Some New Product Cordial Graphs

S K Vaidya\(^1\) and K K Kanani\(^2\)

\(^1\) Saurashtra University, Rajkot-360005, GUJARAT (INDIA)
E-mail: samirkvaidya@yahoo.co.in

\(^2\) L E College, Morbi-363642, GUJARAT (INDIA)
E-mail: kananikk@yahoo.co.in

Abstract

Here we investigate product cordial labeling for the path union of \(C_n(C_n)\) as well as Petersen graph. We also prove that the graph obtained by joining two copies of \(C_n(C_n)\) by a path of arbitrary length admits product cordial labeling. Similar result is also obtained for two copies of Petersen graph.

Keywords: Cordial labeling, Product cordial labeling, Path union.

AMS subject classification number(2000): 05C78.

1. Introduction

We begin with finite, connected and undirected graph \(G = (V(G), E(G))\) without loops and multiple edges. Here elements of sets \(V(G)\) and \(E(G)\) are known as vertices and edges respectively. In the present work \(C_n\) and \(P_n\) denote cycle and path of length \(n - 1\) respectively. For all standard terminology and notations we follow Gross and Yellen\(^4\). We will give brief summary of definitions which are useful for the present investigations.

**Definition 1.1.** If the vertices of graph are assigned values subject to certain conditions then it is known as **graph labeling**.

For detailed survey on graph labeling we refer to *A Dynamic Survey of Graph Labeling* by Gallian\(^3\).

**Definition 1.2.** Let \(G = (V(G), E(G))\) be a graph. A mapping \(f : V(G) \rightarrow \{0,1\}\) is called **binary vertex labeling** of \(G\) and \(f(v)\) is called the **label** of the vertex \(v\) of \(G\) under \(f\).

**Notations:**

- \(v_f(0)\) = Number of vertices with label 0.
- \(v_f(1)\) = Number of vertices with label 1.
- \(e_f(0)\) = Number of edges with label 0.
- \(e_f(1)\) = Number of edges with label 1.

**Definition 1.3.** A binary vertex labeling of graph \(G\) with induced edge labeling \(f^* : E(G) \rightarrow \{0,1\}\) defined by \(f^*(e = uv) = |f(u) - f(v)|\) is called a **cordial** labeling if \(|v_f(0) - v_f(1)| \leq 1\) and \(|e_f(0) - e_f(1)| \leq 1\). A graph which admits cordial labeling is called **cordial graph**.
The concept of cordial labeling was introduced by Cahit[2] and he investigated several results on this concept. Motivated through the concept of cordial labeling the product cordial labeling was introduced by Sundaram et al[6] where absolute difference of vertex labels is replaced by product of vertex labels.

**Definition 1.4.** A binary vertex labeling of graph $G$ with induced edge labeling $f^* : E(G) \to \{0, 1\}$ defined by $f^*(e = uv) = f(u)f(v)$ is called a product cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph which admits product cordial labeling is called product cordial graph.

In [6] it is proved that tree, unicyclic graph of odd order, triangular snakes, dragons and helms admit product cordial labeling. It is also proved that a graph $G$ with $p$ vertices and $q$ edges with $p \geq 4$ is product cordial then $q < \frac{p^2 - 1}{4}$.

**Definition 1.5.** (Shee and Ho[5]) Let $G_1, G_2, \ldots, G_n, n \geq 2$ be $n$ copies of a fixed graph $G$. The graph obtained by adding an edge between $G_i$ and $G_{i+1}$ for $i = 1, 2, \ldots, n - 1$ is called the path union of $G$.

**Definition 1.6.** Let $G = (V(G), E(G))$ be a graph. If every edge of graph $G$ is subdivided then the resulting graph is called barycentric subdivision of $G$. In other words barycentric subdivision is the graph obtained by inserting a vertex of degree 2 into every edge of the original graph. The barycentric subdivision of any graph $G$ is denoted by $S(G)$. It is easy to observe that $|V(S(G))| = |V(G)| + |E(G)|$ and $|E(S(G))| = 2|E(G)|$.

**Definition 1.7.** Consider barycentric subdivision of cycle $C_n$ and join each newly inserted vertices of incident edges by an edge. We denote the new graph by $C_n(C_n)$ as it looks like $C_n$ inscribed in $C_n$. We note that $|V(C_n(C_n))| = 2n$ and $|E(C_n(C_n))| = 3n$.

Vaidya and Kanani[7] have derived following results on product cordial labeling.

- The path union of $k$ copies of cycle $C_n$ is a product cordial graph except for odd $k$ and even $n$.
- The graph obtained by joining two copies of cycle $C_n$ by path $P_k$ admits product cordial labeling.
- The path union of $k$ copies of shadow graph $D_2(C_n)$ is a product cordial graph except for odd $k$.
- The graph $G$ obtained by joining two copies of shadow graph $D_2(C_n)$ by a path of arbitrary length is a product cordial graph.

The present work is aimed to investigate product cordial labeling for the path union of $C_n(C_n)$ and Petersen graph. Moreover we prove that the graph obtained by joining two copies of $C_n(C_n)$ by a path of arbitrary length admits product cordial labeling. Analogous result is derived for Petersen graph.

### 2. Main Results

**Theorem 2.1.** The path union $k$ copies of $C_n(C_n)$ is a product cordial graph except for odd $k$. 

Proof. Let \( G \) be the path union of \( k \) copies of \( C_n(C_n) \). Let \( G_1, G_2, \ldots, G_k \) be \( k \) copies of cycle \( C_n \) and \( G'_1, G'_2, \ldots, G'_k \) be \( k \) copies of cycle \( C_n \) which are obtained by joining each newly inserted vertices of adjacent edges by an edge. Next denote the successive vertices of \( G_i \) by \( u_{i1}, u_{i2}, \ldots, u_{in} \) and corresponding vertices of \( G'_i \) by \( u'_{i1}, u'_{i2}, \ldots, u'_{in} \). Let \( e_i = u_{i1}u_{(i+1)1} \) be the edge joining \( i^{th} \) copy and \( (i+1)^{th} \) copy of \( C_n(C_n) \) for \( i = 1, 2, \ldots, k-1 \).

Here we note that \( |V(G)| = 2nk \) and \( |E(G)| = 3nk + k - 1 \).

To define binary vertex labeling \( f : V(G) \to \{0, 1\} \) we consider following cases.

Case 1. \( k \equiv 0(\mod 2) \)

\[
\begin{align*}
  f(u_{ij}) &= 0; 1 \leq j \leq n & 1 \leq i \leq \frac{k}{2} \\
  f(u'_{ij}) &= 0; 1 \leq j \leq n \\
  f(u_{ij}) &= 1; 1 \leq j \leq n & \frac{k}{2} < i \leq k \\
  f(u'_{ij}) &= 1; 1 \leq j \leq n
\end{align*}
\]

In view of above defined labeling pattern \( v_f(0) = v_f(1) = nk \) and \( e_f(0) = e_f(1) + 1 = \frac{3nk + k}{2} \). Thus the graph under consideration is a product cordial graph when \( k \) is even.

Case 2. \( k \equiv 1(\mod 2) \)

In this case \( |V(G)| = 2nk \) is even. Therefore in order to satisfy the vertex condition for product cordiality and to minimize the edge labels with label 0, we label vertices of first \( \frac{k-1}{2} \) copies of \( C_n(C_n) \) by 0 and last \( \frac{k-1}{2} \) copies of \( C_n(C_n) \) by 1. Now for the \( (\frac{k+1}{2})^{th} \) copy of \( C_n(C_n) \) we label \( n \) adjacent vertices by 0 and remaining \( n \) vertices by 1 then \( |e_f(0) - e_f(1)| > 2 \). It is easy to verify that any other pattern to assign vertex labels satisfying the vertex condition will increase the difference between \( e_f(0) \) and \( e_f(1) \). Thus the graph under consideration is not a product cordial graph when \( k \) is odd.

Illustration 2.2. In the following Figure 1 the product cordial labeling of the path union of two copies of \( C_6(C_6) \) is demonstrated (The hollow vertices are newly inserted vertices for barycentric subdivision of \( C_6 \)).

Theorem 2.3. The graph obtained by joining two copies of \( C_n(C_n) \) by a path length is a product cordial graph.

Proof. Let \( G \) be the graph obtained by joining two copies of \( C_n(C_n) \) by a path \( P_k \) of length \( k - 1 \). Let \( u_1, u_2, \ldots, u_n \) be the vertices of cycle \( C_n \) and \( u'_1, u'_2, \ldots, u'_n \) be the corresponding vertices of the cycle which is obtained by joining newly inserted vertices of adjacent edges in cycle \( C_n \). Next denote the corresponding vertices in second copy of
$C_n(C_n)$ by $v_1, v_2, \ldots, v_n$ and $v'_1, v'_2, \ldots, v'_n$ respectively. Let $w_1, w_2, \ldots, w_k$ be the vertices of path $P_k$ with $u_1 = w_1$ and $v_1 = w_k$. Here we note that $|V(G)| = 4n + k - 2$ and $|E(G)| = 6n + k - 1$.

To define binary vertex labeling $f : V(G) \rightarrow \{0, 1\}$ we consider following cases.

**Case 1.** $k \equiv 0 \pmod{2}$

\[
\begin{align*}
f(u_i) &= 0 & 1 \leq i \leq n \\
f(u'_i) &= 0 \\
f(u_i) &= 1 & 1 \leq i \leq n \\
f(u'_i) &= 1 \\
f(w_j) &= 0 & 1 < j \leq \frac{k}{2} \\
f(w_j) &= 1 & \frac{k}{2} < j \leq k
\end{align*}
\]

**Case 2.** If $k \equiv 1 \pmod{2}$

\[
\begin{align*}
f(u_i) &= 0 & 1 \leq i \leq n \\
f(u'_i) &= 0 \\
f(u_i) &= 1 & 1 \leq i \leq n \\
f(u'_i) &= 1 \\
f(w_j) &= 0 & 1 < j \leq \frac{k-1}{2} \\
f(w_j) &= 1 & \frac{k-1}{2} < j \leq k
\end{align*}
\]

The labeling pattern defined above includes all possible arrangement of vertices. In each case the graph $G$ under consideration satisfies the conditions for product cordiality as shown in Table 1. That is, the graph obtained by joining two copies of $C_n(C_n)$ by a path of arbitrary length is a product cordial graph.

In the following table $n = 2a + b$, $k = 2c + d$ where $a, c \in \mathbb{N}$.

<table>
<thead>
<tr>
<th>$b$</th>
<th>$d$</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$v_f(0) = v_f(1) = \frac{4n+k-2}{2}$</td>
<td>$e_f(0) = e_f(1) + 1 = \frac{6n+k}{2}$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$v_f(0) + 1 = v_f(1) = \frac{4n+k-1}{2}$</td>
<td>$e_f(0) = e_f(1) = \frac{6n+k-1}{2}$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$v_f(0) = v_f(1) = \frac{6n+k-2}{2}$</td>
<td>$e_f(0) = e_f(1) + 1 = \frac{6n+k}{2}$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$v_f(0) + 1 = v_f(1) = \frac{6n+k-1}{2}$</td>
<td>$e_f(0) = e_f(1) = \frac{6n+k-1}{2}$</td>
</tr>
</tbody>
</table>

**Table 1**

**Illustration 2.4.** Consider a graph $G$ obtained by joining two copies of $C_4(C_4)$ by a path $P_3$. The labeling pattern is shown in Figure 2. (The hollow vertices are newly inserted vertices for barycentric subdivision of $C_4$.)

![Figure 2](image-url)
**Theorem 2.5.** The path union of \( k \) copies of Petersen graph is a product cordial graph except for odd \( k \).

**Proof.** Let \( G \) be the path union of \( k \) copies \( G_1, G_2, \ldots, G_k \) of Petersen graph. Let us denote the external vertices of \( G_i \) by \( u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5} \) and internal vertices by \( u_{i6}, u_{i7}, u_{i8}, u_{i9}, u_{i10} \). Let \( e_i = u_{i1}u_{i(i+1)} \) be the edge joining \( G_i \) and \( G_{i+1} \) for \( i = 1, 2, \ldots, k-1 \). We note that \( |V(G)| = 10k \) and \( |E(G)| = 16k - 1 \).

To define binary vertex labeling \( f : V(G) \to \{0, 1\} \) we consider following cases.

**Case 1.** \( k \equiv 0 \text{ (mod 2)} \)

\[
f(u_{ij}) = 0; 1 \leq i \leq \frac{k}{2}, 1 \leq j \leq 10
\]

\[
f(u_{ij}) = 1; \frac{k}{2} < i \leq k, 1 \leq j \leq 10
\]

In view of above defined labeling pattern \( v_f(0) = v_f(1) = 5k \) and \( e_f(0) = e_f(1)+1 = 8k \). Thus the graph under consideration is a product cordial graph when \( k \) is even.

**Case 2.** \( k \equiv 1 \text{ (mod 2)} \)

In this case \( |V(G)| = 10k \) is even. Therefore, in order to satisfy the vertex condition for product cordiality and to minimize the edge labels with label 0, we label the vertices of first \( \frac{k-1}{2} \) copies of \( G \) by 0 and last \( \frac{k-1}{2} \) copies of \( G \) by 1. Now for the \( \left( \frac{k+1}{2}\right) \)th copy of \( G \), we label 5 vertices of degree three by 0 and remaining 5 vertices by 1 then \( |e_f(0) - e_f(1)| > 2 \). It is easy to verify that any other pattern to assign vertex labels satisfying the vertex condition will increase the difference between \( e_f(0) \) and \( e_f(1) \).

Thus the graph under consideration is not a product cordial graph when \( k \) is odd.

**Illustration 2.6.** Consider a graph \( G \) obtained by the path union of two copies of Petersen graph. The product cordial labeling is shown in Figure 3.

**Theorem 2.7.** The graph \( G \) obtained by joining two copies of Petersen graph by a path of arbitrary length is a product cordial graph.

**Proof.** Let \( G_1 \) and \( G_2 \) be two copies of a Petersen graph. Let \( G \) be the graph obtained by joining \( G_1 \) and \( G_2 \) by a path \( P_k \) of length \( k - 1 \). Let \( u_1, u_2, u_3, u_4, u_5 \) be the external vertices of \( G_1 \) and \( u_6, u_7, u_8, u_9, u_{10} \) be the internal vertices of \( G_1 \). Let \( v_1, v_2, v_3, v_4, v_5 \) be the external vertices of \( G_2 \) and \( v_6, v_7, v_8, v_9, v_{10} \) be the internal vertices of \( G_2 \). Let \( w_1, w_2, \ldots, w_k \) be the vertices of path \( P_k \) with \( u_1 = w_1 \) and \( v_1 = w_k \). We note that \( |V(G)| = 20 + k - 2 \) and \( |E(G)| = 30 + k - 1 \).

To define binary vertex labeling \( f : V(G) \to \{0, 1\} \) we consider following cases.

**Case 1.** \( k \equiv 0 \text{ (mod 2)} \)

\[
f(u_i) = 0; 1 \leq i \leq 10
\]
\[ f(v_i) = 1; 1 \leq i \leq 10 \]
\[ f(w_j) = 0; 1 < j \leq \frac{k}{2} \]
\[ = 1; \frac{k}{2} < j < k \]

**Case 2.** \( k \equiv 1 \text{(mod2)} \)
\[ f(u_i) = 0; 1 \leq i \leq 10 \]
\[ f(v_i) = 1; 1 < i \leq 10 \]
\[ f(w_j) = 0; 1 < j \leq \frac{k-1}{2} \]
\[ = 1; \frac{k-1}{2} < j < k \]

The labeling pattern defined above includes all possible arrangement of vertices. In each case the graph \( G \) under consideration satisfies the conditions for product cordiality as shown in Table 2. That is, the graph \( G \) obtained by joining two copies of Petersen graph by a path of arbitrary length is a product cordial graph.

In the following table \( k = 2c + d \) where \( c \in N \).

<table>
<thead>
<tr>
<th>( d )</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( v_f(0) = v_f(1) = 10 + \frac{k-2}{2} )</td>
<td>( e_f(0) = e_f(1) + 1 = 15 + \frac{k}{2} )</td>
</tr>
<tr>
<td>1</td>
<td>( v_f(0) + 1 = v_f(1) = 10 + \frac{k-1}{2} )</td>
<td>( e_f(0) = e_f(1) = 15 + \frac{k-1}{2} )</td>
</tr>
</tbody>
</table>

Table 2

**Illustration 2.8.** Consider a graph \( G \) obtained by joining two copies of Petersen graph by a path \( P_6 \). The product cordial labeling is shown in Figure 4.

---

**3. Concluding Remarks**

As every graph does not admit product cordial labeling it is very interesting to find out graphs or graph families which admit product cordial labeling. Shee and Ho[5] investigated cordial labeling for the path union of various graphs while we have investigated product cordial labeling for the path union various graphs.
References


