High Accuracy 2D Angle Estimation With Extended Aperture Vector Sensor Arrays

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ABSTRACT

A novel ESPRIT-based 2D angle estimation scheme is proposed involving the use of a right-triangular array of three vector sensors spaced much farther apart than a half-wavelength. A vector sensor is composed of six co-located antennas distinctly measuring all six electromagnetic field components of a multi-component incident wavefield. Information on each source's respective electromagnetic field components is obtained by decoupling the signal eigenvectors via lower dimensional eigenvectors derived from TLS-ESPRIT. This facilitates estimation of each source's respective Poynting vector thereby enabling one to resolve the phase ambiguities in ESPRIT's eigenvalues when the inter-vector-sensor spacing is greater than a half-wavelength. Simulations are presented showing the sample variance of the direction cosine estimates decreasing linearly on a log-log scale as the inter-vector-sensor spacing is increased from a half-wavelength to 30 wavelengths, with a factor of 80 reduction in the latter case relative to the former case. The proposed scheme and attendant vector sensor array also outperform a uniformly-spaced array of scalar sensors with the same aperture and same number of component antennas whenever the inter-vector sensor spacing in the former case is greater than 3 half-wavelengths.

1. INTRODUCTION

Array resolution and accuracy are limited by the effective size of the array aperture. In general, a larger array aperture yields correspondingly more accurate DOA estimates [1]. The array aperture may be enlarged by adding more antenna elements in the case of uniform half-wavelength spacing, by spacing the elements nonuniformly over a larger aperture, or by increasing the separation between elements in the case of uniform spacing. Adding more antennas has the obvious drawbacks of increasing hardware costs plus expanding the already considerable computational load required by eigenstructure methods. Nonuniform inter-element spacing would generally violate ESPRIT's requirement of two identical but translated subarrays. The spatial version of the Nyquist Sampling Theorem also poses an upper limit on the distance between elements in the case of uniform spacing without causing aliasing. Two identical sensors spaced over a half-wavelength apart will lead to a set of ambiguous direction-cosine estimates. With no a priori information, this ambiguity cannot be resolved using unpolarized scalar sensors.

A novel scheme is proposed employing a sparse array of six-component electromagnetic vector-sensors [2] that is able to resolve the aforementioned ambiguity despite inter-vector sensor spacings much greater than a half-wavelength. Exceptional estimation accuracy is facilitated by a large effective aperture and exploitation of differences in sources' respective polarizations.

1.1. Basic Principles Underlying New Algorithm

A vector-sensor consists of six spatially co-located antennas measuring all three electrical-field components and all three magnetic-field components of the incident wavefield [2]. The proposed array geometry is that of three vector-sensors located at (0, 0), (∆, 0), and (0, ∆) in (x, y) Cartesian coordinates. The inter-vector sensor spacing ∆ is assumed to be much greater than a half-wavelength to effect a large array aperture and correspondingly achieve highly accurate source direction estimates. A pair of identical vector-sensors effectively represent two identical six-element subarrays displaced in space thereby facilitating the use of ESPRIT [3]. TLS-ESPRIT is applied to the data from the vector sensors located at (0, 0) and (∆, 0) to ultimately yield estimates of the direction cosines of each source relative to the x-axis,

\[ \{u_k = \cos \phi_k \sin \theta_k, \ k = 1, \ldots, K\} \]

where \(\phi_k\) and \(\theta_k\) are the azimuth and elevation arrival angles of the \(k\)-th incident signal. Simultaneously, TLS-ESPRIT is applied to the data from the vector sensors located at (0, 0) and (0, ∆) to ultimately yield estimates of the direction cosines of each source relative to the y-axis,

\[ \{v_k = \cos \phi_k \sin \theta_k, \ k = 1, \ldots, K\} \]

However, when the inter-vector sensor spacing is greater than a half-wavelength there is an integer multiple of \(\pi\) ambiguity in the phase of each eigenvalue generated in the final stage of TLS-ESPRIT. This leads to ambiguous DOA estimates equi-spaced by \(\lambda/\Delta\) in the interval \(-1 \leq u_k < 1\) (or \(-1 \leq v_k < 1\) as the case may be.) If one could resolve this ambiguity, the resulting DOA estimate would be highly accurate due to the large aperture. The key idea of this paper is to use the Poynting vector information for each source embedded in the output of each vector sensor to resolve this ambiguity. The lower dimensional eigenvectors generated in the final stage of TLS-ESPRIT are used to decouple the element space signal eigenvectors to yield both the electric field vector and the magnetic field vector for each source. The cross product between these two vectors for a given source yields the Poynting vector for that source. Normalizing the Poynting vector for a given source to have unit length, its components are the direction cosines of that source relative to the x, y, and z axes. That is,

\[ p_k = \begin{bmatrix} p_{x_k} \\ p_{y_k} \\ p_{z_k} \end{bmatrix} = \begin{bmatrix} u_k \\ v_k \end{bmatrix} \]

\[ \|e(\theta_k, \phi_k, \gamma_k, \eta_k)\| \times \|h(\theta_k, \phi_k, \gamma_k, \eta_k)\| \]

\[ \Re \left\{ \frac{e(\theta_k, \phi_k, \gamma_k, \eta_k) \times h(\theta_k, \phi_k, \gamma_k, \eta_k)}{\|e(\theta_k, \phi_k, \gamma_k, \eta_k)\| \times \|h(\theta_k, \phi_k, \gamma_k, \eta_k)\|} \right\} \]

(1)
where \( w_k = \sqrt{1 - u_k^2 - v_k^2} \), \( * \) denotes conjugation, and \( \gamma_k \) and \( \eta_k \) are polarization parameters defined in Section 2. The idea is to use these as " coarse" estimates to resolve the integer multiple of \( 1/\Delta \) ambiguity in the DOA estimates yielded by TLS-ESPRIT's eigenvalues.

The direction cosine estimates produced by the Poynting vector estimation procedure are characterized as high variance but unambiguous in comparison to the direction cosine estimates extracted from the TLS-ESPRIT eigenvalues which are characterized as low variance but ambiguous. Intuitively, the lower variance of the latter relative to the former is due to the fact that the direction cosine estimates extracted from the TLS-ESPRIT eigenvalues depend on the size of the array aperture as measured by \( \Delta \); the larger \( \Delta \), the smaller the variance of \( L \mu_k / \pi \Delta \), where \( \mu_k \) is a TLS-ESPRIT eigenvalue associated with the \( k \)-th source. This is confirmed by simulations presented in Section 4. The relative high variance of the direction cosine estimates extracted from the Poynting vector estimates is intuitively due to the fact that they are inherently extracted from information provided by a single vector sensor which has no effective aperture.

2. VECTOR SENSOR ARRAY DATA MODEL

The propagation model is transverse electromagnetic planewaves traveling through a non-conductive homogeneous isotropic medium. Under these conditions, the wavefront incident upon the array has an electric-field vector, \( e \), and a corresponding magnetic-field vector, \( h \), that can be expressed in Cartesian coordinates as [2]

\[
e = e_x \mathbf{v}_x + e_y \mathbf{v}_y + e_z \mathbf{v}_z \tag{2}
\]

\[
= (\sin \gamma \cos \theta \cos \phi \, e^{j \phi} - \cos \gamma \sin \phi) \mathbf{v}_x \\
+ (\sin \gamma \cos \theta \sin \phi \, e^{j \phi} + \cos \gamma \sin \phi) \mathbf{v}_y \\
- \sin \gamma \sin \theta \, e^{j \phi} \mathbf{v}_z
\]

\[
h = h_x \mathbf{v}_x + h_y \mathbf{v}_y + h_z \mathbf{v}_z \tag{3}
\]

\[
= (\cos \gamma \cos \phi + \sin \gamma \sin \phi \, e^{j \phi}) \, Z \mathbf{v}_x \\
- (\cos \gamma \sin \phi - \sin \gamma \cos \phi \, e^{j \phi}) \, Z \mathbf{v}_y \\
+ \cos \gamma \sin \theta \, Z \mathbf{v}_z
\]

where \( 0 \leq \gamma \leq \pi/2 \) is the auxiliary polarization angle, \( -\pi < \eta \leq \pi \) is the polarization phase difference, \( 0 \leq \theta < \pi \) is the signal's elevation angle measured from the vertical \( z \)-axis, \( 0 \leq \phi < 2\pi \) is the azimuth angle, \( Z \) is the transmission medium's intrinsic impedance, and \( \mathbf{v} \) is a unit-vector along the coordinate specified by its subscript.

\( K \) co-channel signal sources are assumed to be in the far field and narrowband in the sense that the bandwidth is very small relative to the carrier frequency. Given a right-triangular array of three vector sensors with the geometry described previously, the \( 18 \times 1 \) snapshot vector at time \( t \) may be expressed as

\[
z(t) = \sum_{k=1}^{K} s_k(t)e^{-j\omega t}\mathbf{v}(\theta_k, \phi_k) + n(t). \tag{6}
\]

The first, middle, and last \( 6 \times 1 \) sub-vectors of \( z(t) \) are the outputs of the six component-antennas comprising the vector sensor located at the Cartesian coordinates \((0, 0)\), \((\Delta, 0)\), and \((0, \Delta)\), respectively, relative to the \( x \)-y plane. The output quantities are defined as follows. \( \mathbf{v}(\theta_k, \phi_k) \) is the \( 3 \times 1 \) manifold describing the relative phases between vector sensors for the \( k \)-th source due to the propagation delays

\[
\mathbf{v}(\theta_k, \phi_k) = \begin{bmatrix} e^{j 2\Delta \Delta \Delta_k} \\ e^{j 2\Delta \Delta \Delta_k} \end{bmatrix} \tag{7}
\]

where \( \lambda \) is the wavelength associated with the center frequency, \( \omega_c \), of the passband of the front end bandpass filter. \( \odot \) is the Kronecker product. \( a_k \overset{def}{=} \mathbf{v}(\theta_k, \phi_k, \gamma_k, \eta_k) \) is the \( 6 \times 1 \) composite electromagnetic field vector for the \( k \)-th source sensed at each vector sensor, i.e.,

\[
a_k = \begin{bmatrix} e_{x_k} \\ e_{y_k} \\ e_{z_k} \\ h_{x_k} \\ h_{y_k} \\ h_{z_k} \end{bmatrix} = \begin{bmatrix} \sin \gamma_k \cos \theta_k \cos \phi_k e^{j \phi_k} - \cos \gamma_k \sin \phi_k \\ \sin \gamma_k \cos \theta_k \sin \phi_k e^{j \phi_k} + \cos \gamma_k \cos \phi_k \\ - \sin \gamma_k \sin \theta_k e^{j \phi_k} \\ - \sin \gamma_k \sin \phi_k e^{j \phi_k} - \sin \gamma_k \sin \phi_k e^{j \phi_k} Z_\phi \\ - \sin \gamma_k \sin \phi_k e^{j \phi_k} + \sin \gamma_k \cos \phi_k e^{j \phi_k} Z_\phi \\ \cos \gamma_k \sin \theta_k e^{j \phi_k} \end{bmatrix}
\]

\( s_k(t) \) is the complex baseband signal comprising the \( k \)-th signal arrival. Finally, \( n(t) \) is an \( 18 \times 1 \) additive white noise vector.

3. 2D ANGLE ESTIMATION VIA AN EXTENDED VECTOR SENSOR ARRAY

3.1. Adapting ESPRIT to Vector Sensor Arrays

ESPRIT [1] is applicable to the right triangular array of three vector sensors because any two vector sensors represent a pair of identical six element subarrays. The first step in ESPRIT is to compute the signal eigenvectors. Let \( \mathbf{E} \) be an \( 18 \times K \) matrix composed of the \( K \) "largest" eigenvectors of the \( 18 \times 18 \) sample covariance matrix

\[
\mathbf{R} = \frac{1}{N} \sum_{i=1}^{N} z(t)z^H(t). \tag{8}
\]

Due to the invariance associated with the two vector sensors along the \( y \)-axis as well as the invariance associated with the two vector sensors along the \( z \)-axis, it follows from the snapshot model in (6), that \( \mathbf{E} \) asymptotically approaches the following form

\[
\mathbf{E} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \mathbf{AT} = \begin{bmatrix} A_1 \\ A_1 \Phi^u \\ A_1 \Phi^v \end{bmatrix} T, \tag{9}
\]

where \( A_1 \) is the \( 6 \times K \) matrix

\[
A_1 = [a_1, a_2, \ldots, a_K] = \begin{bmatrix} e_1 & e_2 & \cdots & e_K \\ h_1 & h_2 & \cdots & h_K \end{bmatrix}, \tag{10}
\]

\( \Phi^u \) and \( \Phi^v \) are the \( K \times K \) diagonal matrices,

\[
\Phi^u = \text{diag} \{ e^{j 2\Delta \Delta \Delta_1}, e^{j 2\Delta \Delta \Delta_2}, \ldots, e^{j 2\Delta \Delta \Delta_K} \}, \tag{11}
\]

\[
\Phi^v = \text{diag} \{ e^{j 2\Delta \Delta \Delta_1}, e^{j 2\Delta \Delta \Delta_2}, \ldots, e^{j 2\Delta \Delta \Delta_K} \}, \tag{12}
\]

respectively, and \( T \) is an unknown \( K \times K \) matrix.

It is easily shown that in the no noise or asymptotic cases the solution to the matrix equation \( \mathbf{E}_1 \Psi = \mathbf{E}_2 \) is \( \Psi^u = T^{-1} \Psi^v T \). Let the eigenvalues of \( \Psi^u \) be denoted \( \mu_k, k = 1, \ldots, K \). It follows that \( \mu_k = \Psi_{kk} = e^{j 2\Delta \Delta \Delta_k}, k = 1, \ldots, K \), and the corresponding eigenvectors are the columns of \( T^{-1} \). If the inter-vector sensor spacing \( \Delta \) is greater than a half wavelength, the candidates for the direction cosine of the \( k \)-th source relative to the \( x \)-axis are

\[
\phi_k^{(m)} = \frac{\lambda}{2\pi \Delta} \mu_k + \pi \frac{\Delta}{\lambda}, \quad \left[ -\Delta \frac{\lambda}{2\pi} \mu_k \right] \leq m \leq \left[ \Delta \frac{\lambda}{2\pi} \mu_k \right] \tag{13}
\]

where \( \mu_k \) is the principal argument of \( \mu_k \) in the range \( -\pi < \mu_k \leq \pi \). [\( x \)] is the closest integer to \( x \) that is greater
than \( x \), and \( \lfloor x \rfloor \) is the closest integer to \( x \) that is less than \( x \). (13) represents a set of low variance but ambiguous estimates of the direction cosine of the \( k \)-th source relative to the \( x \)-axis.

Similarly, in the asymptotic or no noise cases the solution to the matrix equation \( \hat{E}_i \hat{S} = \hat{E}_k \) is \( \hat{S} = \hat{S} = \hat{T}^{-1} \hat{S} \hat{T} \).

Let the eigenvalues of \( \hat{S} \) be denoted \( \mu_t \), \( t = 1, \ldots, K \). It follows that \( \nu_t = \lambda_t \hat{S} \hat{S}^T \), \( t = 1, \ldots, K \), and the corresponding eigenvectors are the columns of \( \hat{T}^{-1} \).

With \( \Delta \) greater than a half wavelength, the candidates for the direction cosine of the \( t \)-th source relative to the \( y \)-axis are

\[
\nu_t^{(n)} = \frac{\lambda}{2\pi \Delta} \hat{S} \hat{S}^T + n \frac{\lambda}{\Delta} \left[ \frac{-\Delta}{\lambda} - \frac{\hat{S} \hat{S}^T}{2\pi} \right] \leq n \leq \frac{\Delta}{\lambda} \left[ \frac{-\hat{S} \hat{S}^T}{2\pi} \right]
\]

(14) represents a set of low variance but ambiguous estimates of the direction cosine of the \( k \)-th source relative to the \( y \)-axis.

Note that different indices are used to enumerate the respective eigenvalues of \( \hat{S} \) and \( \hat{S} \). Although \( \hat{S} \) and \( \hat{S} \) have common eigenvectors, the ordering of the eigenvectors of \( \hat{S} \) will in general be permuted relative to the ordering of the eigenvectors of \( \hat{S} \).

van der Veen et al. [4] have developed a 2D extension of ESPRIT that exploits the fact that \( \hat{S} \) and \( \hat{S} \) commute, since they have common eigenvectors, to automatically pair the eigenvalues of \( \hat{S} \) with those of \( \hat{S} \) so that each resulting pair of eigenvalues is one-to-one associated with an incident waveform thereby automatically pairing the direction cosine estimates with respect to the two array axes.

Effectively, van der Veen et al. propose a scheme for perturbing both \( \hat{S} \) and \( \hat{S} \) so that the resulting perturbed matrices commute. \( \hat{T}^{-1} \) is then constructed from the eigenvectors of either perturbed matrix. Pre- and post-multiplying both the perturbed \( \hat{S} \) and the perturbed \( \hat{S} \) by \( \hat{T} \) and \( \hat{T}^{-1} \), respectively, yields a pair of approximately diagonal matrices whose diagonal elements are automatically paired. At this point, we will assume that the respective eigenvalues of \( \hat{S} \) and \( \hat{S} \) have been correctly paired.

The matrix whose columns are one-to-one related to the electromagnetic field vectors for each source defined by (10) may be estimated as

\[
\hat{A}_k = \frac{1}{3} \left\{ \hat{E}_i \hat{T}^{-1} + \hat{E}_k \hat{T}^{-1} \hat{S}^* \right\} (15)
\]

Now, since the \( k \)-th column of \( \hat{T}^{-1} \) is the eigenvector associated with the \( k \)-th eigenvalue of \( \hat{S} \), \( \mu_k \), the \( k \)-th column of \( \hat{A}_k \) above is associated with \( \mu_k \). Given the expression for the asymptotic form of \( \hat{A}_k \) in the far right hand side of (10), the Poynting vector for the \( k \)-th source may be estimated via the vector cross-product between the top and bottom \( 3 \times 1 \) subvectors of the \( k \)-th column of \( \hat{A}_k \) according to (10). Normalizing the resulting cross-product vector to have unit length, the first and second components are high-variance but unambiguous estimates of the direction cosine of the \( k \)-th source relative to the \( x \) and \( y \) axes, respectively.

3.2. Disambiguation of ESPRIT Eigenvectors

Let \( \bar{\hat{S}} = [\hat{S} \hat{S}^T \hat{S} \hat{S}^T] \) denote the estimate of the normalized Poynting vector obtained by using the eigenvectors of the perturbed \( \hat{S} \) to decouple the signal eigenvectors comprising \( \hat{E}_k \) and the subsequent column-wise vector cross

\[1\text{In actuality, van der Veen et al. employ a Schur decomposition as opposed to an EVD and there is much detail on how they approximately enforce commutativity that is not included here due to space limitations.} \]
1 obtained by the Poynting vector estimation procedure proposed in this paper. Both curves are observed to be relatively constant as the inter-vector sensor spacing is increased from $\Delta = \lambda/2$ to $\Delta = 32\lambda$. Both curves are observed to be well above the corresponding counterpart curves associated with the disambiguated direction cosine estimates, with the separation between the two widening as the inter-vector sensor spacing is increased. This is seen clearly from $\Delta = R/2$ to $\Delta = 32\lambda$. This subdivides the claim that the Poynting vector estimation procedure yields high-variance but unambiguous direction cosine estimates in comparison to the direction cosine estimates extracted from TLS-ESPRIT's eigenvalues according to (13) and (14). Intuitively, the lower variance of the latter relative to the former is due to the fact that the direction cosine estimates extracted from the TLS-ESPRIT's eigenvalues depend on the size of the array aperture as measured by $\Delta$: the larger $\Delta$, the smaller the variance of $\bar{\mathbf{r}}_i / 2\pi\Delta$. The relative high variance of the direction cosine estimates extracted from the Poynting vector estimates is intuitively due to the fact that they are inherently extracted from information provided by a single vector sensor which has no effective aperture.

A breakdown phenomenon is observed at a threshold inter-vector sensor spacing of $\Delta = 32\lambda$ (64 half-wavelengths). The cause of this breakdown is still under investigation. The breakdown of the Poynting vector estimation procedure may be attributed to either $\mu_1 = e^\Delta v_1$ and $\mu_2 = e^\Delta v_2$ becoming nearly identical, or $\nu_1 = e^\Delta u_1$ and $\nu_2 = e^\Delta u_2$ becoming nearly identical, when $\Delta$ is near 32 wavelengths. The breakdown of the disambiguation procedure at and above $\Delta = 32\lambda$ may be attributed to picking the wrong grid value in a few of the 300 trial runs. It is well known that the sample variance and sample bias estimators employed in these simulations are not robust and are highly sensitive to a few outliers.

For purposes of comparison, simulations were conducted with an L-shaped array of 18 unpolarized scalar sensors, with 9 elements uniformly spaced at a half-wavelength along both the $x$ and $y$ axes. This results in an $18 \times 1$ array-manifold (same length as the 3 vector sensor case) and an array aperture of roughly 8 half-wavelengths along either axis. Employing the same signal parameters presented previously, the unpolarized scalar sensor array yielded an RMS standard deviation of $5 \times 10^{-4}$ for source 1. The 3 vector sensor array and attendant algorithm achieved the same RMS standard deviation with $\Delta = 1.5\lambda$ (3 half-wavelengths), i.e., with less than half the aperture of the unpolarized L-shaped array! This performance gain is due to the polarization diversity inherent in a vector sensor array. The performance gain of the vector sensor array and attendant algorithm proposed in this paper relative to the L-shaped array with the same number of component antennas becomes even more dramatic as $\Delta$ is increased. Increasing the inter-element spacing in the latter case is not feasible since the ambiguities induced would not be resolvable.

5. CONCLUSION

A novel ESPRIT-based 2D angle estimation scheme was proposed involving the use of a right-triangular array of three vector sensors spaced much farther apart than a half-wavelength. Simulations were presented showing the exceptional estimation accuracy one can achieve with a large extended aperture allowed by the disambiguation facilitated by the electromagnetic field information provided by the vector sensors. Processing three vector sensors according to the method developed in this paper allows one to estimate the azimuth and elevation angle of up to a maximum of six sources. In order to handle more sources, the proposed scheme may be extended for a larger number of vector sensors spaced according to a rectangular array geometry with inter-vector spacings much greater than a half-wavelength. This will be presented in the journal version of the paper.

6. REFERENCES


Figure 1: RMS standard deviation of $\hat{\theta}_1$ and $\hat{\theta}_2$ vs. inter-vector sensor spacing.