar way.

\[\square\]

Lemma 3.9. Let \( f \) be a right generalized derivation of a \( \Gamma \)-near ring \( N \) associated with \( d \). Then

(i) \((f(x)ay + xad(y))\beta z = f(x)ay\beta z + xad(y)\beta z, \) for all \( x, y, \alpha, \beta \in \Gamma \).

(ii) \((d(x)ay + xaf(y))\beta z = d(x)ay\beta z + xaf(y)\beta z, \) for all \( x, y, \alpha, \beta \in \Gamma \).

Proof. (i) For any \( x, y, z \in N, \alpha, \beta \in \Gamma \), we get \( f((xay)\beta z) = f(xay)\beta z + xay\beta d(z) \). On the other hand,

$$f(xa(y\beta z)) = f(xa(y\beta z) + xad(y\beta z) = f(x)ay\beta z + xad(y)\beta z + xay\beta d(z).$$

(3.6)

From these two expressions of \( f(xay\beta z) \), we obtain that, for all \( x, y, z \in N, \alpha, \beta \in \Gamma \),

$$(f(x)ay + xad(y))\beta z = f(x)ay\beta z + xad(y)\beta z.$$

(3.7)

(ii) The proof is similar.

\[\square\]

Lemma 3.10. Let \( N \) be a prime \( \Gamma \)-near-ring, \( f \) a nonzero generalized derivation of \( N \) associated with the nonzero derivation \( d \) and \( a \in N \). (i) If \( af(N) = 0 \), then \( a = 0 \). (ii) If \( f(N) = 0 \), then \( a = 0 \).

Proof. (i) For any \( x, y \in N, \alpha, \beta \in \Gamma \), we get \( 0 = a\beta f(xay) = a\beta f(x)ay + a\beta xad(y) = a\beta xad(y) \). Hence \( d(N) = 0 \). Since \( N \) is a prime \( \Gamma \)-near-ring and \( d \neq 0 \), we obtain \( a = 0 \).

(ii) A similar argument works if \( f(N) = 0 \).

\[\square\]

Lemma 3.11. Let \( N \) be a prime \( \Gamma \)-near-ring. Let \( f \) be a generalized derivation of \( N \) associated with the nonzero derivation \( d \). If \( N \) is a 2-torsion free \( \Gamma \)-near-ring and \( f^2 = 0 \), then \( f = 0 \).

Proof. (i) For any \( x, y \in N, \alpha \in \Gamma \), we get

$$0 = f^2(xay) = f(f(xay)) = f(f(x)ay + xad(y)) = f^2(x)ay + 2f(x)ad(y) + xad^2(y).$$

(3.8)

By the hypothesis,

$$2f(x)ad(y) + xad^2(y) = 0 \quad \forall x, y \in N, \ a \in \Gamma.$$

(3.9)

Writing \( f(x) \) by \( x \) in \( (3.9) \), we get \( f(x)ad^2(y) = 0 \), for all \( x, y \in N, \alpha \in \Gamma \).
By Lemma 3.9(ii), we obtain that \( d^2(N) = 0 \) or \( f = 0 \). If \( d^2(N) = 0 \) then \( d = 0 \) from Lemma 3.4(iv), a contradiction. So we find \( f = 0 \).

\[ \text{Theorem 3.12.} \quad \text{Let } N \text{ be a prime } \Gamma\text{-near-ring with a nonzero generalized derivation } f \text{ associated with } d. \text{ If } f(N) \subseteq Z(N), \text{ then } (N, +) \text{ is abelian. Moreover, if } N \text{ is 2-torsion free, then } N \text{ is commutative } \Gamma\text{-ring.} \]

\[ \text{Proof.} \quad \text{Suppose that } a \in N, \text{ such that } f(a) \neq 0. \text{ So, } f(a) \in Z(N) - \{0\} \text{ and } f(a) + f(a) \in Z(N) - \{0\}. \text{ For all } x, y \in N, a \in \Gamma, \text{ we have } (x + y)a(f(a + f(a))) = (f(a + f(a)))a(x + y).

That is, \( xaf(a) + yaf(a) + yaf(a) = f(a)ax + f(a)ax + f(a)ay + f(a)ay \), for all \( x, y \in N, a \in \Gamma \).

Since \( f(a) \in Z(N) \), we get \( f(a)ax + f(a)ay = f(a)ay + f(a)ax \), and so, \( f(a)a(x, y) = 0 \) for all \( x, y \in N, a \in \Gamma \).

Since \( f(a) \in Z(N) - \{0\} \) and \( N \) is a prime \( \Gamma \)-near-ring, it follows that \( (x, y) = 0 \), for all \( x, y \in N \). Thus \( (N, +) \) is abelian.

Using the hypothesis, for any \( x, y, z \in N, a, b \in \Gamma, \) \( zaf(x^\beta y) = f(x^\beta y)zx \). By Lemma 3.4(ii), we have \( zaf(x)bf(y) + xaf(y) = d(x)af(z)bf(z) \). Using \( f(N) \subset Z(N) \) and \( (N, +) \) being abelian, we obtain that

\[ zaf(x)bf(y) - d(x)af(z)bf(z) = [x, z]_\alpha bf(y), \quad \forall x, y \in N, a, b \in \Gamma. \quad (3.10) \]

Substituting \( f(z) \) for \( z \) in (3.10), we get \( f(z)bf([x, y]_\alpha) = 0 \) for all \( x, y \in N, a, b \in \Gamma \).

Since \( f(z) \in Z(N) \) and \( f \) a nonzero generalized derivation with associated with \( d \), we get \( d(N) \subset Z(N) \). So, \( N \) is a commutative \( \Gamma \)-ring by Theorem 3.3.

\[ \text{Theorem 3.13.} \quad \text{Let } N \text{ be a prime } \Gamma\text{-near-ring with a nonzero generalized derivation } f \text{ associated with } d. \text{ If } [f(N), f(N)]_\alpha = 0, \alpha \in \Gamma, \text{ then } (N, +) \text{ is abelian. Moreover, if } N \text{ is 2-torsion free, then } N \text{ is commutative } \Gamma\text{-ring.} \]

\[ \text{Proof.} \quad \text{By the same argument as in Theorem 3.12, it is shown that if both } z \text{ and } z + z \text{ commute elementwise with } f(N), \text{ then we have}

\[ zaf(x, y) = 0 \quad \forall x, y \in N, \alpha \in \Gamma. \quad (3.11) \]

Substituting \( f(t), t \in N \) for \( z \) in (3.11), we get \( f(t)af(x, y) = 0, \alpha \in \Gamma \). By Lemma 3.9(i), we obtain that \( f(x, y) = 0 \) for all \( x, y \in N, \alpha \in \Gamma \). For any \( w \in N, \beta \in \Gamma \), we have \( 0 = f(w_\beta x, w_\beta y) = f(w_\beta(x, y)) = d(w_\beta x, y) + w_\beta f(x, y) \) and so, we obtain \( d(w)\beta(x, y) = 0 \), for any \( w \in N, \beta \in \Gamma \). From Lemma 3.4(iii), we get \( (x, y) = 0 \) for any \( x, y \in N \).

Now we assume that \( N \) is 2-torsion free. By the assumption \([f(N), f(N)]_\alpha = 0, \alpha \in \Gamma\), we have

\[ f(z)af(f(x)bf(y) = f(f(x)bf(z)af(z) \quad \forall x, y, z \in N, \alpha, \beta \in \Gamma. \quad (3.12) \]

Hence we get

\[ f(z)ad(f(x))bf(y) + f(z)af(x)bf(y) = d(f(x)bf(z) + f(x)af(y)bf(z), \]

\[ f(z)af(f(x)bf(y) = d(f(x)bf(z) + f(x)af(z)bf(y), \quad (3.13) \]
and so,

$$f(z)ad(f(x)βy) = d(f(x))αyβf(z) \quad ∀x, y, z ∈ N, α, β ∈ Γ.$$  \hspace{1cm} (3.14)

If we take $yδw$ instead of $y$ in (3.14), then

$$d(f(x))αyδwβf(z) = f(z)ad(f(x)βyδw) = d(f(x))αyδf(z)βw$$

$$∀x, y, z ∈ N, α, β, δ ∈ Γ,$$  \hspace{1cm} (3.15)

and so,

$$d(f(x))αyδwβf(z) - d(f(x))αyδf(z)βw = d(f(x))αyδ[f(z), w]_δ = 0$$

$$∀x, y, z ∈ N, α, β, δ ∈ Γ.$$  \hspace{1cm} (3.16)

Thus we get $d(f(x))ΓNΓ[f(z), w]_δ = 0$, for all $x, y, z ∈ N, α, β, δ ∈ Γ$. Since $N$ is a prime $Γ$-near-ring, we have $d(f(N)) = 0$ or $f(N) ⊂ Z(N)$. Let us assume that $d(f(N)) = 0$. Then

$$0 = d(f(xay)) = d(d(x)ay + xaf(y))$$  \hspace{1cm} (3.17)

and so,

$$d^2(x)ay + d(x)ad(y) + d(x)af(y) = 0, \quad ∀x, y ∈ N, α ∈ Γ.$$  \hspace{1cm} (3.18)

Replacing $y$ by $yβz, β ∈ Γ$, in (3.18), we get

$$0 = d^2(x)ayβz + d(x)ad(yβz) + d(x)af(yβz)$$

$$= d^2(x)ayβz + d(x)ad(y)βz + d(x)adyβd(z) + d(x)af(y)βz + d(x)ayβd(z)$$

$$= \left\{d^2(x)ay + d(x)ad(y) + d(x)af(y)\right\}βz + 2d(x)ayβd(z) \quad ∀x, y, z ∈ N, α, β ∈ Γ.$$  \hspace{1cm} (3.19)

Using (3.18) and $N$ being 2-torsion free $Γ$-near-ring, we get $d(N)ΓNΓd(N) = 0$. Thus we obtain that $d = 0$. It contradicts by $d ≠ 0$. The theorem is proved. \hfill □

4. **Generalized Derivations of $Γ$-Near-Rings**

We denote a generalized derivation $f : N → N$ determined by a derivation $d$ of $N$ by $(f, d)$. We assume that $d$ is a nonzero derivation of $N$ unless stated otherwise.

**Theorem 4.1.** Let $(f, d)$ be a generalized derivation of $N$. If $f([x, y]_d) = 0$ for all $x, y ∈ N, α ∈ Γ$, then $N$ is a commutative $Γ$-ring.
Proof. Assume that \( f([x,y]_\alpha) = 0 \) for all \( x, y \in N, \alpha \in \Gamma \). Substitute \( x\beta y \) instead of \( y \), obtaining
\[
f([x,x\beta y]_\alpha) = f(x\beta[x,y]_\alpha) = d(x)\beta[x,y]_\alpha + x\beta f([x,y]_\alpha) = 0. \tag{4.1}
\]
Since the second term is zero, it is clear that
\[
d(x)ax\beta y = d(x)ay\beta x \quad \forall x, y \in N, \alpha, \beta \in \Gamma. \tag{4.2}
\]
Replacing \( y \) by \( y\delta z \) in (4.2) and using this equation, we get
\[
d(x)a\gamma\beta[\alpha, z]_\delta = 0 \quad \forall x, y, z \in N, \alpha, \beta, \delta \in \Gamma. \tag{4.3}
\]
Hence either \( x \in Z(N) \) or \( d(x) = 0 \). Let \( L = \{x \in N \mid d(x) = 0\} \). Then \( Z(N) \) and \( L \) are two additive subgroups of \( (N, +) = Z(N) \cup L \). However, a group cannot be the union of proper subgroups, hence either \( N = Z(N) \) or \( N = L \). Since \( d \neq 0 \), we are forced to conclude that \( N \) is a commutative \( \Gamma \)-ring.

**Theorem 4.2.** Let \((f, d)\) be a generalized derivation of \( N \). If \( f([x,y]_\alpha) = \pm [x,y]_\alpha \) for all \( x, y \in N, \alpha \in \Gamma \), then \( N \) is a commutative \( \Gamma \)-ring.

Proof. Assume that \( f([x,y]_\alpha) = \pm [x,y]_\alpha \) for all \( x, y \in N, \alpha \in \Gamma \). Replacing \( y \) by \( x\beta y, \beta \in \Gamma \), in the hypothesis, we have
\[
f([x,x\beta y]_\alpha) = \pm(xax\beta y - xay\beta x) = \pm x\beta[x,y]_\alpha. \tag{4.4}
\]
On the other hand,
\[
f([x,x\beta y]_\alpha) = f(x\beta[x,y]_\alpha) = d(x)\beta[x,y]_\alpha + x\beta f([x,y]_\alpha) = d(x)\beta[x,y]_\alpha + x\beta(\pm [x,y]_\alpha). \tag{4.5}
\]
It follows from the two expressions for \( f([x,x\beta y]_\alpha) \) that
\[
d(x)ax\beta y = d(x)ay\beta x \quad \forall x, y \in N, \alpha, \beta \in \Gamma. \tag{4.6}
\]
Using the same argument as in the proof of Theorem 4.1, we get that \( N \) is a commutative \( \Gamma \)-ring.

**Theorem 4.3.** Let \((f, d)\) be a nonzero generalized derivation of \( N \). If \( f \) acts as a homomorphism on \( N \), then \( f \) is the identity map.

Proof. Assume that \( f \) acts as a homomorphism on \( N \). Then one obtains
\[
f(xy) = f(x)af(y) = d(xy) + xaf(y) \quad \forall x, y \in N, \alpha \in \Gamma. \tag{4.7}
\]
Replacing \( y \) by \( y\beta z \) in (4.7), we arrive at
\[
f(x)\alpha f(y\beta z) = d(x)\alpha y\beta z + x\alpha f(y\beta z). \tag{4.8}
\]

Since \((f, d)\) is a generalized derivation and \( f \) acts as a homomorphism on \( N \), we deduce that
\[
f(x\alpha y)\beta f(z) = d(x)\alpha y\beta z + x\alpha d(y)\beta z + x\alpha y\beta f(z). \tag{4.9}
\]

By Lemma 3.9(ii), we get
\[
d(x)\alpha y\beta f(z) + x\alpha f(y)\beta f(z) = d(x)\alpha y\beta z + x\alpha d(y)\beta z + x\alpha y\beta f(z), \tag{4.10}
\]
and so
\[
d(x)\alpha y\beta f(z) + x\alpha f(y\beta z) = d(x)\alpha y\beta z + x\alpha d(y)\beta z + x\alpha y\beta f(z). \tag{4.11}
\]
That is,
\[
d(x)\alpha y\beta f(z) + x\alpha d(y)\beta z + x\alpha y\beta f(z) = d(x)\alpha y\beta z + x\alpha d(y)\beta z + x\alpha y\beta f(z). \tag{4.12}
\]

Hence, we deduce that
\[
d(x)\alpha y\beta (f(z) - z) = 0 \quad \forall x, y, z \in N, \; \alpha, \beta \in \Gamma. \tag{4.13}
\]

Because \( N \) is prime and \( d \neq 0 \), we have \( f(z) = z \) for all \( z \in N \). Thus, \( f \) is the identity map. \( \square \)

**Theorem 4.4.** Let \((f, d)\) be a nonzero generalized derivation of \( N \). If \( f \) acts as an antihomomorphism on \( N \), then \( f \) is the identity map.

**Proof.** By the hypothesis, we have
\[
f(x\alpha y) = f(y)\alpha f(x) = d(x)\alpha y + x\alpha f(y) \quad \forall x, y \in N, \; \alpha \in \Gamma. \tag{4.14}
\]
Replacing \( y \) by \( x\beta y \) in the last equation, we obtain
\[
f(x\beta y)\alpha f(x) = d(x)\beta x\alpha y + x\beta f(x\alpha y). \tag{4.15}
\]

Since \((f, d)\) is a generalized derivation and \( f \) acts as an antihomomorphism on \( N \), we get
\[
(d(x)\beta y + x\beta f(y))\alpha f(x) = d(x)\alpha x\beta y + x\alpha f(y)\beta f(x). \tag{4.16}
\]

By Lemma 3.9(ii), we conclude that
\[
d(x)\alpha y\beta f(x) + x\alpha f(y)\beta f(x) = d(x)\alpha x\beta y + x\alpha f(y)\beta f(x), \tag{4.17}
\]

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and so
\[ d(x)ay\beta f(x) = d(x)ax\beta y \quad \forall x, y \in N, \ a, \beta \in \Gamma. \quad (4.18) \]
Replacing \( y \) by \( y\delta z \) and using this equation, we have
\[ d(x)ay\beta [f(x), z]_a = 0 \quad \forall x, z \in N, \ a, \beta \in \Gamma. \quad (4.19) \]

Hence we obtain the following alternatives: \( d(x) = 0 \) or \( f(x) \in Z(N) \), for all \( x \in N \). By a standard argument, one of these must hold for all \( x \in N \). Since \( d \neq 0 \), the second possibility gives that \( N \) is commutative \( \Gamma \)-ring by Theorem 3.12, and so we deduce that \( f \) is the identity map by Theorem 4.3.

**Theorem 4.5.** Let \((f, d)\) be a generalized derivation of \( N \) such that \( d(Z(N)) \neq 0 \), and \( a \in N \). If \([f(x), a]_a = 0 \) for all \( x \in N, \ a, \beta \in \Gamma, \) then \( a \in Z(N) \).

**Proof.** Since \( d(Z(N)) \neq 0 \), there exists \( c \in Z(N) \) such that \( d(c) \neq 0 \). Furthermore, as \( d \) is a derivation, it is clear that \( d(c) \in Z(N) \). Replacing \( x \) by \( c\beta x, \ \beta \in \Gamma \), in the hypothesis and using Lemma 3.9(ii), we have
\[ f(c\beta x)aa = aa f(c\beta x), \]
\[ d(c)ax\beta a + c\alpha f(x)\beta a = aad(c)\beta x + aac\beta f(x). \]

Since \( c \in Z(N) \) and \( d(c) \in Z(N) \), we get
\[ d(c)ax\beta [y, a]_a = 0 \quad \forall y \in N, \ a, \beta, \delta \in \Gamma. \quad (4.21) \]
By the primeness of \( N \) and \( 0 \neq d(c) \in Z(N) \), we obtain that \( a \in Z(N) \). \( \square \)

**Theorem 4.6.** Let \((f, d)\) be a generalized derivation of \( N \), and \( a \in N \). If \([f(x), a]_a = 0 \) for all \( x \in N \), then \( d(a) \in Z(N) \).

**Proof.** If \( a = 0 \), then there is nothing to prove. Hence, we assume that \( a \neq 0 \). Replacing \( x \) by \( a\beta x \) in the hypothesis, we have
\[ f(a\beta x)aa = aa f(a\beta x), \]
\[ d(a)ax\beta a + aaf(x)\beta a = aad(a)\beta x + aac\beta f(x). \]
Using \( f(x)aa = aa f(x) \), we have
\[ d(a)ax\beta a = aad(a)\beta x \quad \forall x \in N, \ a, \beta \in \Gamma. \quad (4.23) \]
Taking \( x\delta y \) instead of \( x \) in the last equation and using this, we conclude that
\[ d(a)\alpha N\beta [a, y]_a = 0 \quad \forall y \in N, \ a, \beta \in \Gamma. \quad (4.24) \]
Since $N$ is a prime $\Gamma$-near-ring, we have either $d(a) = 0$ or $a \in Z(N)$. If $0 \neq a \in Z(N)$, then $(N, +)$ is abelian by Lemma 3.2(ii). Thus

\begin{equation}
\begin{aligned}
f(xaa) &= f(aax) \\
f(x)ax + xad(a) &= d(a)ax + aa f(x)
\end{aligned}
\end{equation}

and so

\[ [d(a), x]_a = 0 \quad \forall x \in N, \; a \in \Gamma. \]  

That is, $d(a) \in Z(N)$. Hence in either case we have $d(a) \in Z(N)$. This completes the proof. 

**Theorem 4.7.** Let $(f, d)$ be a generalized derivation of $N$. If $N$ is a 2-torsion free $\Gamma$-near-ring and $f^2(N) \subset Z(N)$, then $N$ is a commutative $\Gamma$-ring.

**Proof.** Suppose that $f^2(N) \subset Z(N)$. Then we get

\begin{equation}
\begin{aligned}
f^2(xay) &= f^2(x)ay + 2f(x)ad(y) + xad^2(y) \in Z(N) \quad \forall x, y \in N, \; a \in \Gamma.
\end{aligned}
\end{equation}

In particular, $f^2(x)ac + 2f(x)ad(c) + xad^2(c) \in Z(N)$ for all $x \in N$, $c \in Z(N)$, $a \in \Gamma$. Since the first summand is an element of $Z(N)$, we have

\begin{equation}
2f(x)ad(c) + xad^2(c) \in Z(N) \quad \forall x \in N, \; c \in Z(N), \; a \in \Gamma.
\end{equation}

Taking $f(x)$ instead of $x$ in (4.28), we obtain that

\begin{equation}
2f^2(x)ad(c) + f(x)ad^2(c) \in Z(N) \quad \forall x \in N, \; c \in Z(N), \; a \in \Gamma.
\end{equation}

Since $d(c) \in Z(N)$, $f^2(x) \in Z(N)$, and so $f^2(x)ad(c) \in Z(N)$ for all $x \in N$, $c \in Z(N)$, $a \in \Gamma$, we conclude $f(x)ad^2(c) \in Z(N)$ for all $x \in N$, $c \in Z(N)$, $a \in \Gamma$.

Since $N$ is prime, we get $d^2(Z(N)) = 0$ or $f(N) \subseteq Z(N)$. If $f(N) \subseteq Z(N)$, then $N$ is a commutative $\Gamma$-ring by Lemma 3.8. Hence, we assume $d^2(Z) = 0$. By (4.28), we get $2f(x)ad(c) \in Z(N)$ for all $x \in N$, $c \in Z(N)$, $a \in \Gamma$.

Since $N$ is a 2-torsion free near-ring and $d(c) \in Z(N)$, we obtain that either $f(N) \subset Z(N)$ or $d(Z(N)) = 0$. If $f(N) \subset Z(N)$, then we are already done. So, we may assume that $d(Z(N)) = 0$. Then

\begin{equation}
\begin{aligned}
f(cax) &= f(xac), \\
f(c)ax + cad(x) &= f(x)ac + xad(c),
\end{aligned}
\end{equation}

and so

\begin{equation}
\begin{aligned}
f(c)ax + cad(x) &= f(x)ac \quad \forall x \in N, \; c \in Z(N).
\end{aligned}
\end{equation}
Now replacing $x$ by $f(x)$ in (4.31), and using the fact that $f^2(N) \subset Z(N)$, we get

$$f(c)af(x) + cad(f(x)) = f^2(x)ac \quad \forall x \in N, \ c \in Z(N). \quad (4.32)$$

That is,

$$f(c)af(x) + cad(f(x)) \in Z \quad \forall x \in N, \ c \in Z(N), \ a \in \Gamma. \quad (4.33)$$

Again taking $f(x)$ instead of $x$ in this equation, one can obtain

$$f(c)af^2(x) + cad(f^2(x)) \in Z \quad \forall x \in N, \ c \in Z(N), \ a \in \Gamma. \quad (4.34)$$

The second term is equal to zero because of $d(Z) = 0$. Hence we have $f(c)af^2(x) \in Z(N)$ for all $x \in N, \ c \in Z(N), \ a \in \Gamma$.

Since $f^2(N) \subset Z(N)$ by the hypothesis, we get either $f^2(N) = 0$ or $f(Z(N)) \subset Z(N)$. If $f^2(N) = 0$, then the theorem holds by Definition 3.7. If $f(Z) \subset Z(N)$, then $f(xaf(c)) = f(f(c)ax)$ for all $x \in N, \ c \in Z(N)$, and so

$$d(x)af(c) = f(c)af(x) \quad \forall x \in N, \ c \in Z(N). \quad (4.35)$$

Using $f(c) \in Z(N)$, we now have $f(c)af(x) = f(x)af(c) = 0$ for all $x \in N, \ c \in Z(N), \ a \in \Gamma$. Since $f(Z(N)) \subset Z(N)$, we have either $f(Z(N)) = 0$ or $d = f$. If $d = f$, then $f$ is a derivation of $N$ and so $N$ is commutative $\Gamma$-ring by Lemma 3.11.

Now assume that $f(Z(N)) = 0$. Returning to the equation (4.31), we have

$$ca(d(x) - f(x)) = 0 \quad \forall x \in N, \ c \in Z(N), \ a \in \Gamma. \quad (4.36)$$

Since $c \in Z(N)$, we have either $d = f$ or $Z(N) = 0$. Clearly, $d = f$ implies the theorem holds. If $Z(N) = 0$, then $f^2(N) = 0$ by the hypothesis, and so $N$ is a commutative $\Gamma$-ring by Lemma 3.4(iv). Hence, the proof is completed. \qed

**Corollary 4.8.** Let $N$ be a 2-torsion free near-ring, and $(f, d)$ a nonzero generalized derivation of $N$. If $[f(f), f(N)]_d = 0, \ a \in \Gamma$, then $N$ is a commutative $\Gamma$-ring.

**Lemma 4.9.** Let $(f, d)$ and $(g, h)$ be two generalized derivations of $N$. If $h$ is a nonzero derivation on $N$ and $f(x)ah(y) = -g(x)ad(y)$ for all $x, y \in N$, then $(N, +)$ is abelian.

**Proof.** Suppose that $f(x)ah(y) + g(x)ad(y) = 0$ for all $x, y \in N, \ a \in \Gamma$.

We substitute $y + z$ for $y$, thereby obtaining

$$f(x)ah(y) + f(x)ah(z) + g(x)ad(y) + g(x)ad(z) = 0. \quad (4.37)$$

Using the hypothesis, we get

$$f(x)ah(y, z) = 0 \quad \forall x, y, z \in N, \ a \in \Gamma. \quad (4.38)$$
It follows by Lemma 3.10(ii) that \( h(y, z) = 0 \) for all \( y, z \in N \). For any \( w \in N \), we have \( h(w ay, wz) = h(way, wz) = h(way, z) = 0 \) and so \( h(w ay, z) = 0 \) for all \( w, y, z \in N, a, \in \Gamma \).

An appeal to Lemma 3.4(iii) yields that \((N, +)\) is abelian.

**Theorem 4.10.** Let \((f, d)\) and \((g, h)\) be two generalized derivations of \(N\). If \(N\) is 2-torsion free and \(f(x)ah(y) = -g(x)ad(y)\) for all \(x, y \in N, a \in \Gamma\), then \(f = 0\) or \(g = 0\).

**Proof.** If \(h = 0\) or \(d = 0\), then the proof of the theorem is obvious. So, we may assume that \(h \neq 0\) and \(d \neq 0\). Therefore, we know that \((N, +)\) is abelian by Lemma 4.9.

Now suppose that

\[
f(x)ah(y) + g(x)ad(y) = 0 \quad \forall x, y \in N, \ a \in \Gamma.
\] (4.39)

Replacing \(x\) by \(u^{\beta}v\) in this equation and using the hypothesis, we get

\[
f(u^{\beta}v)ah(y) + g(u^{\beta}v)ad(y)
\]

\[
= uaf(v)\beta h(y) + d(u)\alpha v\beta h(y) + uag(v)\beta d(y) + h(u)\alpha v\beta d(y)
\]

\[
= 0,
\] (4.40)

and so

\[
d(u)\alpha v\beta h(y) = -h(u)\alpha v\beta d(y) \quad \forall u, v, y \in N, \ a \in \Gamma.
\] (4.41)

Taking \(y^\delta t\) instead of \(y\) in the above relation, we obtain

\[
d(u)\alpha v\beta h(y)^\delta t + d(u)\alpha v\beta y^\delta h(t) = -h(u)\alpha v\beta d(y)^\delta t - h(u)\alpha v\beta y^\delta d(t).
\] (4.42)

That is,

\[
d(u)\alpha v\beta y^\delta h(t) = -h(u)\alpha v\beta y^\delta d(t) \quad \forall u, v, y, t \in N, \ a, \beta, \delta \in \Gamma.
\] (4.43)

Replacing \(y\) by \(h(y)\) in (4.43) and using this relation, we have

\[
h(u)\alpha N\beta (d(y)^\delta h(t) - h(y)ad(t)) = 0 \quad \forall u, y, t \in N.
\] (4.44)

Since \(N\) is a prime \(\Gamma\)-near-ring and \(h \neq 0\), we obtain that

\[
d(y)ah(t) = h(y)ad(t), \quad \forall y, t \in N.
\] (4.45)

Now again taking \(u^{\lambda}v\) instead of \(x\) in the initial hypothesis, we get

\[
f(u^{\lambda}v)ah(y) + uad(v)\beta h(y) + g(u)\alpha v\beta d(y) + uah(v)\beta d(y) = 0.
\] (4.46)
Using (4.45) yields that
\[
f(u)\alpha v\beta h(y) + 2uah(v)\beta d(y) + g(u)\alpha v\beta d(y) = 0 \quad \forall u, v, y \in N,
\]
Taking \( h(v) \) instead of \( v \) in this equation, we arrive at
\[
f(u)ah(v)\beta h(y) + 2uah^2(v)\beta d(y) + g(u)ah(v)\beta d(y) = 0.
\]
By the hypothesis and (4.45), we have
\[
0 = -g(u)d\alpha v\beta h(y) + 2uah^2(v)\beta d(y) + g(u)ah(v)\beta d(y)
\]
\[
= -g(u)ah(v)\beta d(y) + 2uah^2(v)\beta d(y) + g(u)ah(v)\beta d(y),
\]
and so
\[
2uah^2(v)\beta d(y) = 0 \quad \forall u, v, y \in N, \alpha, \beta \in \Gamma.
\]

Since \( N \) is a 2-torsion free prime \( \Gamma \)-near-ring, we obtain that \( h^2(N)\Gamma d(N) = 0 \). An appeal to Lemma 3.4(iii) and (iv) gives that \( h = 0 \). This contradicts by our assumption. Thus the proof is completed.

**Theorem 4.11.** Let \((f, d)\) and \((g, h)\) be two generalized derivations of \( N \). If \((fg, dh)\) acts as a generalized derivation on \( N \), then \( f = 0 \) or \( g = 0 \).

**Proof.** By calculating \( fg(xay) \) in two different ways, we see that \( g(x)ad(y) + f(x)ah(y) = 0 \) for all \( x, y \in N, \alpha \in \Gamma \). The proof is completed by using Theorem 4.10.

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