Stochastic Control of Event-Driven Feedback in Multi-Antenna Interference Channels

Kaibin Huang, Vincent K. N. Lau, and Dongku Kim

Abstract

Spatial interference avoidance is a simple and effective way of mitigating interference in multi-antenna wireless networks. The deployment of this technique requires channel-state information (CSI) feedback from each receiver to all interferers, resulting in substantial network overhead. To address this issue, this paper proposes the method of distributive feedback control that intelligently allocates feedback bits over multiple feedback links and adapts feedback to channel dynamics. For symmetric channel distributions, it is optimal for each receiver to equally allocate the average sum-feedback rate for different feedback links, thereby decoupling their control. For low mobility and using the criterion of minimum sum-interference power, the optimal feedback-control policy is shown using stochastic optimization theory to exhibit opportunism. Specifically, a specific feedback link is turned on only when the corresponding transmit-CSI error is significant or interference-channel gain is large, and the optimal number of feedback bits increases with this gain. For high mobility and considering the sphere-cap-quantized-CI model, the optimal feedback-control policy is shown to perform water-filling in time, where the number of feedback bits increases logarithmically with the corresponding interference-channel gain. Furthermore, we consider asymmetric channel distributions with heterogeneous path loses and high mobility, and prove the existence of a unique optimal policy for jointly controlling multiple feedback links. Given the sphere-cap-quantized-CI model, this policy is shown to be of the water-filling type, where the number of feedback bits per feedback link increases logarithmically not only with the corresponding interference-channel gain but also with its distance. Finally, simulation demonstrates that feedback-control yields significant throughput gains compared with the conventional fixed-feedback method especially at low mobility.

Index Terms

Interference channels, array signal processing, stochastic optimal control, feedback communication, time-varying channels, dynamic programming, Markov processes

I. INTRODUCTION

Interference limits the performance of decentralized wireless networks but can be effectively mitigated by multi-antenna techniques, namely spatial interference cancelation and avoidance. In a frequency-
division-duplex network, spatial interference avoidance at interferers requires feedback of interference channel state information (CSI) from all interfered receivers, called cooperative feedback. Given finite-rate cooperative feedback, CSI quantization errors result in residual interference. Suppressing such interference requires high-resolution feedback over a network of feedback links, resulting in overwhelming network overhead. This demands research on intelligent feedback control that allocates feedback bits over multiple feedback links and adapts feedback to channel dynamics, which is the main theme of this paper.

A. Prior Work

Extensive research has been carried out on designing feedback-CSI-quantization algorithms for multi-antenna systems, called limited feedback [1], based on different approaches including line packing [2] and Lloyd’s algorithm [3]. Besides quantization, another effective approach for compressing feedback CSI is to explore the CSI redundancy due to the wireless-channel correlation in time [4], [5], frequency [6], and space [7]. Though a few feedback bits suffice in a point-to-point multi-antenna system, the feedback requirement is more stringent in multi-antenna downlink where CSI errors cause multiuser interference [8]. This motivates the jointly design of CSI feedback and scheduling algorithms to exploit multiuser diversity for reducing the required numbers of feedback bits [9]–[12]. Both high-resolution feedback for multi-antenna downlink and progressive feedback for correlated channels require CSI feedback with adjustable resolutions. This motivates the designs of hierarchical CSI-quantizer codebooks [13], [14] or methods for systematic codebook generation [15], [16]. The current work also concerns variable-rate feedback but focuses on the feedback-rate control rather than codebook designs.

Recent research on limited feedback explores more complex network topologies. In [17], the decentralized wireless networks based on interference alignment [18] are considered, and the required scaling of the numbers of feedback bits with respect to the signal-to-noise ratio (SNR) is derived such that the channel capacity is achieved for high SNR’s. The Grassmannian codebooks designed for point-to-point beamforming systems with limited feedback is shown in [19] to be suitable for multiple-input-multiple-output (MIMO) amplify-and-forward relay systems. The algorithms for cooperative feedback from the primary user to the secondary user are designed in [20] for implementing cognitive beamforming in two-user cognitive-radio systems. The above prior work does not explicitly address the issue of the optimal tradeoff between the network performance and the amount of CSI overhead.

In wireless networks, excessive CSI feedback yields marginal performance gain per additional feedback bit but insufficient feedback causes unacceptable performance degradation. Therefore, the optimal resource allocation for CSI feedback is a pertinent issue for designing efficient wireless networks. In [21], CSI feedback rates are optimized for maximizing the sum throughput in a two-way beamforming system.
where a pair of transceivers exchange both data and CSI. For a transmit beamforming system, bandwidth is optimally partitioned for CSI feedback and data transmission [22]. The problem of splitting the sum-feedback rate by a mobile for multiple cooperative-feedback links to interferers is studied in [23] in the context of base-station collaboration. It was shown that more feedback bits should be sent to nearer interfering base stations so as to reduce the throughput loss caused by feedback quantization. The feedback-bit allocation considered in prior work is static, targeting dedicated feedback channels in cellular networks [24]. In decentralized networks where a feedback channel is shared by multiple users, more efficient feedback-allocation should be adapted to channel dynamics, motivating event-driven feedback and stochastic feedback control.

B. Contributions and Organization

This work adopts the approach of stochastic feedback control proposed in [25] but targets more complex systems. Specifically, this paper concerns the $K$-user multiple-input-single-output (MISO) interference channel where there is an event-driven feedback controller at each receiver. The feedback controller dynamically and distributively determines the CSI feedback rate for each feedback link according to local CSI. As a result, each feedback controller serves multiple cooperative-feedback links in our system rather than a single feedback link to the intended transmitter as in [25]. Furthermore, we generalize the on/off feedback control in [25] to the variable-rate feedback control. This work establishes a novel approach of using stochastic optimization to achieve an optimal tradeoff between the CSI-feedback overhead and the throughput in the $K$-user multi-antenna interference network. For tractability, this paper focuses on the MISO interference channel where each receiver employs a distributive feedback controller for allocating CSI bits over multiple feedback links. The controller is designed based on several key assumptions. Channel coefficients are assumed to be independent and identically distributed (i.i.d.). The expectation of a CSI quantization error is assumed to be a monotone decreasing and convex function of the number of feedback bits, which is consistent with the popular CSI-quantizer models in [26], [27]. Moreover, for low mobility, the channel parameters, namely channel gains and transmit CSI (CSIT) errors, are assumed to vary in time following Markov chains and have the property that samples conditioned on large past realizations stochastically dominate those conditioned on small ones. The channels are assumed to follow independent block fading for high mobility. Based on these assumptions, this work has the following main contributions:

1. In this paper, we consider only cooperative feedback but not feedback from receivers to their intended transmitters. Hereafter, cooperative feedback is referred to simply as feedback.
For low mobility and under an average sum-feedback-rate constraint, a feedback controller is designed as a Markov decision process with average cost. By channel symmetry, it is optimal for each controller to equally split the average sum-feedback rate for all feedback links, reducing the problem of optimizing the multiple-feedback-link control policy to the single-feedback-link-policy optimization. The optimal policy for minimizing average sum-interference power is shown to exhibit opportunism. Specifically, feedback should be performed only when the corresponding interference-channel gain is large or the CSIT error is significant. Upon feedback, the optimal number of feedback bits for each feedback link increases with the corresponding interference-channel gain but is independent with the observed CSIT error.

For the same settings but for high mobility, the properties of the optimal feedback-control policy are similar to those of the low-mobility counterpart but are more explicit thanks to the simpler channel and sphere-cap-quantized-CSI models. Specifically, the number of feedback bits for each feedback link follows water-filling in time and is proportional to the logarithm of the corresponding interference-channel gain.

We also consider asymmetric channel distributions where interference-channel gains are scaled by heterogeneous path losses. For high mobility, the feedback-control policy optimization problem is decomposed into a master problem that optimally allocates average feedback rates for multiple feedback links, and a sub-problem that optimizes the policy for controlling the feedback-bit allocation in time for a particular feedback link given an allocated average feedback rate. This decomposed optimization problems are proved to yield a unique optimal policy.

Furthermore, given the sphere-cap-quantized-CSI model, the optimal feedback-control policy is shown to remain as the water-filling type. However, the optimal number of feedback bits is proportional not only with the logarithm of the corresponding interference-channel gain but also with the logarithm of the corresponding interferer’s distance.

The remainder of this paper is organized as follows. The system model is described in Section II. The problem formulation for the optimal feedback control is presented in Section III. The optimal feedback-control policies for low and high mobility are designed and analyzed in Section IV and V, respectively. In Section VI, the design of the feedback controller for asymmetric channel distributions is discussed. Simulation results are presented in Section VII, followed by concluding remarks in Section VIII.

II. SYSTEM MODEL

We consider the $K$-user MISO interference channels as illustrated in Fig. 1, where the $n$-th transmitter is denoted as $T^{[n]}$ and the $m$-th receiver as $R^{[m]}$. Provisioned with $L$ antennas, each transmitter sends
a single data stream to an intended receiver using beamforming. As illustrated in Fig. 2, time is slotted and each slot is divided into the *feedback phase* (feedback control and cooperative CSI feedback) and the *data phase* (data transmission). Each system parameter affected by feedback is represented by the same symbol without and with the accent “´”, corresponding to the beginnings of the feedback and data phases, respectively. Moreover, the subscript \( t \) denotes the slot index.

A. Zero-Forcing Transmit Beamforming

Each transmitter steers its beam to avoid interfering with \((K - 1)\) unintended receivers. Let \( \mathbf{h}_{t}[mn] \) denote the \( L \times 1 \) vector representing the channel from the \( n \)-th transmitter to the \( m \)-th receiver. For exposition, we decompose \( \mathbf{h}_{t}[mn] \) as \( \mathbf{h}_{t}[mn] = \sqrt{g_{t}[mn]} \mathbf{s}_{t}[mn] \) where \( g_{t}[mn] = \| \mathbf{h}_{t}[mn] \|^2 \) is the channel gain and \( \mathbf{s}_{t}[mn] = \mathbf{h}_{t}[mn] / \| \mathbf{h}_{t}[mn] \| \) specifies the channel direction. Node \( T^n \) applies zero-forcing beamforming, namely choosing its beamformer \( \mathbf{f}_{t}[mn] \) to be orthogonal to the interference-channel directions. As a result, \( K \) links are decoupled if all transmitters have perfect CSIT of the channels to their interfered receivers.

Consider the scenario where transit beamforming at a transmitter relies on finite-rate CSI feedback from interfered receivers. Let \( \mathbf{u}_{t}[mn] \) denote the CSIT at \( T^n \) updated by the feedback of \( \mathbf{s}_{t}[mn] \) from \( R^m \). Then the zero-forcing beamformer \( \mathbf{f}_{t}[n] \) at \( T^n \) satisfies the constraints: \((\mathbf{f}_{t}[n])^\dagger \mathbf{u}_{t}[mn] = 0 \) for all \( m \neq n \), which requires \( L \geq K \). Under the finite-rate feedback constraints, imperfect CSIT results in residual interference between links. The interference from \( T^n \) to \( R^m \) has the power

\[
I_{t}[mn] = g_{t}[mn] \left| (\mathbf{f}_{t}[n])^\dagger \mathbf{s}_{t}[mn] \right|^2, \quad m \neq n
\]  

where unit transmission power is used by all transmitters.
B. Variable-Rate Feedback Control

In the feedback phase of every slot, each receiver, say \( R[m] \), sends the quantized version \( \hat{s}_t^{[mn]} \) of \( s_t^{[mn]} \) to an interferer \( T[n] \) in a variable-length packet comprising \( B_t^{[mn]} \) bits. The variable-rate feedback is modeled as \( B_t^{[mn]} \in \mathbb{B} \), where \( \mathbb{B} \) is a set of nonnegative integers including 0 that corresponds to no feedback. As illustrated in Fig. 2, a feedback controller at each receiver controls the number of feedback bits sent to a particular interferer by observing the interference-channel gain and the CSIT error that is defined as follows. To this end, the dynamics of the CSIT \( u_t^{[mn]} \) at \( T[n] \) can be specified as

\[
 u_{t+1}^{[mn]} = \begin{cases} 
   s_t^{[mn]}, & B_t^{[mn]} > 0 \\
   u_t^{[mn]}, & B_t^{[mn]} = 0.
\end{cases}
\]  

The CSIT error is defined as \( \delta_t^{[mn]} = 1 - \left| (s_t^{[mn]})^t u_t^{[mn]} \right|^2 \) with \( \delta_t^{[mn]} = 0 \) for the case of perfect CSIT: \( u_t^{[mn]} = s_t^{[mn]} \) [2]. The feedback controller at \( R[m] \) observes the state \( \{ (g_t^{[mn]}, \delta_t^{[mn]} ) \mid n \neq m \} \) and generates the feedback decision \( \{ B_t^{[mn]} \mid n \neq m \} \).

Similarly, we define the quantization error \( \epsilon_t^{[mn]} = 1 - \left| (s_t^{[mn]})^t \hat{s}_t^{[mn]} \right|^2 \).

Assumption 1. The expected quantization error \( \mathbb{E} \left[ \epsilon_t^{[mn]} \mid B_t^{[mn]} \right] \) is a monotone decreasing and convex function of \( B_t^{[mn]} \).

Example 1 (Sphere-cap-quantized-CSI model). The quantization error \( \epsilon^{[mn]} \) is modeled in [9], [26] to be uniformly distributed on a sphere-cap in \( \mathbb{C}^L \) with the following distribution function

\[
 \Pr \left( \epsilon^{[mn]} \leq \tau \mid B^{[mn]} \right) = \begin{cases} 
  2B^{[mn]} \tau^{L-1}, & 0 \leq \tau \leq 2^{- \frac{n^{[mn]}}{B^{[mn]}}} \\
  1, & \text{otherwise}.
\end{cases}
\]  

Fig. 2. Variable-rate feedback control
Using this model, the expectation of $\epsilon^{[mn]}$ is obtained as
\[
E \left[ \epsilon^{[mn]} \mid B^{[mn]} \right] = \frac{L - 1}{L} 2^{-\frac{B^{[mn]}}{L-1}}
\]
which is a monotone decreasing and convex function of $B^{[mn]}$, consistent with Assumption 1.

**Example 2** (Random-vector quantization). As shown in [27], the use of a random beamformer codebook of i.i.d. and isotropic vectors results in the following distribution of $\epsilon^{[mn]}$
\[
Pr \left( \epsilon^{[mn]} \geq \tau \mid B^{[mn]} \right) = (1 - \tau) \left( \frac{L}{L - 1} \right)^{2B^{[mn]}}
\]
and the expectation
\[
E \left[ \epsilon^{[mn]} \mid B^{[mn]} \right] = 2^{B^{[mn]}} \text{beta} \left( \frac{2B^{[mn]}}{L}, \frac{L}{L - 1} \right)
\approx ae^{-\frac{1}{L-1}B^{[mn]}}, \quad B^{[mn]} \gg 1
\]
where $\text{beta}(\cdot, \cdot)$ denotes the beta function and $a$ is a constant. The last expression is a monotone decreasing and convex function of $B^{[mn]}$, justifying Assumption 1.

Finally, it follows from (2) that
\[
\delta^{[mn]}_t = \begin{cases} 
\epsilon^{[mn]}_t, & B^{[mn]}_t > 0 \\
\delta^{[mn]}_t, & B^{[mn]}_t = 0
\end{cases}
\]
given that channels remain constant within each slot as assumed in the sequel. Note that $\delta^{[mn]}_t = g^{[mn]}_t$ since it is unaffected by feedback.

**C. Channel Model**

For simplicity, all channel coefficients are assumed to be samples of i.i.d circularly-symmetric complex Gaussian processes with unit variance, which is denoted as $CN(0,1)$ (asymmetric channel distributions are considered in Section VI). Moreover, the channel random processes are assumed stationary and ergodic. Note that as a result of channel isotropicity, the two channel parameters $g^{[mn]}_t$ and $\delta^{[mn]}_t$ are independent conditioned on $B^{[mn]}_t$. We consider both low and high mobility, corresponding to temporally correlated and independent block-fading channels, respectively. For low mobility, the channel temporal correlation is modeled using the following two assumptions.

**Assumption 2.** For low mobility, each channel coefficient evolves as a Markov chain. Given $B^{[mn]}_{t-1} = 0$, the distributions of $\left( g^{[mn]}_t, \delta^{[mn]}_t \right)$ conditioned on $\left( g^{[mn]}_{t-1}, \delta^{[mn]}_{t-1} \right)$ satisfy
\[
\Pr \left( \delta^{[mn]}_t \geq \tau_1 \mid \delta^{[mn]}_{t-1} = a_1 \right) \geq \Pr \left( \delta^{[mn]}_t \geq \tau_1 \mid \delta^{[mn]}_{t-1} = b_1 \right)
\]
\[
\Pr \left( g^{[mn]}_t \geq \tau_2 \mid g^{[mn]}_{t-1} = a_2 \right) \geq \Pr \left( g^{[mn]}_t \geq \tau_2 \mid g^{[mn]}_{t-1} = b_2 \right)
\]
if \( a_1 \geq b_1 \) and \( a_2 \geq b_2 \), where \( 0 \leq \tau_1 \leq 1 \) and \( \tau_2 \geq 0 \).

The above assumption states that given no feedback, large CSIT error and channel power in the current time slot are likely to stay large in the next time slot thanks to channel temporal correlation.

**Assumption 3.** For low mobility and \( B_{t-1}^{[mn]} > 0 \), the conditional distribution \( \Pr \left( \delta_t^{[mn]} \geq \tau \mid B_{t-1}^{[mn]} \right) \) is a monotone decreasing and convex function of \( B_{t-1}^{[mn]} \).

Note that upon feedback, \( \delta_t^{[mn]} \) is independent of \( \delta_{t-1}^{[mn]} \) as a result of (7). The property in the above assumption depends on both channel dynamics and CSI quantization.

**Example 3.** Define the angles \( \theta_t^{[mn]} = \angle(s_t^{[mn]}, u_t^{[mn]}) \) and \( \theta_t^{[mn]} = \angle(s_t^{[mn]}, s_t^{[mn]}) \). Given \( B_t^{[mn]} > 0 \), \( \delta_t^{[mn]} = \epsilon_t^{[mn]} = \sin^2 \theta_t^{[mn]} \) and \( \delta_t^{[mn]} = \sin^2 \theta_t^{[mn]} \). At low mobility, the temporal angular variation can be modeled as \( \delta_t^{[mn]} = \theta_t^{[mn]} + \Delta \theta_t^{[mn]} \) where \( \Delta \theta_t^{[mn]} \) is a small angular increment. Given this model and \( \Delta \theta_t^{[mn]} \to 0 \)

\[
\delta_t^{[mn]} \approx \frac{1 - \cos \Delta \theta_t^{[mn]}}{2} + \epsilon_t^{[mn]} \cos \Delta \theta_t^{[mn]}.
\]

It follows that

\[
\Pr \left( \delta_t^{[mn]} \geq \tau \mid \delta_{t-1}^{[mn]}, B_{t-1}^{[mn]} \right) = \Pr \left( \delta_t^{[mn]} \geq \tau \mid B_{t-1}^{[mn]} \right), \quad B_{t-1}^{[mn]} > 0 \\
\approx \Pr \left( \epsilon_{t-1}^{[mn]} \geq \tau' \mid B_{t-1}^{[mn]} \right)
\]

where \( \tau' = \frac{2\tau - 1 + \cos \Delta \theta_{t-1}^{[mn]}}{2 \cos \Delta \theta_{t-1}^{[mn]}} \). From Assumption 1, the expression in (9) is a monotone decreasing and convex function of \( B_{t-1}^{[mn]} \), which justifies Assumption 3.

**D. Performance Metric**

The objective for designing the distributed feedback controller at each receiver is to minimize the average interference power. For receiver \( R^{[m]} \), this metric is given as

\[
\bar{I}^{[m]} = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{n=1}^{K} I_t^{[mn]}
\]

with \( I_t^{[mn]} \) given in (1). Minimizing \( \bar{I}^{[m]} \) suppresses the system performance degradation caused by quantized CSI feedback as illustrated by the following examples.

**Example 4 (Throughput loss).** Let \( S^{[m]} \) denote the received-signal power at receiver \( R^{[m]} \). Assuming
Gaussian signaling, the throughput loss of the $m$-th data link is given as [8]

$$ \Delta R = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \left[ \log_2 \left( 1 + \frac{S_t^{[m]}}{\sigma^2} \right) - \log_2 \left( 1 + \frac{S_t^{[m]} - I_t^{[m]}}{\sigma^2 + I_t^{[m]}} \right) \right] $$

$$ \leq \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \left[ \log_2 \left( 1 + \frac{I_t^{[m]}}{\sigma^2} \right) \right] $$

$$ \leq \log_2 \left( 1 + \frac{I_t^{[m]}}{\sigma^2} \right) $$

where $\sigma^2$ is the variance of a sample of the additive-white-Gaussian-noise process and (11) uses Jensen’s inequality. It can be observed from (11) that minimizing an upper bound on the throughput loss is equivalent to minimizing $\bar{I}^{[m]}$.

**Example 5** (Outage degradation). Consider the current system model with $K = L$ for which $S^{[m]}$ follows the exponential distribution [8]. Let $P_{\text{out}}$ and $P'_{\text{out}}$ denote the outage probabilities for the $m$-th link corresponding to quantized and perfect CSI feedback, respectively. Given a target signal-to-interference-plus-noise ratio (SINR) $\psi$, the outage-performance degradation is quantified by the ratio [28]

$$ \frac{P_{\text{out}}}{P'_{\text{out}}} = \frac{\Pr(S^{[m]} < (\sigma^2 + I^{[m]})\psi)}{\Pr(S^{[m]} < \sigma^2 \psi)} $$

$$ = \frac{\mathbb{E}[1 - e^{-(\sigma^2 + I^{[m]})\psi}]}{1 - e^{-\sigma^2 \psi}}. $$

For small outage probabilities, it follows from the last equation that

$$ \frac{P_{\text{out}}}{P'_{\text{out}}} \approx \frac{\sigma^2 + \bar{I}^{[m]}}{\sigma^2}. $$

Therefore, this ratio can be reduced by minimizing $\bar{I}^{[m]}$, justifying this performance metric.

### III. Problem Formulation

The design of the feedback controller is formulated as an optimization problem under an average sum-feedback constraint.

The cost function and state space for feedback control are defined as follows. To this end, the channel shape $s_t^{[mn]}$ is decomposed as

$$ s_t^{[mn]} = \sqrt{1 - \delta_t^{[mn]} q_t^{[mn]}} u_t^{[mn]} + \sqrt{\delta_t^{[mn]} q_t^{[mn]}} q_t^{[mn]} $$

$$ = \sqrt{1 - \delta_t^{[mn]} u_t^{[mn]}} u_t^{[mn]} + \sqrt{\delta_t^{[mn]} q_t^{[mn]}} q_t^{[mn]} $$

where $q_t^{[mn]}$ and $q_t^{[mn]}$ are unitary vectors orthogonal to $u_t^{[mn]}$ and $u_t^{[mn]}$, respectively. Based on the above decomposition, we can define the channel parameters $\beta_t^{[mn]} = |(f_t^{[mn]})^\dagger q_t^{[mn]}|^2$ and $\beta_t^{[mn]} =$
These parameters allow $\bar{I}[m]$ in (10) to be written as

$$\bar{I}[m] = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{n \neq m} \mathbb{E} \left[ g_t[m] \beta_t[m] \delta_t[m] \mid x_t[m], B_t[m] \right],$$

where $x_t[m]$ and $B_t[m] = \{B_t[n] \mid n \neq m\}$ denote the state and decision of the feedback controller at receiver $R[m]$, respectively. From (14), the controller’s state should be intuitively chosen to comprise all channel parameters $\{\beta_t[m], g_t[m], \delta_t[m] \mid n \neq m\}$. However, this results in the coupling of feedback control at different receivers. Specifically, by definition, the state parameter $\beta_t[m]$ depends on the beamformer $f_t[m]$ that in turn is computed based the feedback CSI from the receivers $\{R[n] \mid m \neq n\}$, and each of these receivers also controls other beamformers. Therefore, to enable distributive feedback control, $\beta_t[m]$ is excluded from the controller’s state and hence $x_t[m] = \{g_t[m], \delta_t[m] \mid n \neq m\}$ where each parameter pair $(g_t[m], \delta_t[m])$ depends only on the single channel $h_t[m]$. Since all channel vectors are isotropy and that feedback control is independent of $\{g_t[m], \delta_t[m]\}$ and $\beta_t[m]$ can be shown to be beta(1, $L-2$) random variables and independent with $(g_t[m], \delta_t[m])$ [9]. This simplifies (14) as

$$\bar{I}[m] = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{n \neq m} \mathbb{E} \left[ g_t[m] \delta_t[m] \mid x_t[m], B_t[m] \right].$$

Consider a stationary feedback-control policy and an average sum-feedback constraint where the total average feedback rate for each receiver is no more than $\bar{b} > 0$. The optimal policy $P^*_m : x_t[m] \to B_t[m]$ at receiver $R[m]$ solves the following optimization problem:

$$\begin{align*}
\text{minimize:} & \quad \bar{I}[m](P_m) \\
\text{subject to:} & \quad \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \sum_{n \neq m} B_t[m] \right] \leq \bar{b}.
\end{align*}$$

Due to symmetric channel distributions, it is optimal for receiver $R[m]$ to equally split $\bar{b}$ for $(K-1)$ feedback links. Consequently, the optimization of $P_m$ reduces to that of the policy $P$ for controlling an arbitrary single feedback link. To simplify notation, define the random process $(g_t, \delta_t, \hat{\delta}_t, B_t) \sim (g_t[m], \delta_t[m], \hat{\delta}_t[m], B_t[m])$ where “~” represents equality in distribution, and the metric

$$J = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ g_t \hat{\delta}_t \mid x_t, B_t \right]$$

where $x_t = (g_t, \delta_t)$ is the state of a single-feedback-link controller. Then $P : x_t \to B_t$ can be designed by solving the following optimization problem:

$$\begin{align*}
\text{minimize:} & \quad J(P) \\
\text{subject to:} & \quad \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} [B_t] \leq \frac{\bar{b}}{K-1}.
\end{align*}$$
IV. EVENT-DRIVEN FEEDBACK CONTROL: LOW MOBILITY

Given channel Markovity, the optimization problem in (18) can be transformed into a stochastic optimization problem as follows. By applying Lagrangian-multiplier theory, there exists a Lagrangian multiplier $\lambda > 0$ such that the optimal policy $P^*$ that solves (18) also minimizes the following Lagrangian function:

$$
L(P) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \left\{ E \left[ g_t \delta_t \mid x_t, B_t \right] + \lambda B_t \right\}.
$$

Minizing $L(P)$ is an average-cost stochastic optimization problem with a continuous state space. Though there exists no systematic method for solving this problem, it can be approximated by a discrete-space counterpart whose solution can be computed efficiently using dynamic programming [29]. The required state-space discretization is discussed and the resultant optimal feedback-control policy analyzed in the following subsections.

A. State-Space Discretization

The spaces of the feedback-controller’s state parameters $g_t$ and $\delta_t$ are discretized separately. The set $\mathcal{G} = \{g_t \geq 0\}$ is partitioned into $M$ line segments $[\bar{g}_1, \bar{g}_2], [\bar{g}_2, \bar{g}_3], \ldots, [\bar{g}_M, \infty)$ with $\bar{g}_1 = 0$ and $0 < \bar{g}_1 < \bar{g}_2 < \cdots < \bar{g}_M$. These line segments are represented by a set of $M$ grid points $\hat{\mathcal{G}} = \{\bar{g}_m\}$. Specifically, $g_t \in \mathcal{G}$ is mapped to $\bar{g}_m$ if $g_t$ lies in the $m$-th line segment. Similarly, we divide the set $\mathcal{D} = \{0 \leq \delta_t \leq 1\}$ into $N$ line segments $[\bar{\delta}_1, \bar{\delta}_2], [\bar{\delta}_2, \bar{\delta}_3], \ldots, [\bar{\delta}_N, 1]$ with $\bar{\delta}_1 = 0$ and $0 < \bar{\delta}_1 < \bar{\delta}_2 < \cdots < \bar{\delta}_N < 1$ and represent these segments using a set of $N$ grid points $\hat{\mathcal{D}} = \{\bar{\delta}_n\}$. The optimization of the grid points $\hat{\mathcal{G}}$ and $\hat{\mathcal{D}}$ is out the scope of this paper. Last, the discrete state space is represented by $\hat{X} = \hat{\mathcal{G}} \times \hat{\mathcal{D}}$.

The discretized version of the controller state $x_t$ is denoted as $\hat{x}_t = \{\hat{g}_t, \hat{\delta}_t\}$. Given Assumption 2, $\{\hat{g}_t\}$ and $\{\hat{\delta}_t\}$ are two Markov chains whose transition probabilities are obtained as follows. Let $P_{m,n}(B)$ denote the probability for the transition of $\hat{\delta}$ from the state $m$ to $n$ given the feedback decision $B$. Then $P_{m,n}$ can be written as

$$
P_{m,n}(B) = \text{Pr}(\delta_{t+1} \in [\bar{\delta}_n, \bar{\delta}_{n+1}) \mid \delta_t \in [\bar{\delta}_m, \bar{\delta}_{m+1}), B_t = B)
$$

where $1 \leq m, n \leq N$. Similarly, let $\hat{P}_{m,n}$ denote the transition probability for $\{\hat{g}_t\}$, which is given as

$$
\hat{P}_{m,n} = \text{Pr}(g_{t+1} \in [\bar{g}_n, \bar{g}_{n+1}) \mid g_t \in [\bar{g}_m, \bar{g}_{m+1}])
$$

where $1 \leq m, n \leq M$. Note that given $B$, $P_{m,n}(B)$ and $\hat{P}_{m,n}$ are independent as a result of channel isotropicity. Last, the transition kernel for the controller-state Markov chain $\{x_t\}$ can be readily defined as $P(B) = \{P_{m,n}(B)\} \times \{\hat{P}_{m,n}\}$. 
B. Optimal Feedback-Control Policy with a Discrete State Space

The stochastic optimization problems for feedback control with the discrete state space \( \hat{X} \) are formulated as follows. Define the corresponding feedback-control policy as \( \hat{P} : \hat{X} \to \mathbb{B} \). The matching average cost function \( \hat{L} \) is modified from (19) as

\[
\hat{L}(\hat{P}) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} G(\hat{x}_t, B_t)
\]

(20)

where \( G(\hat{x}_t, B_t) \) is the cost-per-stage and obtained using (7) as

\[
G(\hat{x}_t, B_t) = \begin{cases} 
\hat{g}_t E[\epsilon_t | B_t] + \lambda B_t, & B_t \geq 0 \\
\hat{g}_t \delta_t, & B_t = 0.
\end{cases}
\]

(21)

The problem of optimizing \( \hat{P} \) can be readily written as:

\[
\begin{align*}
\text{minimize:} & \quad \hat{L}(\hat{P}) \\
\text{subject to:} & \quad \hat{P} : \hat{X} \to \mathbb{B}.
\end{align*}
\]

(22)

Note that the minimum cost \( \hat{L}^* \) converges to \( \mathcal{L}^* = \min_P \mathcal{L}(P) \) as \( N, M \to \infty \) provided that the grid points are suitably chosen \([25],[30]\). The optimal policy \( \hat{P}^* \) depends on the state-transition kernel \( P \).

Given \( P \), \( \hat{P}^* \) can be computed efficiently using policy iteration \([29]\). The analysis of \( \hat{P}^* \) is simplified by considering the discounted-cost problem. Specifically, given a discount factor \( \rho \in (0,1) \) and the discounted cost function

\[
\hat{L}_\rho(\hat{P}_\rho, \hat{x}_0) = \sum_{t=0}^{\infty} \rho^t G(\hat{x}_t, B_t)
\]

(23)

with the initial state \( \hat{x}_0 \), a stationary feedback-control policy \( \hat{P}_\rho : \hat{X} \to \mathbb{B} \) is designed by solving the following stochastic optimization problem

\[
\begin{align*}
\text{minimize:} & \quad \hat{L}_\rho(\hat{P}_\rho, \hat{x}_0), \quad \forall \hat{x}_0 \in \hat{X} \\
\text{subject to:} & \quad \hat{P}_\rho : \hat{X} \to \mathbb{B}.
\end{align*}
\]

(24)

The optimal policy \( \hat{P}_\rho^* \) and minimum cost \( \hat{L}_\rho^* \) converge to their average-cost counterparts as: \( \hat{P}^* = \lim_{\rho \to 1} \hat{P}_\rho^* \) and \( \hat{L} = \lim_{\rho \to 1} (1 - \rho) \hat{L}_\rho^*(\hat{x}_0) \) for arbitrary \( \hat{x}_0 \) \([29]\).

The discounted-cost problem allows simpler analysis as \( \hat{L}_\rho^* \) satisfies the following Bellman’s equation:

\[
\hat{L}_\rho^*(\hat{x}_t) = F\hat{L}_\rho^*(\hat{x}_t)
\]

(25)

where \( F \) is the dynamic-programming operator and defined for a given function \( q : \hat{X} \to \mathbb{R} \) as

\[
Fq(\hat{x}_t) = \min_{B \in \mathbb{B}} \{ G(\hat{x}_t, B) + \rho E[q(\hat{x}_{t+1}) | \hat{x}_t, B] \}.
\]

(26)
Though solving Bellman’s equation analytically is infeasible, we can derive from this equation some properties of the optimal policy as follows. Some auxiliary results are obtained as shown in the following two lemmas. First, the monotonicity of $\hat{L}_p^*(\hat{x})$ depends on if the following function is negative or nonnegative

$$f(\hat{g}_k, \hat{\delta}_\ell) = \hat{L}_p^*(\hat{g}_k, \hat{\delta}_\ell) - \hat{L}_p^*(\hat{g}_k, \hat{\delta}_{\ell - 1}) - \hat{L}_p^*(\hat{g}_{k - 1}, \hat{\delta}_\ell) + \hat{L}_p^*(\hat{g}_{k - 1}, \hat{\delta}_{\ell - 1}).$$

(27)

**Lemma 1.** The function $f(\hat{x})$ is nonnegative for all $\hat{x} \in \hat{X}$.

*Proof:* The proof uses the value iteration, namely that for an arbitrary function $q : \hat{X} \rightarrow \mathbb{R}$, the minimum discounted cost is [29]

$$\hat{L}_p^*(\hat{x}_t) = \lim_{n \rightarrow \infty} F^n q(\hat{x}_t).$$

(28)

We show that if $q$ is chosen to have the property in the lemma statement, this property also holds for $Fq$ or in other words, remains unchanged by the dynamic-programming operation $F$. Combining this fact and the value iteration in (28) proves the lemma. The details are provided in Appendix A. ■

The above lemma shows that $f(\hat{g}, \hat{\delta})$ is a monotone increasing function of $(\hat{g}, \hat{\delta}) \in \hat{X}$. Next, define the function

$$Z(\hat{x}_t, B) = G(\hat{x}_t, B) + \rho \mathbb{E} \left[ \hat{L}_p^*(\hat{x}_{t+1}) \mid \hat{x}_t, B \right].$$

(29)

Given the relation $\hat{P}_p^*(\hat{x}_t) = \arg \min_B Z(\hat{x}_t, B)$, the structure of $\hat{P}_p^*$ depends directly on the characteristics of $Z$, which are specified in the following lemma.

**Lemma 2.** The function $Z(\hat{x}_t, B)$ has the following properties:

1) With $\hat{x}_t$ fixed and for $B > 0$, $Z(\hat{x}_t, B)$ is a monotone decreasing and convex function of $B$;

2) With $B$ fixed, $Z(\hat{g}_t, \hat{\delta}_t, B)$ is a monotone increasing function of $\hat{g}_t$ and $\hat{\delta}_t$ if $B > 0$, and of $\hat{g}_t$ if $B = 0$.

*Proof:* See Appendix B. ■

Note that conditioned on $B > 0$, $Z(\hat{g}_t, \hat{\delta}_t, B)$ is independent with $\hat{\delta}_t$. Using Lemma 1 and 2, we obtain the properties of $\hat{P}_p^*$ as stated in the following theorem.

**Theorem 6.** The optimal feedback-control policy $\hat{P}_p^*$ has the following properties:

1) If there exists $(a, b) \in \hat{X}$ such that $\hat{P}(a, b) = 0$, $\hat{P}(a, \hat{\delta}) = 0$ for all $\hat{\delta} \in \hat{D}$ and $\hat{\delta} \leq b$.

2) If there exists $(a, b) \in \hat{X}$ such that $\hat{P}(a, b) > 0$, $\hat{P}(a, \hat{\delta}) = \hat{P}(a, b)$ for all $\hat{\delta} \in \hat{D}$ and $\hat{\delta} \geq b$.

3) If there exist $(a, b), (c, b) \in \hat{X}$ such that $\hat{P}(a, b) > 0$ and $\hat{P}(c, b) > 0$, $\hat{P}(c, b) \geq \hat{P}(a, b)$ if $c \geq a$ and vice versa.
Fig. 3. The structure of the optimal feedback-control policy $\hat{P}^*$ for high mobility where $0 \leq \bar{B}_2 \leq \cdots \leq \bar{B}_{A-1} \leq \bar{B}_A$ and \{\bar{B}_k\} \in \mathbb{B}$.

Proof: See Appendix C.

The structure of $\hat{P}^*$ as specified in Theorem 6 is illustrated in Fig. 3, from which $\hat{P}^*$ is observed to be opportunistic in nature. To be specific, CSI feedback over a particular feedback link is performed only when the corresponding CSIT error and/or interference channel gain are large. As a result, the optimal policy partitions the state space into the feedback and no-feedback regions similar to the on/off-feedback policy in [25]. The current policy that supports variable-rate feedback further partitions the feedback region into smaller regions and assigns them different numbers of feedback bits. Upon feedback, the number of feedback bits increases with the interference-channel gain. The CSIT error observed prior to feedback affects the decision on if feedback should be performed but has no influence on the number of feedback bits upon feedback. The reason is that the CSIT error after feedback is equal to the quantization error that is independent of CSIT error prior to feedback. Last, note that $\hat{P}^*$ converges to $P^*$ with a continuous state space as the number of grid points $M \times N \to \infty$.

Intuitively, the feedback-link should be turned off less frequently when the interference-channel gain is large. In other words, the feedback-threshold function separating the feedback and no-feedback regions should map larger values of $\hat{g}$ to smaller ones of $\hat{\delta}$. However, proving this property requires more restrictive assumptions on the channel temporal correlation than the current ones.

The feedback control can be treated as the dual of bit loading (or adaptive modulation) over forward data links [31], [32]. Both feedback control and bit loading opportunistically allocate (CSI or data) bits over (feedback or forward) channels based on instantaneous (interference or data) CSI. Furthermore, both
functions share the same objective of enhancing the system throughput.

V. EVENT-DRIVEN FEEDBACK CONTROL: HIGH MOBILITY

The feedback-control policy is designed by solving the optimization problem in (18) for high mobility. To simplify the solution, we consider the sphere-cap-quantized-CSI model in Example 1, resulting in the optimal feedback-control policy of the water-filling type. This property is expected to also hold for the random-vector quantization in Example 2 since the quantization-error expectations for both models have the similar exponential forms (compare (4) and (6)).

Given independent block fading and a stationary feedback-control policy, the optimal feedback decisions in different slots are made independently. Consequently, \((g_t, \delta_t, B_t)\) have stationary distributions and are i.i.d. in different lots. To simplify notation, let \((g, \delta, B)\) represent a sample of \(\{g_t, \delta_t, B_t\}\) in an arbitrary slot. Using this notation and (4), (18) can be rewritten as follows:

\[
\begin{align*}
\text{minimize: } & \quad \mathbb{E}\left[ g_{\text{min}} \left( \frac{L - 1}{L} 2^{-\frac{\alpha}{L-1}}, \delta \right) \right] \\
\text{subject to: } & \quad \mathbb{E}[B] \leq \frac{\bar{b}}{K - 1} \\
& \quad B \in \mathcal{B}
\end{align*}
\]

where the \(\text{min}\) operator in the objective function accounts for the fact that feedback from a receiver to a particular interferer should be performed only if it reduces the expected CSIT error. Solving the above optimization problem analytically is difficult due to the constraint \(B \in \mathcal{B}\). This constraint is replaced with \(B \geq 0\) which approximates the case where many quantization resolutions are supportable. The above optimization problem is modified accordingly as:

\[
\begin{align*}
\text{minimize: } & \quad \mathbb{E}\left[ g_{\text{min}} \left( \frac{L - 1}{L} 2^{-\frac{\alpha}{L-1}}, \delta \right) \right] \\
\text{subject to: } & \quad \mathbb{E}[B] \leq \frac{\bar{b}}{K - 1} \\
& \quad B \geq 0
\end{align*}
\]

Solving the above problem yields insights into the structure of the optimal feedback-control policy as described in the following proposition.

**Proposition 1.** For high mobility, the optimal feedback-control policy \(\mathcal{P}^* : \mathcal{X} \rightarrow \mathbb{R}^+\) resulting from solving (31) is of the water-filling type and obtained as

\[
\mathcal{P}^*(g, \delta) = \begin{cases} 
\Upsilon - (L - 1) \log_2 \frac{1}{g}, & \delta \geq \Psi(g) \\
0, & \text{otherwise}
\end{cases}
\]
where \( \Upsilon \) is the water level given as
\[
\Upsilon = \frac{\bar{b}}{(K-1) \Pr(\delta \geq \Psi(g))} + (L-1) \mathbb{E} \left[ \log_2 \frac{1}{g} \mid \delta \geq \Psi(g) \right].
\] (33)

The feedback-threshold function \( \Psi : G \to D \) solves the following optimization problem: 
\[
\begin{align*}
\text{minimize:} & & \bar{I}^*(\Psi) \\
\text{subject to:} & & \Upsilon(\Psi) - (L-1) \log_2 \frac{1}{\Psi^{-1}(1)} \geq \left( (L-1) \log_2 \frac{L-1}{L \Psi^{-1}(1)} \right)^+.
\end{align*}
\] (34)

where \( \bar{I}^*(\Psi) \) given below is the sum-interference power at any receiver achieved by \( P^* \) given \( \Psi \)
\[
\bar{I}^* = \frac{1}{L-1} 2^{-\frac{\Psi^{-1}(1)}{L-1}} \Pr(\delta \geq \Psi(g)) + \frac{1}{L-1} \mathbb{E}[g \delta \mid \delta < \Psi(g)].
\] (35)

In addition, \( \Psi(g) \) is a monotone decreasing function of \( g \).

**Proof:** See Appendix D.

The optimal feedback-control policy for high mobility has the same opportunistic properties as the low-mobility counterpart, which are described after Theorem 6. In addition, for high mobility, the optimal feedback-control is similar to the classic adaptive modulation algorithm that allocates data bits in time also based on water-filling [31].

For a large average feedback rate \( \bar{b} \gg 1 \), \( \Pr(\delta \geq \Psi(g)) \approx 1 \) and thus the minimum average sum-interference power at an arbitrary receiver follows from (35) as

\[
\bar{I}^* \approx \frac{1}{L-1} 2^{-\frac{\Psi^{-1}(1)}{L-1}} = c 2^{-\frac{b}{(K-1)(L-1)}}
\] (36)

where \( c \) is a constant. It can be observed from (36) that \( \bar{I}^* \) decreases exponentially with increasing \( \bar{b} \), where the slope is smaller for a larger number of links or transmit antennas per transmitter. In addition, the optimal number of feedback bits given in (32) need be rounded to the nearest and smaller integer for implementation and this operation increases \( \bar{I}^* \) by a multiplicative factor no larger than \( 2^{-\frac{1}{(K-1)(L-1)}} \).

It is infeasible to obtain the feedback-threshold function \( \Psi \) analytically by solving the optimization problem in Proposition 1. Thus computing \( \Psi \) has to rely on a numerical search, which is used to obtain the relevant simulation results in Section VII.

\(^2\)The operation \((a)^+\) for \( a \in \mathbb{R} \) gives \( a \) if \( a \geq 0 \) or otherwise 0.
VI. EVENT-DRIVEN FEEDBACK CONTROL: ASYMMETRIC CHANNEL DISTRIBUTIONS

In the preceding sections, all interference channels are assumed to following identical distributions. In this section, we discuss feedback control for asymmetric interference channel distributions in terms of heterogeneous path losses and assuming high mobility for mathematical tractability. Let \( d_{mn} \) denote the distance between receiver \( R^m \) and transmitter \( T^n \). The average interference power at receiver \( R^m \) can be written as

\[
\bar{I}^m = \sum_{n \neq m} (d_{mn})^{-\alpha} \bar{I}_{mn}^{[\nu]}
\]  

(37)

where \( \alpha \) is the path-loss exponent and

\[
\bar{I}_{mn}^{[\nu]} = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} I_{t}^{[\nu]}
\]  

\[
= \frac{1}{L-1} \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} g_{t}^{[\nu]} \min \left( \mathbb{E} \left[ \epsilon_{t}^{[\nu]} \mid B_{t}^{[\nu]} \right], g_{t}^{[\nu]} \right).
\]

(38)

Given heterogeneous path losses, the uniform allocation of average feedback rates by each receiver to different feedback channels is no longer optimal. Consequently, we should optimize the average feedback-rate allocation besides feedback-control over time. Specifically, the feedback-control optimization problem can be decomposed as:

– Master problem (average feedback-rate allocation)

\[
\text{minimize: } \sum_{n=1}^{K} \left( \frac{1}{L} \sum_{t=1}^{T} I_{t}^{[\nu]} \right) \left( \bar{I}_{mn}^{[\nu]} \right)
\]

subject to:

\[
\sum_{n=1}^{K} \bar{b}_{m,n} \leq \bar{b}
\]

\[
\bar{b}_{m,n} \geq 0 \ \forall \ m \neq n.
\]

(39)

where \( \bar{I}_{mn}^{[\nu]} \) solves the following sub-problem.

– Sub-problem (stochastic feedback control)

\[
\text{minimize: } \bar{I}_{mn}^{[\nu]} \left( \mathcal{P}_{mn} \right)
\]

subject to:

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \left[ B_{t}^{[\nu]} \right] \leq \bar{b}_{m,n}
\]

(40)

where \( \bar{I}_{mn}^{[\nu]} \) is given in (38) and \( \mathcal{P}_{mn} \) denotes the policy for controlling the feedback link from receiver \( R^m \) to transmitter \( T^n \).

Note that the sub-problem is identical to (18) except for the difference in the maximum average feedback rates. The above decomposed optimization problems have a unique solution as shown below.
Lemma 3. $\overline{I}_{mn}(\overline{b}_{m,n})$ is a convex and monotone decreasing function over $\overline{b}_{m,n} \geq 0$.

Proof: See Appendix E.

The following result holds given the convexity of the master problem as a result of Lemma 3 and that of the sub-problem follows from the discussion in Section V.

Proposition 2. Solving the master problem and sub-problem gives a unique optimal stationary feedback-control policy.

Next, we characterize the optimal feedback-control policy based on the quantizer model in Example 1 and for a large average sum-feedback rate per user. For this case, using (36) and given $\overline{b}_{m,n}$, the average interference power from node $T_n$ to $R_m$ can be approximated as

$$\overline{I}_{mn} \approx \frac{1}{L-1}2^{-E\left[\log_2 \frac{1}{g}ight]}2^{-\frac{b_{m,n}}{r}}$$

where $g$ follows the chi-square distribution. This approximation reduces the master problem as:

$$\text{minimize: } \sum_{n=1, n \neq m}^{K} (d_{mn})^{-\alpha/2} 2^{-\frac{b_{m,n}}{r}}$$

subject to:

$$\sum_{n=1, n \neq m}^{K} \overline{b}_{m,n} \leq \overline{b}$$

$$\overline{b}_{m,n} \geq 0 \forall m \neq n.$$

Solving the above constrained optimization problem using Lagrangian method yields that the optimal allocation of average feedback rates is of the water-filling type:

$$\overline{b}_{m,n}^{\star} = \eta - \alpha(L-1) \log_2 d_{mn}, \quad n \neq m$$

where $\eta$ is the water-level given as

$$\eta = \frac{\overline{b}}{K-1} + \frac{\alpha(L-1)}{K-1} \sum_{n \neq m} \log_2 d_{mn}.$$  \hspace{1cm} (43)

For a sanity check, the substitution of equal distances $d_{m1} = d_{m2} = \cdots = d_{mK}$ into (42) gives equal-rate splitting: $\overline{b}_{m,n}^{\star} = \frac{\overline{b}}{K-1}$ for all $n \neq m$. It can be observed from (42) that the optimal average feedback rate allocated by a receiver for suppressing the interference-power of a particular interferer decreases logarithmically with the increasing distance between the interferer and the receiver. Relaxing the integer constraint on the numbers of feedback bits and combining (32) and (42), we can approximate the optimal number of feedback bits $B_{m,n}^{\star}$ sent from node $R_m$ to $T_n$ with $n \neq m$ as

$$B_{m,n}^{\star} = \eta' - \alpha(L-1) \log_2 d_{mn} - (L-1) \log_2 \frac{1}{g_{mn}}.$$  \hspace{1cm} (44)
The above expression shows two-tier water-filling for allocating average feedback rates over multiple feedback links and for each link distributing feedback bits over different slots.

VII. SIMULATION RESULTS

The simulation has the following settings unless specified otherwise. The number of antennas $L = 4$, the number of users $K = 3$, and the set of available numbers of feedback bits is $B = \{2^n \mid 0 \leq n \leq 15\}$. All channel fading coefficients are modeled as i.i.d. $CN(0, 1)$ Gaussian processes. For low mobility, the temporal correlation of each process is specified by Clark’s function [33]. The values of Doppler frequency are normalized by the symbol rate. The state space for feedback control at low mobility is discretized to have $M = 16$ grid points for the interference-channel gain and $N = 16$ points for the CSIT error. The set $\hat{G}$ is chosen based on the equal-probability criterion such that $\Pr(\tilde{g}_k \leq g^{[mn]} < \tilde{g}_{k+1}) = \frac{1}{M}$ for $1 \leq k \leq M$, $\tilde{g}_1 = 0$ and $\tilde{g}_{M+1} = \infty$. The CSI quantization error is generated based on the sphere-cap-quantized-CSI model in Example 1. Correspondingly, the grid points for the CSIT error are chosen to be the expected quantization errors for different numbers of feedback bits in $B$, namely $\hat{D} = \left\{ \frac{L-1}{L} 2^{-\frac{n}{L-1}} \mid B \in \mathbb{B} \right\}$.

Fig. 4 to 6 concern stochastic feedback control for low mobility. Fig. 4 shows the optimal feedback-control policies computed using policy iteration for different combinations of (normalized) Doppler frequency $f_d$ and average sum-feedback rates $\bar{b}$. Both Fig. 4(a) and 4(b) are consistent with Theorem 6. Specifically, it can be observed from the figures that given the optimal policy, the state space is partitioned into the feedback and no feedback regions. Moreover, in the feedback region, $B^*$ is independent of the
Fig. 5. Throughput-per-user versus transmit SNR for low mobility and CSI feedback under an average sum-feedback constraint of 12 bit/slot.

CSIT error $\hat{\delta}$; given $\hat{\delta}$, $B^*$ is a monotone non-decreasing function of $\hat{g}$. Comparing Fig. 4(a) and 4(b), increasing Doppler frequency and the average sum-feedback rate enlarge the feedback region as well as the numbers of feedback bits in the feedback region.

Fig. 5 shows the throughput-per-user versus transmit SNR for optimally controlled feedback given $\bar{b} = 12$ bit/slot. For comparison, the curve for the conventional case of fixed feedback is also plotted, for which the number of CSI bits sent to each feedback link is fixed at $\bar{b}/(K - 1)$. The throughput-per-user for fixed feedback is observed to saturate as the transmit SNR increases and residual interference becomes dominant over noise. The use of feedback control alleviates this performance degradation and doubles the throughput-per-user for high SNRs. Moreover, the throughput-per-user given feedback control increases rapidly as the Doppler frequency decreases, corresponding to growing redundancy in CSI. In particular, reducing $f_d$ from $6 \times 10^{-3}$ to $2 \times 10^{-3}$ increases the throughput-per-user by more than 1 bit/s/Hz.

Fig. 6 shows the throughput-per-user versus average sum-feedback rate $\bar{b}$ for both controlled and fixed feedback. It can be observed that as $\bar{b}$ increases, the throughput-per-user for optimally controlled feedback converges to the upper bound corresponding to perfect CSIT much more rapidly than that for fixed feedback. Consequently, the throughput gains of feedback control with respect to fixed feedback are significant for small sum-feedback rates and small Doppler frequencies. For example, feedback control
Fig. 6. Throughput-per-user versus average sum-feedback rate for the optimal feedback control with low mobility and the transmit SNR equal to 13 dB.

increases the throughput-per-user of fixed feedback by more than 4 times given $\bar{b} = 2$ bit/slot and $f_d = 2 \times 10^{-3}$. In addition, note that the humps on the curves for feedback control are due to the discretization of the state space.

Finally, we consider the optimal feedback control for high mobility. Fig. 7 displays the curves of throughput-per-user versus average sum-feedback rate $\bar{b}$ for controlled feedback as well as fixed feedback. These results are based on $B \in \mathbb{N}^+$, aligned with the analysis in Section V. It is observed that the throughput gain of the optimal feedback control with respect to fixed feedback is marginal given symmetric channel distributions and high mobility, namely no redundancy in CSI. However, this gain is significant in the presence of asymmetric channel distributions modeled by heterogeneous path losses. For this case, substantial throughput gains rise from unequal distribution of average feedback rates over different feedback links.

VIII. CONCLUSION

This work has proposed the new approach of distributive and stochastic control of event-driven CSI feedback in multi-antenna interference networks. For low-mobility and symmetric channel distributions, the optimal feedback-control policy for each feedback link has been proved to be opportunistic. Specifi-
Huang et al.: Stochastic Control of Event-Driven Feedback in Multi-Antenna Interference Channels

Fig. 7. Throughput-per-user versus average sum-feedback rate for the optimal feedback control with high mobility. All interference channels have unit propagation distances for the case of symmetric channel distributions. For the asymmetric case, the \((K - 1) = 2\) interferers for each receiver are located at distances of 1 and 3 units away. The path-loss exponent is \(\alpha = 3\) and transmit SNR 13 dB.

cally, feedback is performed only if the corresponding interference-channel gain is large or the CSI at the transmitter is significantly outdated; the number of feedback bits increases with the interference-channel gain. For high-mobility and symmetric channel distributions, by considering a specific CSI quantization model, the optimal feedback policy has been shown to be of the water-filling type that also has the above opportunistic properties. For high-mobility and heterogeneous path-losses for the interference channels, the optimization of the feedback controller has been decomposed into a master problem and a sub-problem. We have proved the existence of a unique solution for the decomposed optimization problems.

To the best of our knowledge, this is the first work on applying stochastic-optimization theory to design feedback controllers in multi-antenna interference networks. This work opens several issues for future investigation. First, in the case of bursty traffic, the queues and feedback-links can be jointly controlled to achieve the optimal tradeoff between transmission delay and feedback overhead. Second, the event-driven feedback targets shared feedback channels where feedback collisions are inevitable. Collisions and the resultant feedback delay are omitted in the current work but important issues to consider in designing practical feedback controllers and protocols. Last, it is challenging to generalize the current feedback-controller designs to more complex settings such as MIMO channels and spatial multiplexing.
A. Proof for Lemma 1

Define \( \Phi(\bar{g}_k \mid \bar{\delta}_b, B^*) = \mathbb{E}_{\hat{\delta}_{t+1}} \left[ \hat{L}^*_\rho(\hat{g}_{t+1} = \bar{g}_k, \hat{\delta}_{t+1}) \mid \hat{\delta}_t = \bar{\delta}_b, B^* \right] \). We can write that

\[
\Phi(\bar{g}_k \mid \bar{\delta}_b, B^*) = \sum_{n=1}^N \hat{L}^*_\rho(\bar{g}_k, \bar{\delta}_n) P_{b,n}(B^*)
\]

\[
= \sum_{n=1}^N \left[ \hat{L}^*_\rho(\bar{g}_k, \bar{\delta}_n) - \hat{L}^*_\rho(\bar{g}_k, \bar{\delta}_{n-1}) \right] \sum_{m=1}^N P_{b,m}(B^*).
\]

Similarly, we can obtain that

\[
\mathbb{E}\left[ \hat{L}^*_\rho(\hat{g}_{t+1}, \hat{\delta}_{t+1}) \mid \hat{g}_t = \bar{g}_a, \hat{\delta}_t = \bar{\delta}_b, B^* \right] = \mathbb{E}\left[ \Phi(\hat{g}_{t+1} \mid \bar{\delta}_b, B^*) \mid \hat{g}_t = \bar{g}_a \right]
\]

\[
= \sum_{k=1}^M \left[ \Phi(\bar{g}_k \mid \bar{\delta}_b, B^*) - \Phi(\bar{g}_{k-1} \mid \bar{\delta}_b, B^*) \right] \sum_{\ell=k}^M \tilde{P}_{a,\ell}
\]

\[
= \sum_{n=1}^N \sum_{k=1}^M f(\bar{\delta}_n, \bar{g}_k) \sum_{m=1}^N P_{b,m}(B^*) \sum_{\ell=k}^M \tilde{P}_{a,\ell}.
\]

For \( B^* > 0 \),

\[
\hat{L}^*_\rho(\bar{g}_a, \bar{\delta}_b) = \bar{g}_a \mathbb{E}[\varepsilon \mid B^*] + B^* + \mathbb{E}[\hat{L}^*_\rho(\hat{g}_{t+1}, \hat{\delta}_{t+1}) \mid \hat{g}_t = \bar{g}_a, B^*].
\]

If \( f(\bar{g}, \bar{\delta}) > 0 \) for all \((\bar{g}, \bar{\delta}) \in \hat{X}\) and \( B^* > 0 \), it follows from (46) and (47) that \( \mathbb{F}^n f(\bar{g}, \bar{\delta}) = 0 \) for all \( n > 1 \). For \( B^* = 0 \),

\[
\hat{L}^*_\rho(\bar{g}_a, \bar{\delta}_b) = \bar{g}_a \bar{\delta}_b + \mathbb{E}[\hat{L}^*_\rho(\hat{g}_{t+1}, \hat{\delta}_{t+1}) \mid \hat{g}_t = \bar{g}_a, \hat{\delta}_t = \bar{\delta}_b]
\]

\[
= \bar{g}_a \bar{\delta}_b + \sum_{n=1}^N \sum_{k=1}^M f(\bar{\delta}_n, \bar{g}_k) \sum_{m=1}^N P_{b,m}(0) \sum_{\ell=k}^M \tilde{P}_{a,\ell}.
\]

We obtain from (48) that

\[
\mathbb{F} f(\bar{g}_a, \bar{\delta}_b) = (\bar{g}_a - \bar{g}_{a-1})(\bar{\delta}_b - \bar{\delta}_{b-1}) + \sum_{n=1}^N \sum_{k=1}^M f(\bar{\delta}_n, \bar{g}_k) \times
\]

\[
\left[ \sum_{m=1}^N P_{b,m}(0) - \sum_{m=1}^N P_{b-1,m}(0) \right] \left[ \sum_{\ell=k}^M \tilde{P}_{a,\ell} - \sum_{\ell=k}^M \tilde{P}_{a-1,\ell} \right] .
\]

Given Assumption 2, it follows that \( \mathbb{F} f(\bar{g}, \bar{\delta}) \geq 0 \) for all \((\bar{g}, \bar{\delta})\) if \( f(\hat{g}, \hat{\delta}) \geq 0 \) for all \((\hat{g}, \hat{\delta})\) and \( B^* = 0 \). By combining above results, we conclude that the policy iteration retains the property \( f(\hat{g}, \hat{\delta}) \geq 0 \) if its initialization has such a property (e.g., \( f(\hat{g}, \hat{\delta}) = 1 \) for all \((\hat{g}, \hat{\delta})\)). This completes the proof.
B. Proof for Lemma 2

Following the same procedure for deriving (46), we obtain that

\[
Z(a, b, B) = \begin{cases} 
  aE[\varepsilon | B] + B + \sum_{n=1}^{N} \sum_{k=1}^{M} f(\overline{\delta}_n, \overline{g}_k) \sum_{m=n}^{N} P_{b,m}(B) \sum_{\ell=k}^{M} \tilde{P}_{a,\ell}, \quad B > 0 \\
  ab + \sum_{n=1}^{N} \sum_{k=1}^{M} f(\overline{\delta}_n, \overline{g}_k) \sum_{m=n}^{N} \tilde{P}_{b,m}(0) \sum_{\ell=k}^{M} \tilde{P}_{a,\ell}, \quad B = 0.
\end{cases}
\]

(50)

Given Assumption 1 and 3 and using Lemma 1, Property 1) in the lemma statement holds since \(Z(a, b, B)\) is a nonnegative combination of monotone decreasing and convex functions of \(B\) as can be observed from (50). Property 2) follows from Assumption 2, Lemma 1 and (50).

C. Proof for Theorem 6

Since \(\hat{P}^*_\rho \to \hat{P}^*\) as \(\rho \to 1\), it is sufficient to prove that given an arbitrary \(\rho \in (0, 1)\), \(\hat{P}^*_\rho\) has the properties of \(\hat{P}^*\) as described in the theorem statement.

Assume that there exists \((a, b) \in \hat{X}\) such that \(\hat{P}_\rho(a, b) = 0\). Using (25) and (48), \((a, b)\) satisfies the following condition

\[
Z(a, b, 0) \leq \min_{B > 0} Z(a, b, B).
\]

(51)

It follows from Assumption 2 and (50) that with \(a\) fixed, \(Z(a, b, 0)\) is a monotone increasing function of \(b\). As a result, we obtain from (51) that

\[
Z(a, \delta, 0) \leq \min_{B > 0} Z(a, b, B), \quad \forall \delta \leq b.
\]

It follows that \(\hat{P}_\rho\) has Property 1) in the theorem statement.

Next, assume that there exists \((a, b) \in \hat{X}\) such that \(\hat{P}_\rho(a, b) > 0\). Using Property 1) in the theorem statement,

\[
\hat{L}^*_\rho(a, \delta) = \min_{B > 0} Z(a, \delta, B), \quad \forall \delta \geq b.
\]

(52)

It follows that

\[
\hat{P}_\rho(a, \delta) = \arg \min_{B > 0} Z(a, \delta, B), \quad \forall \delta \geq b.
\]

(53)

By observing from (50) that \(Z(a, \delta, B)\) with \(B > 0\) is independent of \(\delta\), the above equation gives Property 2) in the theorem statement.

Last, assume that there exist \((\bar{g}_a, \bar{\delta}_b), (\bar{g}_c, \bar{\delta}_b) \in \hat{X}\) such that \(\bar{g}_a \leq \bar{g}_c\), \(\hat{P}_\rho(\bar{g}_a, \bar{\delta}_b) > 0\), and \(\hat{P}_\rho(\bar{g}_c, \bar{\delta}_b) > 0\). To facilitate the proof, we arrange the available numbers of feedback bits in the ascending order:
\( B = \{ \Bar{B}_1, \Bar{B}_2, \ldots , \Bar{B}_A \} \) with \( \Bar{B}_1 \leq \Bar{B}_2 \cdots \leq \Bar{B}_A \). Moreover, define the differences
\[
\Delta^+ Z(\Bar{g}_a, \Bar{\delta}_b, \Bar{B}_u) = Z(\Bar{g}_a, \Bar{\delta}_b, \Bar{B}_u) - Z(\Bar{g}_a, \Bar{\delta}_b, \Bar{B}_{u+1}), \quad u < A \quad (54)
\]
\[
\Delta^- Z(\Bar{g}_a, \Bar{\delta}_b, \Bar{B}_u) = Z(\Bar{g}_a, \Bar{\delta}_b, \Bar{B}_u) - Z(\Bar{g}_a, \Bar{\delta}_b, \Bar{B}_{u-1}), \quad u > 1. \quad (55)
\]
Using (50) and the definition in (54), we obtain that
\[
\Delta^+ Z(\Bar{g}_a, \Bar{\delta}_b, \Bar{B}_u) = \bar{g}_a \{ E[\epsilon \mid \Bar{B}_u] - E[\epsilon \mid \Bar{B}_{u+1}] \} + \sum_{n=1}^{N} \sum_{k=1}^{M} f(\delta_n, g_k) \sum_{\ell=k}^{M} \bar{P}_{a,\ell} \times \left[ \sum_{m=n}^{N} p_{b,m}(\Bar{B}_u) - \sum_{m=n}^{N} p_{b,m}(\Bar{B}_{u+1}) \right] - (\Bar{B}_{u+1} - \Bar{B}_u). \quad (56)
\]
It follows that
\[
\Delta^+ Z(\Bar{g}_a, \Bar{\delta}_b, \Bar{B}_u) - \Delta^+ Z(\Bar{g}_c, \Bar{\delta}_b, \Bar{B}_u) = (\bar{g}_a - \bar{g}_c) \{ E[\epsilon \mid \Bar{B}_u] - E[\epsilon \mid \Bar{B}_{u+1}] \} + \sum_{n=1}^{N} \sum_{k=1}^{M} f(\delta_n, g_k) \times \left[ \sum_{\ell=k}^{M} \bar{P}_{a,\ell} - \sum_{\ell=k}^{M} \bar{P}_{c,\ell} \right] \left[ \sum_{m=n}^{N} p_{b,m}(\Bar{B}_u) - \sum_{m=n}^{N} p_{b,m}(\Bar{B}_{u+1}) \right] \leq 0 \quad (57)
\]
where (57) is obtained using Assumption 1 and 2, and Lemma 1. Similarly, it can be shown that
\[
\Delta^- Z(\Bar{g}_a, \Bar{\delta}_b, \Bar{B}_u) - \Delta^- Z(\Bar{g}_c, \Bar{\delta}_b, \Bar{B}_u) \geq 0. \quad (58)
\]
Define \( \Bar{B}_\alpha = \arg \min_{B > 0} Z(\bar{g}_a, \bar{\delta}_b, B) \) and \( \Bar{B}_\beta = \arg \min_{B > 0} Z(\bar{g}_c, \bar{\delta}_b, B) \). Since \( Z(\cdot, \cdot, B) \) is a convex function of \( B \) for \( B > 0 \) according to Lemma 2, \( \Delta^+ Z(\bar{g}_a, \bar{\delta}_b, \Bar{B}_\alpha) \leq 0 \) and \( \Delta^- Z(\bar{g}_a, \bar{\delta}_b, \Bar{B}_\alpha) \leq 0 \). Combining these facts with (57) and (58) gives that
\[
\Delta^+ Z(\bar{g}_c, \bar{\delta}_b, \Bar{B}_\alpha) \geq \Delta^+ Z(\bar{g}_a, \bar{\delta}_b, \Bar{B}_\alpha) \quad (59)
\]
\[
\Delta^- Z(\bar{g}_c, \bar{\delta}_b, \Bar{B}_\alpha) \leq 0. \quad (60)
\]
The inequality in (59) ensures that \( \Bar{B}_\beta \geq \Bar{B}_\alpha \). The other inequality in (60) gives \( \Delta^+ Z(\bar{g}_c, \bar{\delta}_b, \Bar{B}_\alpha) \geq 0 \), corresponding to \( \Bar{B}_\beta \geq \Bar{B}_\alpha \). This proves that \( \Bar{P}_\rho \) satisfies Property 3) in the theorem statement.

D. Proof for Proposition 1

We claim that there exists a threshold \( \Psi : g \to \delta \) such that the optimization problem in (31) is equivalent to the following one:
\[
\text{minimize: } \quad E \left[ g^{2 - \frac{n}{\ell + \tau}} \mid \delta \geq \Psi(g) \right] \quad (61)
\]
subject to: \( E[B] \leq \frac{\bar{b}}{K - 1} \)
\[
B \geq 0
\]
which is proved as follows. Let \( B^* \) denote the solution of (31). Given \( g \) and from (31), if there exists \( \delta_a \in \mathcal{D} \) such that \( \frac{1}{L}2^{-\frac{\mu}{L}} < \delta_a \), \( \frac{L}{L}2^{-\frac{\mu}{L}} < \delta \) for all \( \delta \geq \delta_a \); if there exists \( \delta_b \in \mathcal{D} \) such that \( \frac{L}{L}2^{-\frac{\mu}{L}} \geq \delta_b \), \( B^* \) satisfies \( B^* = 0 \) for all \( \delta \leq \delta_b \). This proves the claim.

The above optimization problem can be solved as follows. First, by neglecting the positivity constraint on \( B \), the convex optimization problem in (61) can be solved using Lagrangian method [34]. The resultant policy is specified in (32). Next, we design \( \Psi \) to suppress the expected interference power. The choice of \( \Psi \) must enforce two constraints in (31): i) feedback reduces the expected CSI error, namely \( \frac{L}{L}2^{-\frac{\mu}{L}} < \delta \) if \( B > 0 \) and ii) \( B \geq 0 \ \forall \ (g, \delta) \). It follows that the problem of optimizing \( \Psi \) is as given in (34) with \( \Psi^{-1}(1) \) replaced with \( \min_{\delta} \Psi^{-1}(\delta) \). Next, it can be observed from (35) that given \( \Pr(\delta \geq \Psi(g)) \), minimizing \( I^* \) requires \( \Psi(g) \) to be a monotone decreasing function of \( g \), proving the stated monotonicity of \( \Psi(g) \). As a result, \( \min_{\delta} \Psi^{-1}(\delta) = \Psi^{-1}(1) \) and (34) follows. This completes the proof.

E. Proof for Lemma 3

Let \( \mathcal{P} \) denote the space of optimal feedback-control policies. Consider \( B_a, B_b \in \bigcup_{\mathcal{P}} \mathcal{P}^*(g^{[mn]}_t, \delta^{[mn]}_t) \) for given \( (g^{[mn]}_t, \delta^{[mn]}_t) \in \mathcal{X} \). Note that \( \mathcal{E} [\epsilon^{[mn]} | B] < \delta^{[mn]}_t \) if \( B \in \{B_a, B_b\} \) and \( B > 0 \). Using this fact and for \( \mu \in [0, 1] \), the term \( q(B) = \min \left( \mathcal{E} [\epsilon^{[mn]} | B], \delta^{[mn]}_t \right) \) in the objective function of the sub-problem is proved to be convex as follows

\[
\mu q(B_a) + (1 - \mu) q(B_b) = \begin{cases} 
\mu E \left[ \epsilon^{[mn]}_t | B_a \right] + (1 - \mu) E \left[ \epsilon^{[mn]}_t | B_b \right], & B_a \geq 0, B_b \geq 0 \\
\mu \delta^{[mn]}_t + (1 - \mu) E \left[ \epsilon^{[mn]}_t | B_b \right], & B_a = 0, B_b > 0 \\
\mu E \left[ \epsilon^{[mn]}_t | B_a \right] + (1 - \mu) \delta^{[mn]}_t, & B_a > 0, B_b = 0 \\
\delta^{[mn]}_t, & B_a = 0, B_b = 0 
\end{cases}
\]

\[ (62) \]

\[
\geq \begin{cases} 
E \left[ \epsilon^{[mn]}_t | \mu B_a + (1 - \mu) B_b \right], & B_a > 0, B_b > 0 \\
E \left[ \epsilon^{[mn]}_t | B_b \right], & B_a = 0, B_b > 0 \\
E \left[ \epsilon^{[mn]}_t | B_a \right], & B_a > 0, B_b = 0 \\
\delta^{[mn]}_t, & B_a = 0, B_b = 0 
\end{cases}
\]

\[ = q(\mu B_a + (1 - \mu) B_b) \]

where (62) uses the convexity of \( \mathcal{E} [\epsilon^{[mn]}_t | B] \) over \( B \) as assumed in Assumption 1. Thus, \( q(B) \) is a convex function.

Next, we prove that \( \bar{\epsilon}^{[mn]}_t(x) \) is a convex function for \( x > 0 \) using the sample-path method. Consider two average sum-feedback rates \( \bar{b}_x, \bar{b}_y > 0 \). Let \( \mathcal{P}^*_x \) and \( \mathcal{P}^*_y \) denote the optimal feedback-control policies
that yield \( \tilde{I}_{\text{min}}^{[mn]}(\tilde{b}_x) \) and \( \tilde{I}_{\text{min}}^{[mn]}(\tilde{b}_y) \), respectively. Given the sample paths \( \{g_t^{[mn]}\}_{t=1}^{\infty} \) and \( \{\delta_t^{[mn]}\}_{t=1}^{\infty} \), let \( \{B_t^x\}_{t=1}^{\infty} \) and \( \{B_t^y\}_{t=1}^{\infty} \) denote the sequences of numbers of feedback bits generated by \( \mathcal{P}^*_x \) and \( \mathcal{P}^*_y \), respectively. Moreover, given \( \mu \in [0,1] \), define the sequence \( \{B_t^z\}_{t=1}^{\infty} = \mu \{B_t^x\}_{t=1}^{\infty} + (1-\mu)\{B_t^y\}_{t=1}^{\infty} \). Given \( \{g_t^{[mn]}\}_{t=1}^{\infty} \) and \( \{\delta_t^{[mn]}\}_{t=1}^{\infty} \) and using the function \( q(B) \) defined earlier, we can write

\[
\mu \tilde{I}_{\text{min}}^{[mn]}(\tilde{b}_x) + (1-\mu) \tilde{I}_{\text{min}}^{[mn]}(\tilde{b}_y) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \frac{1}{L-1} g_t^{[mn]} [\mu q(B_t^x) + (1-\mu)q(B_t^y)]
\]

\[
\geq \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \frac{1}{L-1} g_t^{[mn]} q(B_t^z)
\]

(63)

\[
\geq \tilde{I}_{\text{min}}^{[mn]}(\mu \tilde{b}_x + (1-\mu)\tilde{b}_y)
\]

(64)

where (63) uses the convexity of \( q(B) \) as proved earlier. The desired result follows from (64).

**ACKNOWLEDGMENT**

The authors are thankful to Ying Cui and Rang Rong for useful discussion on this work.

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