Large-System Analysis of MIMO Multiple-Access Systems with Pattern Diversity

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Abstract—By considering the large-system regimes, we analyze the asymptotic performance of a multiple-input multiple-output (MIMO) multiple-access (MA) wireless network in the presence of spatial correlation at the transmitters (i.e., mobile stations) and the receiver (i.e., base station) with particular emphasis on the receive antennas utilizing pattern diversity. Using the replica method originally developed in statistical physics, we are able to derive analytical solutions to the spectral efficiency of the MIMO-MA systems for any given arbitrary input signal distributions at the transmitters assuming the information-theoretic optimum decoding at the receivers. Our results are general in that they encompass lots of previously published results for degenerate cases. In addition, based on the asymptotic solution, we propose a computation-efficient algorithm to determine the asymptotic optimum transmit signal covariance matrices that can maximize the spectral efficiency of the MA network when only the slow-varying channel spatial covariance information is available at the transmitters.

I. INTRODUCTION

Over the last few years, most researches have tended to investigate the spectral efficiency of the linear communication channel $y = Hx + u$ where $x$ is the transmit signal vector, $y$ is the receive signal vector, $H$ is the channel matrix, and $u$ is the additive white Gaussian noise vector. Much contributions have been made on understanding the spectral efficiency of the concerned system model, as is typical for communications systems such as code-division multiple-access (CDMA) [1], [2] and multiple-input multiple-output (MIMO) antenna channels [3]–[8].

Of particular interests, the large-system analysis (where the dimensions of the channel matrix, $H$, are considered to be large) is often utilized to overcome mathematical difficulty and obtain useful insights into the system behavior. Along this line, Tse and Hanly [1] and Verdú and Shamai [2] introduced the notion of large random matrix theory [9] in the analysis of synchronous random-spreading CDMA systems. Their success has triggered many studies in the asymptotic spectral efficiency of MIMO channels. However, so far, most results exploiting large random matrix theory (e.g., [5]–[8]) were restricted to systems with Gaussian input distributions.

It is in fact practically more appealing to know the channel spectral efficiency when the input signals are not Gaussian such as B/QPSK- (binary/quaternary phase shift keying) modulated signals. Since the spectral efficiencies of such systems cannot be described by the eigenvalues of channel matrices, large random matrix theory will become not very useful.

Another powerful approach for analyzing the large-system properties of a system is the replica method [10]. Tanaka was the first to use the replica method for solving the large-system spectral efficiency of CDMA systems with antipodal signaling inputs [11]. Later, Guo and Verdú further generalized Tanaka’s results to multiple-access (MA) scenario with users of unequal power [12]. Most recently in [13], Guo and Verdú continued the work of [11], [12] and presented a unified treatment of the spectral efficiency of CDMA systems for arbitrary input distributions. It has been demonstrated in [11]–[13] that the replica method cannot only provide innovative results, but also reproduce lots of known results found by large random matrix theory. Following their works, recent studies can be found in [14]–[18].

Our aim of this paper is to generalize the previous results of MIMO-MA wireless networks in [11]–[17] by considering the presence of the spatial correlations at both the transmitters (or mobile stations) and the receiver (or base station). Unlike [16], this paper considers the receive antennas using pattern diversity [19]. This generalization is original and non-trivial. The spectral efficiency formulae we derive are general and can be reduced to previously published results [1], [2], [5]–[8], [11]–[18], for degenerate cases of a MIMO-MA system. The analytical results clarify the impact of the pattern diversity on the spectral efficiency of a MIMO-MA system. Based on the asymptotic solution, we further propose an efficient algorithm to determine the asymptotic capacity-achieving transmit signal covariance matrices when only the slow-varying channel spatial covariance information is available at the transmitters.

Throughout this paper, for any matrix $A$, $A^*$ denotes the conjugate transpose of $A$, $A^T$ denotes the transpose of $A$, $\text{Tr}(A)$ denotes the trace of $A$, and $\lambda_i(A)$ denotes the $i$-th largest eigenvalues of $A$. In addition, $I$ denotes the identity matrix, $0$ denotes the zero matrix, and $E\{\cdot\}$ represents the expectation operator.

II. CHANNEL MODEL

In this paper, we consider a MIMO-MA system where $K$ users are transmitting to a base station with $B$ antenna sets, and $M_k$ and $N_b$ antennas are located, respectively, at the $k$-th user and the $b$-th antenna set (see Figure 1). Antennas with different rotated versions could have different radiation patterns and provide pattern diversity. Such pattern diversity are usually used to decrease the signal correlation of antennas [19].
In this setting, the channel responses from the $k$-th user to the $b$-th antenna set form a channel matrix $H_{b,k} \in \mathbb{C}^{N_b \times M_k}$. Let $M \triangleq \sum_k M_k$ and $N \triangleq \sum_b N_b$. Also, let $x_k \in \mathbb{C}^{M_k}$ be the transmit signal vector of user $k$, $y_b \in \mathbb{C}^{N_b}$ be the receive signal vector of antenna set $b$, and $u \in \mathbb{C}^{M_b}$ be the noise vector at the receiver, which has zero mean and $\mathbb{E}\{uu^*\} = \sigma^2 I$. The receive signals can then be written in matrix form as

$$
\begin{bmatrix}
y_1 \\
\vdots \\
y_B
\end{bmatrix} = \sum_{k=1}^{K} \frac{1}{\sqrt{M_k}} \begin{bmatrix} H_{1,k} \\
\vdots \\
H_{B,k}
\end{bmatrix} x_k + u
$$

(1)

where $H_k \triangleq \begin{bmatrix} H_{1,k}^T & \cdots & H_{B,k}^T \end{bmatrix}^T$. Also, we define

$$
E\{x_kx_{k'}^*\} \triangleq \begin{cases} 
\Omega_k & \text{if } k = k' \\
0 & \text{if } k \neq k'
\end{cases}
$$

(3)

as the covariance matrix of the transmit signal $x_k$. The total transmit power of the $k$-th user is limited to $P_k$, i.e., $\text{Tr}(\Omega_k) \leq P_k$.

To model the spatial correlation of a MIMO channel, we use the separable correlation model [20], [21] so that we can factorize $H_{b,k}$ as

$$
H_{b,k} = R_{b,k}^z W_{b,k} T_{b,k}^z
$$

(4)

where $R_{b,k} \in \mathbb{C}^{N_b \times N_b}$ is the receive spatial correlation matrix of the $b$-th antenna set responding to the $k$-th user, $W_{b,k} \in \mathbb{C}^{M_k \times M_k}$ is a diagonal matrix representing the transmit power distribution of the $k$-th user directing to the $b$-th antenna set. Likewise, $W_{b,k} \in \mathbb{C}^{N_b \times N_b}$ has zero mean and independent and identically distributed (i.i.d.) circular symmetric Gaussian entries. For systems with Gaussian distribution inputs, our analytical results can be applied to scenarios where $T_{b,k}^z$ represents the transmit spatial correlation matrix of the $k$-th user directing to the $b$-th antenna set.

The degenerate cases of the concerned channel scenario (1) and (4) appear in several areas of wireless communications. Existing large-system spectral efficiencies in [11], [12], [14]–[18] have been derived with either Gaussian or antipodal input distributions. In this paper and in contrast to the previous works, we aim to derive the asymptotic spectral efficiency of the MIMO-MA systems applicable for any arbitrary input distributions.

We focus on the large-system regimes where both $M$ and $N$ tend to infinity, but $\frac{M}{N}$ converges to fixed positive numbers $\rho$. Let $\mu_k \triangleq \frac{M_k}{M}$, $\varrho_b \triangleq \frac{N_b}{N}$, and $\rho_k \triangleq \frac{M_k}{M}$. We thus have $\rho = \sum_k \rho_k$ which is considered as the total system load. Also, we find it useful to define the following empirical distributions for the eigenvalues, $\{\lambda[T_{b,k}\Omega_k]\}$ and $\{\lambda_n[R_{b,k}\Omega_k]\}$

$$
P^{(M_k)}_{\lambda[T_{b,k}\Omega_k]}(\lambda) = \frac{1}{M_k} \sum_{m=1}^{M_k} 1 \left\{ \lambda_m \left[ T_{b,k}^z \Omega_k^z \left( T_{b,k}^z \Omega_k^z \right)^* \right]^2 \right\} \leq \lambda
$$

(5)

and

$$
P^{(N_b)}_{\lambda[R_{b,k}\Omega_k]}(\lambda) = \frac{1}{N_b} \sum_{n=1}^{N_b} 1 \left\{ \lambda_n \left[ R_{b,k} \right] \right\} \leq \lambda
$$

(6)

where $1\{\cdot\}$ is the indicator function. Hereafter, $\lambda_m[T_{b,k}\Omega_k]$ is adopted to replace $\lambda_m[T_{b,k}^z \Omega_k^z \left( T_{b,k}^z \Omega_k^z \right)^*]$. For convenience, we shall assume that the empirical distribution $P^{(M_k)}_{\lambda[T_{b,k}\Omega_k]}(\lambda)$ converges almost everywhere to the distribution $P_{\lambda[T_{b,k}\Omega_k]}(\lambda)$ of the random variable $\lambda[T_{b,k}\Omega_k]$ as $M_k \to \infty$. Similarly, the empirical distribution $P^{(N_b)}_{\lambda[R_{b,k}\Omega_k]}(\lambda)$ converges almost everywhere to the distribution $P_{\lambda[R_{b,k}\Omega_k]}(\lambda)$ of the random variable $\lambda[R_{b,k}\Omega_k]$ as $N_b \to \infty$. In addition, we define a random vector $\lambda[T_{b,k}\Omega_k] \triangleq \left[ \lambda[T_{1,k}\Omega_k] \cdots \lambda[T_{B,k}\Omega_k] \right]^T$.
Theorem 1: Given an input distribution \( \{p(x_k)\} \) with signal covariance matrices \( \{\Omega_k\} \), the asymptotic spectral efficiency of the MIMO-MA system (1) is given by

\[
I_{\text{MA}} = - \sum_{k=1}^{K} \rho_k \cdot E_{\lambda[T_k \Omega_k]} \left\{ \int p(y_k|\lambda[T_k \Omega_k], \varsigma_k) \log p(y_k|\lambda[T_k \Omega_k], \varsigma_k) \, dy_k + \log \frac{\pi e}{\sum_{b=1}^{B} \lambda[T_b \Omega_k][\varsigma_b, k]} \right\}
\]

\[+ \sum_{b=1}^{B} \rho_b \cdot E_{\lambda[R_{b,k}, \ldots, \lambda[R_{b,K}]} \left\{ \log \left( 1 + \sum_{k=1}^{K} \frac{\lambda[R_{b,k}]}{\sigma^2} \varepsilon_{b,k} \right) \right\} - \sum_{k=1}^{K} \sum_{b=1}^{B} \rho_k \varepsilon_{b,k, k}, \tag{9} \]

where \( \varsigma_{b,k} \) satisfies the following fixed-point equation

\[
\varsigma_{b,k} = \frac{\rho_b}{\rho_k} E_{\lambda[R_{b,k}, \ldots, \lambda[R_{b,K}]} \left\{ \frac{\lambda[R_{b,k}]}{\sigma^2 + \sum_{l=1}^{K} \lambda[R_{b,l}] \varepsilon_{b,l}} \right\} \tag{10} \]

in which \( \varepsilon_{b,k} \) is defined as

\[
\varepsilon_{b,k} \triangleq E_{\lambda[T_k \Omega_k]} \left\{ \lambda[T_b \Omega_k] \times \int E_{x_k} \left\{ |x_k - \hat{x}_k|^2 p(y_k|x_k, \lambda[T_k \Omega_k], \varsigma_k) \right\} \, dy_k \right\}, \tag{11} \]

\[
\hat{x}_k \triangleq E_{x_k} \{ x_k | y_k, \lambda[T_k \Omega_k], \varsigma_k \}, \tag{12} \]

and

\[
p(y_k|x_k, \lambda[T_k \Omega_k], \varsigma_k) \triangleq \frac{\sum_{b=1}^{B} \lambda[T_{b,k} \Omega_k][\varsigma_{b,k}, x_k]}{\pi e^{-\sum_{b=1}^{B} \lambda[T_b \Omega_k][\varsigma_{b,k}, y_k-x_k]^2}}. \tag{13} \]

In (12), the expectation is taken over the posteriori distribution of the channel \( p(x_k|y_k, \lambda[T_k \Omega_k], \varsigma_k) \) which can be obtained from the priori distribution \( p(x_k) \) and the channel conditional distribution (13) through the Baye’s formula.

Proof: Omitted [23].

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![Fig. 2. Reconstruction of transmit data using separate detection and decoding scheme.](image)

The fixed-point equation (10), obtained using the tool in statistical physics, may have multiple solutions. Among them, the one that gives the smallest value of the spectral efficiency corresponds to the one maximizing the capacity of communication problems (9) [11].

From Theorem 1, we characterize the MIMO-MA system in great details as follows:

- Theorem 1 provides the asymptotic spectral efficiency of the MIMO-MA system using the information-theoretic optimum decoding. This scheme jointly detects and decodes all the signals received at the receiver. A receive structure that consists of a spatial decorrelator followed by a bank of temporal error-correction decoders, however, is more practical (see Figure 2). We shall refer to this decoding scheme as separate detection and decoding (SDD) scheme. Using a way analogous to [13], it can be shown that the first term of (9) corresponds to the spectral efficiency of the MIMO-MA SDD system employing a minimum mean square error (MMSE) detector followed by the optimum error-correction decoders at the receiver.

- The first term of (9) can also be recognized as the mutual information between \( x_k \) and \( y_k \) for a scalar Gaussian channel

\[
y_k = \sqrt{\frac{\sum_{b=1}^{B} \lambda[T_{b,k} \Omega_k][\varsigma_{b,k}, x_k]}{\pi e^{\sum_{b=1}^{B} \lambda[T_b \Omega_k][\varsigma_{b,k}, y_k-x_k]^2}}} + u_k \tag{14} \]

where \( u_k \) is a standard Gaussian random variable. Therefore, \( \varsigma_{b,k} \) can be understood as the equivalent channel gain contributed by the \( b \)-th pattern diversity to the \( k \)-th user. Apparently, pattern diversity provides a combining-like gain in the spectral efficiency of the MIMO-MA SDD system.

- The information loss resulting from the separate processing can be found by

\[
I_{\text{MA}} - I_{\text{SDD}} = \sum_{b=1}^{B} \rho_b E_{\lambda[R_{b,k}, \ldots, \lambda[R_{b,K}]} \{ (\eta_b - 1) - \log \eta_b \} \tag{15} \]

where

\[
\eta_b = \frac{\sigma^2 + \sum_{k=1}^{K} \lambda[R_{b,k}] \varepsilon_{b,k}}{\lambda[R_b, k][\varsigma_{b,k}, x_k]}.
\]

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IV. APPLICATIONS

Theorem 1 provides not only new findings but a unified formula that encompasses many previous known results. In Section IV-A, we shall show how Theorem 1 can be used...
to obtain the asymptotic spectral efficiency of the MIMO-MA system with Gaussian inputs. Based on the asymptotic solution, we then develop an efficient algorithm to determine the asymptotic optimum transmit signal covariance matrix for each user to maximize the spectral efficiency in Section IV-B.

A. Gaussian Inputs

By Theorem 1, we obtain the asymptotic spectral efficiency of the MIMO-MA system with Gaussian distributed inputs

\[
\mathcal{I}_{\text{MA}} \left( \{ \Omega_k \} \right) = \sum_{k=1}^{K} \log \det \left( I + \frac{1}{M_k} \sum_{b=1}^{B} \mathbf{S}_{b,k} \mathbf{T}_{b,k} \Omega_k \right) + \sum_{b=1}^{B} \log \det \left( I + \frac{1}{\sigma^2} \mathbf{E}_{b,k} \right) - \sum_{k=1}^{K} \sum_{b=1}^{B} \mathbf{M}_{b,k} \mathbf{S}_{b,k} \mathbf{e}_{b,k} \tag{16}
\]

where \( \mathbf{S}_{b,k} \) and \( \mathbf{E}_{b,k} \) satisfy the fixed-point equations

\[
\mathbf{S}_{b,k} = \frac{1}{M_k} \text{Tr} \left\{ \sigma^2 I + \sum_{l=1}^{K} \mathbf{E}_{b,l} \mathbf{R}_{b,l} \right\}^{-1} \mathbf{R}_{b,k},
\]

\[
\mathbf{E}_{b,k} = \frac{1}{M_k} \text{Tr} \left\{ I + \sum_{q=1}^{B} \mathbf{S}_{q,k} \mathbf{T}_{q,k} \Omega_k \right\}^{-1} \mathbf{T}_{b,k} \Omega_k \tag{17}
\]

Note that \( \mathcal{I}_{\text{MA}} \) is now expressed in a quantized format as if the numbers of input and output arrays are finite. Also, the spectral efficiency (16) is exactly the same as the mutual information 1 in [7] obtained using the random matrix theory and 2) in [15] obtained using the replica method under a corresponding single-user MIMO channel with polarization/pattern diversity. However, unlike our expression, the forms in [7] and [15] do not report explicitly the eigenvector structure of the spatial correlation matrices.

B. Asymptotic-Optimum Input Signal Covariance Matrices

Through the simulation results in Section V, we shall see that the asymptotic spectral efficiency of MIMO-MA systems is extremely close to its true one, even with as few as three or four antennas at each transmitter and receiver. This observation offers the asymptotic solution the practicality for developing an efficient algorithm that finds the asymptotic optimal input signal covariance matrices to maximize the spectral efficiency.

Theorem 2: When only channel (spatial) correlation information (CCI) is known, the signal transmission strategy for the \( k \)-th user that maximizes \( \mathcal{I}_{\text{MA}} \) is the water-filling solution to an equivalent system with the MIMO channel \( \sum_{b=1}^{B} \mathbf{S}_{b,k} \mathbf{T}_{b,k} \) where \( \mathbf{s}_{b,k} \)’s are a set of positive roots of the simultaneous equations (17).

Proof: Omitted [23].

Theorem 2 is a generalization of the MIMO-MA system result assuming \( B = 1 \) [23]. We can characterize the theorem in great details as follows:

- The eigen-modes of the optimum signal covariance matrix \( \Omega_k \) can be obtained by using the same eigen-basis of the transmit-side channel correlation matrix \( \sum_{b=1}^{B} \mathbf{S}_{b,k} \mathbf{T}_{b,k} \). If there exists a large \( \mathbf{s}_{b,k} \), then its corresponding channel structure \( \mathbf{T}_{b,k} \) will dominate the eigen-mode of \( \Omega_k \). Otherwise, we can expect that the eigen-modes of \( \Omega_k \) would be by and large structure-less (i.e., \( \Omega_k \approx I \)) or pattern diversity can be used to decrease the signal correlation of antennas. Obviously, Theorem 2 depicts clearly the structure of the optimum signal covariance matrices affected by pattern diversity.
- The interaction among users is through \( \mathbf{s}_{b,k} \)’s. The signal covariance matrix of each user is not affected by the channel structures of the other users. The eigen-modes of \( \Omega_k \) are only affected by the transmit channel structure of its own. \( \{ \mathbf{T}_{b,k} \}_{b=1}^{B} \).

Since all the signal covariance matrices \( \Omega_k \)’s are involved in solving \( \mathbf{s}_{b,k} \)’s [see (17)], we resort to the following iterative algorithm to find the asymptotic optimal signal covariance matrices.

Algorithm 1: Let the superscript \((t)\) denote the \( t \)-th iterate. We have the following algorithm for MIMO-MA systems with only CCI:

1. Initialize \( \mathbf{s}_{b,k}^{(0)} \geq 0 \) \( \forall b,k \).
2. \( \Omega_k^{(t)} = \arg \max_{\Omega_k} \log \det \left[ I + \sum_{b=1}^{B} \mathbf{s}_{b,k}^{(t)} \mathbf{T}_{b,k} \Omega_k \right], \quad k = 1, \ldots, K \).
3. Solve \( \{ \mathbf{s}_{b,k}^{(t+1)} \} \) according to the simultaneous equations (17) with \( \{ \Omega_k^{(t)} \} \).
4. Go back to Step 2 until convergence.

V. Simulation Results

In the simulations, we have assumed that the spatial correlation is generated from an uniform linear array with half wavelength spacing in a wireless environment where there is one propagation path cluster with Gaussian power azimuthal distribution having mean angle of \( \theta_{b,k} \) and angular spread of \( \delta_{b,k} \). In addition, the subscripts \( T \) and \( R \), respectively, refer to the corresponding values at the transmit and receive sides.

Experiment 1: Accuracy of (16). Using (7), the empirical spectral efficiency of the MIMO-MA system with Gaussian distributed inputs can be found by

\[
I(x_1, \cdots, x_K; y) = \mathbb{E}_H \left\{ \log \det \left( I + \frac{1}{K} \sum_{k=1}^{K} \mathbf{H}_k \Omega_k \mathbf{H}_k^H \right) \right\}. \tag{18}
\]

Table 1 compares the spectral efficiency between the analytical results according to (16) and their corresponding simulation results (18) obtained from 10,000 realizations of \( \mathbf{H} \). Without loss of generality, in the experiment, we assume \( \Omega_k = I \) and a two-antenna-set two-user scenario (i.e., \( B = 2 \) and \( K = 2 \)) with \( \theta_{T_{1,1}} = 10^\circ, \theta_{R_{1,1}} = 80^\circ, \theta_{T_{2,1}} = 20^\circ, \theta_{R_{2,1}} = 60^\circ, \theta_{T_{1,2}} = 30^\circ, \theta_{R_{1,2}} = 40^\circ, \theta_{T_{2,2}} = 40^\circ, \theta_{R_{2,2}} = 20^\circ, \delta_{T_{1,k}} = \delta_{R_{1,k}} = 3^\circ \). Results show that regardless of the signal-to-noise ratio (SNR), (16) produces highly accurate results even with only a few antenna elements at each transmitter and receiver.

Experiment 2: Efficiency of the proposed iterative algorithm. With perfect channel state information (CSI), an efficient
TABLE I

Comparison of the sum-rates between (18) and (16) for various $N_b, M_b$ and SNRs.

<table>
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<th>Antennas</th>
<th>$N_1 = N_2 = 4$, $M_1 = M_2 = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SNR(dB)</td>
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<tr>
<td>Analytical (bps/Hz)</td>
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</tr>
<tr>
<td>Empirical (bps/Hz)</td>
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<tr>
<td>Difference (bps/Hz)</td>
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<table>
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<tr>
<th>Antennas</th>
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<tbody>
<tr>
<td>SNR(dB)</td>
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<tr>
<td>Analytical (bps/Hz)</td>
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</tr>
<tr>
<td>Empirical (bps/Hz)</td>
<td>6.88</td>
</tr>
<tr>
<td>Difference (bps/Hz)</td>
<td>2.46%</td>
</tr>
</tbody>
</table>

![Fig. 3. Spectral efficiency of a two-user MIMO-MA system.](image)

VI. Conclusions

In this paper, we have provided a general analytical expression for the asymptotic (in the large-system limit) spectral efficiency of the MIMO-MA system with pattern diversity that is applicable for systems with heterogeneous spatial correlations at both the transmitters and the receive antenna sets. Our results degenerate to many previously published results for special cases. From the asymptotic spectral efficiency, we have proposed an efficient algorithm to find the optimum signal covariance matrices that can maximize the spectral efficiency of the MA network when only the slow-varying channel spatial covariance information is available at the transmitters.

REFERENCES


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