Performance Analysis of Rayleigh-Product MIMO Channels with Optimal Beamforming

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Abstract—This paper presents an analytical performance investigation of a Rayleigh-product multiple-input multiple-output (MIMO) channel using optimum transmit-receive beamforming. By deriving the closed-form expressions for the cumulative distribution function (C.D.F.), and the probability density function (P.D.F.), we provide a complete statistical characterization of the received signal-to-noise ratio (SNR) of the Rayleigh-product MIMO channel. These new statistical results further permit the analysis for the outage probability and the symbol-error-rate (SER), which are important performance indications of MIMO systems. In addition, we examine, in detail, an important special case corresponding to the degenerate keyhole scenario, for which we present insightful closed-form expressions for the diversity order and array gain.

I. INTRODUCTION

Recently, the use of multiple-input multiple-output (MIMO) antenna technologies, as a means for providing capacity and performance improvements over conventional single-antenna systems, has received considerable attention [1]. Many practical signaling algorithms have now been developed to exploit the underlying benefits of MIMO channels, with the optimal transmission strategy depending, among other things, on the specific optimization criteria and the availability of channel state information (CSI) at both ends. This paper will focus on the optimal strategy for maximizing the instantaneous received signal-to-noise ratio (SNR) with CSI at both the transmitter and the receiver [2]: the so-called MIMO beamforming (BF). MIMO BF provides robustness against the severe effects of fading by linearly pre- and post-combining the transmitted and received signals to steer the information symbols along the dominant eigen-direction of the MIMO channel matrix.

The performance of MIMO BF has been well investigated for various channel scenarios, e.g., [3–9], mostly in terms of the outage probability and the symbol-error-rate (SER). All of these prior studies, however, were based on the rich-scattering assumption that renders a full-rank channel matrix. It has been evident recently from field measurements [10] that the channel in practice may exhibit a reduced-rank behavior due to the lack of scatterers around the transmitter and receiver terminals. A more general model which embraces this aspect of the channel phenomenon, as well as allowing for correlation amongst the antennas and scatterers, is the double-scattering model [11], which involves a product of three deterministic matrices (i.e., transmitter, receiver, and scatterer correlation matrices), and two statistically independent random matrices.

In this paper, we consider a particular form of the double-scattering structure (as in, e.g., [12]) for which the transmit antennas, receive antennas, and scatterers are spatially uncorrelated. In this case, the double-scattering model reduces to the product of two independent zero-mean complex Gaussian matrices, which we shall specifically refer to as the Rayleigh-product channel. Closed-form expressions for the cumulative distribution function (C.D.F.), and the probability density function (P.D.F.). These statistical results permit us to evaluate the outage probability and the SER of a Rayleigh-product MIMO BF channel. To gain further insights, we also investigate an important special case corresponding to the degenerate keyhole scenario by deriving closed-form expressions for the diversity order and array gain.

II. SYSTEM MODEL

Consider a MIMO BF system with \( N_t \) transmit antennas and \( N_r \) receive antennas. The \( N_r \)-received samples in the baseband can be written in vector form as
\[
r = Hw + n
\]
where \( x \) denotes the transmitted symbol with \( \mathbb{E}||x||^2 = P \), \( w \) is the transmit BF vector with \( \mathbb{E}||w||^2 = 1 \), \( n \in \mathcal{C}^{N_r \times 1} \) is the additive noise vector containing independent entries \( \sim \mathcal{CN}(0, N_0) \), and \( H \in \mathcal{C}^{N_r \times N_t} \) represents the MIMO channel. For the Rayleigh-product model (i.e., the double-scattering model in [11], but with the transmitter, receiver, and scatterer correlation matrices being an identity matrix), \( H \) admits the following statistical factorization
\[
H = \frac{1}{\sqrt{N_s}} H_1 H_2,
\]
where \( H_1 \in \mathcal{C}^{N_r \times N_s} \) and \( H_2 \in \mathcal{C}^{N_s \times N_t} \) are independent matrices containing independent entries \( \sim \mathcal{CN}(0, 1) \), and \( N_s \) denotes the number of effective scatterers on each side. In the sequel, we shall parameterize the Rayleigh-product channel model by the three-tuple \( (N_t, N_r, N_s) \). This generic model encompasses a broad family of practical channel environments including, for instance, rich-scattering Rayleigh fading \( (N_s \to \infty) \) and a degenerate keyhole channel \( (N_s = 1) \).

The optimal receiver of a MIMO BF system in maximizing the SNR follows the maximal-ratio combining (MRC) princi-
ple to give
\[ \bar{x} = w^H r = w^H H w x + w^H H n, \]
from which the instantaneous output SNR is given by
\[ \gamma = \bar{\gamma} w^H H w \]
where \( \bar{\gamma} \triangleq P / N_0 \) is the average transmit SNR. The optimum transmit BF vector, \( w_{\text{opt}} \), is the eigenvector corresponding to the maximum eigenvalue, or \( \lambda_{\text{max}} \), of \( H^H H \), which yields
\[ \gamma_{\text{max}} = \bar{\gamma} w_{\text{opt}}^H H w_{\text{opt}} = \bar{\gamma} \lambda_{\text{max}}. \]

Clearly the performance of \( \lambda_{\text{max}} \), which we now derive.

III. THE MARGINAL DISTRIBUTIONS OF \( \lambda_{\text{max}} \)

This section will present a number of closed-form exact and asymptotic expressions for the C.D.F. and P.D.F. for all results in this section, and those which follow in the paper, it is convenient to define the following notation: \( s = \min(N_s, N_r) \), \( t = \max(N_s, N_r) \), \( m = \min(N_s, t) \), \( n = \max(N_s, t) \) and \( \tau_{i,j} = n - m + i + j - 1 \).

A. Exact Expressions for the C.D.F.

The following theorem presents the C.D.F. of \( \lambda_{\text{max}} \), which will be useful for deriving the outage probability of MIMO BF systems in Rayleigh-product channels.

**Theorem 1:** The C.D.F. of \( \lambda_{\text{max}} \) is given by
\[ F_{\lambda_{\text{max}}}(x) = \prod_{l=1}^{m} \frac{\Gamma(m-i+1)\Gamma(n-i+1)}{\Gamma(l+1)\Gamma(t-s)}, \]
where \( \Delta(x) \) is an \( m \times m \) matrix function of \( x \) with \( (i,j) \)th entries expressed by (7) (at the top of the next page), \( K_s(z) \) in (7) is the modified Bessel function of the second kind, and \( \Gamma(\cdot) \) is the gamma function.

**Proof:** See Appendix I.

**Corollary 1:** For a keyhole-channel, \( N_s = 1 \) and we have
\[ F_{\lambda_{\text{max}}}(x) = 1 - \sum_{l=0}^{s-1} \frac{2x^{s+l}}{\Gamma(l+1)\Gamma(t-s)} K_{t-l}(2\sqrt{x}). \]

B. Exact Expressions for the P.D.F.

The following theorem presents the P.D.F. of \( \lambda_{\text{max}} \), which will be useful for computing the SER and ergodic capacity of MIMO BF systems in Rayleigh-product channels.

**Theorem 2:** The P.D.F. of \( \lambda_{\text{max}} \) is given by
\[ f_{\lambda_{\text{max}}}(x) = \frac{\sum_{m-s+1}^{m} \det \Delta_l(x)}{\Gamma(m-i+1)\Gamma(n-i+1)}, \]
where \( \Delta_l(x) \) is an \( m \times m \) matrix function of \( x \) with
\[ \{ \Delta_l(x) \}_{i,j} = \begin{cases} \{ \Delta(x) \}_{i,j} & \text{for } i \neq l, \\ 2K_{s+m+i+j-1}(2\sqrt{x}) & \text{for } i = l, \end{cases} \]
in which \( \{ \Delta(x) \}_{i,j} \) has been defined in (7).

**Proof:** See Appendix II.

**Corollary 2:** If \( N_s = 1 \), the P.D.F. of \( \lambda_{\text{max}} \) becomes
\[ f_{\lambda_{\text{max}}}(x) = \frac{2x^{s-1}}{\Gamma(t-s)\Gamma(t)} K_{t-l}(2\sqrt{x}). \]

C. Asymptotic Expansions for the C.D.F. and P.D.F.

Here, we present the first-order expansions for the C.D.F. and P.D.F. of \( \lambda_{\text{max}} \). As such expansions for general Rayleigh-product channels with arbitrary \( N_s, N_r \), and \( N_r \) are extremely difficult to obtain, we consider only the special case \( N_s = 1 \). These results will be crucial for investigating the asymptotic outage probability and for deriving the diversity order and array gain of MIMO BF systems in keyhole channels.

**Theorem 3:** For the case \( N_s = 1 \), the first-order expansions of the C.D.F. and P.D.F. of \( \lambda_{\text{max}} \) are, respectively, given by
\[ F_{\lambda_{\text{max}}}(x) = a_1 x^s + o(x^s), \]
\[ f_{\lambda_{\text{max}}}(x) = a_1 s x^{s-1} + o(x^{s-1}), \]
where
\[ a_1 = \frac{\Gamma(t-s)}{\Gamma(t)\Gamma(s+1)} \quad \text{for } s \neq t, \]
\[ \sum_{l=0}^{s-1} \frac{(-1)^s_l \ln x - \psi(1) - \psi(s-l+1)}{\Gamma(s)\Gamma(l+1)\Gamma(s-l+1)} \quad \text{for } s = t, \]
where \( \psi(n) = -\gamma + \sum_{l=1}^{n} \frac{1}{l} \) is the digamma function with \( \gamma \approx 0.57721566 \) being the Euler’s constant.

**Proof:** Omitted.

Intriguingly, the results indicate that for keyhole channels, both the C.D.F. and P.D.F. of \( \lambda_{\text{max}} \) decay to zero more slowly for symmetric systems (i.e., \( s = t \)) than for nonsymmetric systems, due to the \( \ln x \) term in the leading factor \( a_1 \).

IV. ANALYSIS OF RAYLEIGH-PRODUCT MIMO CHANNELS

A. Outage Probability

Outage probability is an important quality metric, which is defined as the probability that the received SNR drops below a certain target threshold, e.g., \( \gamma_{\text{th}} \). For Rayleigh-product MIMO BF channels, it can be obtained directly from **Theorem 1** as
\[ P_{\text{out}}(\gamma_{\text{th}}) = \Pr(\gamma_{\text{max}} \leq \gamma_{\text{th}}) = F_{\lambda_{\text{max}}}(\gamma_{\text{th}}/\gamma) \]
\[ = \prod_{l=1}^{m} \frac{\Gamma(m-i+1)\Gamma(n-i+1)}{\Gamma(l+1)\Gamma(t-s)}. \]
Note that this expression is general for any \( N_s, N_r \), and \( N_r \). Using (15) and **Corollary 1**, we can work out the outage probability expression for a keyhole MIMO channel to give
\[ P_{\text{out}}(N_s = 1)(\gamma_{\text{th}}) =
\begin{align*}
1 & - \sum_{l=0}^{s-1} \frac{2x^{s+l}}{\Gamma(l+1)\Gamma(t-s)} K_{t-l}(2\sqrt{x}).
\end{align*} \]
Note that this result has been derived indirectly, in the context of antenna selection, and via different methods in [13, **Theorem 4**]. In practice, the outage probabilities of interest would be small (e.g., \( 0.01 \) and \( 0.001 \)), which, in turn, correspond to small values of \( \gamma_{\text{th}} \). In this case, further intuition can be gained by using **Theorem 3** to approximate the outage probability in (17) as (18) (at the top of the next page). Considering the case \( s = t \), we note that for sufficiently small SNR thresholds \( \gamma_{\text{th}} \) (or sufficiently large \( \gamma \)), we can further simplify it by noting
\[ \{ \Delta(x) \}_{i,j} = \begin{cases} \Gamma(\tau_{i,j}) - \sum_{l=0}^{i-m-s-1} \frac{2(xN_s)^{\frac{\tau_{i,j}+l}{s}}}{\Gamma(l+1)} K_{\tau_{i,j}-l} \left( 2\sqrt{xN_s} \right) & \text{for } i \leq m - s, \\ \Gamma(\tau_{i,j}) & \text{for } i > m - s, \end{cases} \]  

(7)

\[ \mathcal{P}_{\text{out}}^{(N_r=1)}(\gamma_{\text{th}}) \approx \begin{cases} \left( \frac{\gamma_{\text{th}}}{\gamma} \right)^s \frac{\Gamma(t-s)}{\Gamma(t)\Gamma(s+1)} & \text{for } s \neq t, \\ \left( \frac{\gamma_{\text{th}}}{\gamma} \right)^{s-1} \frac{(-1)^{s-1} \ln \frac{\gamma_{\text{th}}}{\gamma} \left[ \ln \left( \frac{\gamma_{\text{th}}}{\gamma} \right) - \psi(1) - \psi(s-l+1) \right]}{\Gamma(s)\Gamma(l+1)\Gamma(s-l+1)} & \text{for } s = t, \end{cases} \]  

(18)

that the \( \ln(\cdot) \) term dominates the constant terms, and write

\[ \mathcal{P}_{\text{out}}^{(N_r=1)}(\gamma_{\text{th}}) \approx \left( \frac{\gamma_{\text{th}}}{\gamma} \right)^s \frac{\ln \left( \frac{\gamma_{\text{th}}}{\gamma} \right)}{\Gamma(s)\Gamma(s+1)} \]  

(19)

(18) and (19) show explicitly that for keyhole MIMO channels in the low-outage regime, the outage probability of MIMO BF scales inversely with \( s = \min(N_r, N_t) \). This is in direct contrast to rich scattering channels, whose outage probability scales inversely with the product \( N_r N_t \), in particular for rate-1 OSTBC in keyhole MIMO channels.

B. SER

For typical modulation formats, the average SER can be expressed as \( \text{SER}(\gamma) = E_{\text{max}} \left[ aQ \left( \sqrt{2bG_{\text{max}}} \right) \right] \) where \( a \) and \( b \) are modulation-specific constants and \( Q(\cdot) \) is the Gaussian Q-function. Using Theorem 2, we have

\[ \text{SER}(\gamma) = \int_0^\infty aQ \left( \sqrt{2bG_{\text{max}}} \right) f_{\text{max}}(x)dx \]

(20)

\[ = \frac{a}{\Gamma(t)\Gamma(s)} \int_0^\infty Q \left( \sqrt{2bG_{\text{max}}} \right) \det \Delta_l(x)dx = \frac{a}{\Gamma(t)\Gamma(s)} \int_0^\infty Q \left( \sqrt{2bG_{\text{max}}} \right) \det \Delta_l(x)dx \]

(21)

Although for the most channels, it does not appear that (21) can be evaluated in closed form, numerical integration can be performed to calculate \( \text{SER}(\cdot) \) much more efficiently than via Monte-Carlo simulations. In addition, a closed-form solution is possible for keyhole channels, as we now show.

Using Corollary 2 in (20), and the relation between \( Q(x) \) and the complementary error function \( \text{erfc}(x) \) [15], we write the SER of a keyhole-channel as

\[ \text{SER}^{(N_r=1)}(\gamma_{\text{th}}) = \frac{a}{\Gamma(t)\Gamma(s)} \int_0^\infty \frac{Q \left( \sqrt{2bG_{\text{max}}} \right) \det \Delta_l(x)dx}{\Gamma(m-i+1)\Gamma(m-i+1)} \]  

(22)

Noting that the complementary error function can be written in terms of the Meijer G-function as [16]

\[ \text{erfc} \left( \sqrt{x} \right) = \frac{1}{\sqrt{\pi}} G_{1,2}^{2,0} \left( x \left| \begin{array}{c} 1 \\ 0, 1/2 \end{array} \right. \right) \]  

(23)

and integrating using [15, Eq. 7.821.3] yields the closed-form SER expression

\[ \text{SER}^{(N_r=1)}(\gamma_{\text{th}}) = \frac{aG_{2,3}^{2,0} \left( \gamma_{\text{th}} \left| \begin{array}{c} 1-t, 1-s, 1 \\ 0, 1/2 \end{array} \right. \right)}{2\sqrt{\pi} \Gamma(t)\Gamma(s)} \]  

(24)

To obtain further insights and to derive the diversity order and array gain of a keyhole-channel, we analyze the SER performance in the high SNR regime. To this end, armed with Theorem 3, we can invoke a general parameterized SER result from [18, Proposition 1] and perform some basic algebraic manipulations along the lines of [17, Eq. 29], to obtain

\[ \text{SER}_{\text{high-SNR}}^{(N_r=1)}(\gamma_{\text{th}}) = \begin{cases} \frac{a\Gamma(t-s)}{\Gamma(t)\Gamma(s+1)} \left( \frac{\gamma_{\text{th}}}{\gamma} \right)^{-s} + o \left( \gamma_{\text{th}}^{-s} \right) & \text{for } s \neq t, \\ \frac{2\sqrt{\pi} \Gamma(t)\Gamma(s+1)}{a\Gamma(s)\Gamma(s+1)} \left( \frac{\gamma_{\text{th}}}{\gamma} \right)^{-s} + o \left( \gamma_{\text{th}}^{-s} \right) & \text{for } s = t. \end{cases} \]  

(25)

The diversity order is the key high-SNR parameter of interest and it determines the asymptotic slope of the SER curve when plotted against SNR on a log-log scale. It is defined as [19]

\[ \lim_{\gamma_{\text{th}} \to \infty} \frac{- \log \text{SER}^{(N_r=1)}(\gamma_{\text{th}})}{\log(\gamma_{\text{th}})} = \frac{\psi(s)}{\psi(s-1)} \]  

(26)

which is seen from (25) to be equal to \( s = \min(N_r, N_t) \). The array gain, on the other hand, is determined via the leading simplifications of \( \gamma_{\text{th}}^{-s} \) which therefore provides a measure of the horizontal shift of the SER-SNR curve on a log-log scale, and is dependent on both \( s = \min(N_r, N_t) \) and \( t = \max(N_r, N_t) \).

It is interesting to contrast the result (25) with the corresponding SER expression for MIMO systems employing rate-1 OSTBC (orthogonal space-time block-coding) systems in keyhole channels, given in [20] and [17] as (27) (at the top of the next page). Comparing (25) and (27), we see that both MIMO BF and OSTBC achieve the same diversity order \( s \) in keyhole channels. The main difference, however, is that MIMO BF exploits channel knowledge at the transmitter, whereas OSTBC does not. Therefore, we expect an increase in array gain in the MIMO BF channel. To quantify this, we consider the case \( s \neq t \), and define \( (\gamma_{\text{BF}})_{\text{dB}} \) and \( (\gamma_{\text{OSTBC}})_{\text{dB}} \) as the respective SNRs in dB such that both schemes attain the same SER in the high SNR regime. As such, this increase in array gain of BF over OSTBC in keyhole channels can be found as

\[ \Delta_{\text{dB}} = (\gamma_{\text{OSTBC}})_{\text{dB}} - (\gamma_{\text{BF}})_{\text{dB}} = \log(N_tR_1)_{\text{dB}} \]  

(28)

Therefore, recalling that \( R_1 = 1 \) for \( N_t \leq 2 \), and \( R_1 < 1 \) otherwise, we see that for a given SER the power saving of MIMO BF over OSTBC in keyhole MIMO channels can be very significant, even for small numbers of antennas.
\[ \text{SER}_{\text{high-SNR;OSTBC}}^{(N_s=1)}(\gamma) = \begin{cases} \frac{\alpha \Gamma(t-s) \Gamma\left(s + \frac{1}{2}\right) (N_i R_t)^s}{2\sqrt{\pi t} \Gamma\left(s + 1\right)} \left(b\gamma\right)^{-s} + o\left(\gamma^{-s}\right) & \text{for } s \neq t, \\ \frac{\alpha \Gamma\left(s + \frac{1}{2}\right) \ln\left(\frac{b\gamma}{N_i R_t}\right) (N_i R_t)^s}{2\sqrt{\pi} \Gamma\left(s + 1\right)} \left(b\gamma\right)^{-s} + o\left(\gamma^{-s}\right) & \text{for } s = t, \end{cases} \] (27)

![Fig. 1. C.D.F. of \( \sqrt{\lambda_{\text{max}}} \) of Rayleigh-product channels.](image1)

![Fig. 2. Outage probability of keyhole channels.](image2)

![Fig. 3. SER of Rayleigh-product channels.](image3)

**V. Numerical Results**

Fig. 1 shows the marginal C.D.F.s of the maximum singular values for Rayleigh-product channels with different antenna configurations. The analytical results are generated using (6), and the simulated curves are generated based on \(10^5\) independent channel realizations. Results indicate an absolute agreement between the analytical and simulated results.

The outage probability of MIMO keyhole channels is illustrated in Fig. 2, where we compare the exact outage probability curves based on (17), the low-outage approximations based on (18) and (19), and Monte-Carlo simulations for different antenna configurations. As can be seen, the exact analytical results match exactly with the Monte-Carlo results, and the analytical approximations are very accurate in the low outage regime. It is also apparent that the slopes of the probability curves are determined by the minimum number of transmit and receive antennas.

Results in Fig. 3 show the SER of MIMO Rayleigh-product channels based on (24), the high-SNR approximations (25) [without the \(O(\cdot)\) terms], and Monte-Carlo simulation results. Results are shown for coherent 8-PSK modulation (i.e., \(a = 2\) and \(b = 0.146\)). Exact agreement between the analytical SER results and Monte-Carlo simulated curves are observed. Moreover, we see that the diversity order and array gains are predicted by the high SNR analytical results (25). For further comparison, Fig. 3 also shows the exact SER curves based on (20), high-SNR approximated SER curves based on (27) [without the \(O(\cdot)\) term], and Monte-Carlo simulations for OSTBC G2 [21], for keyhole channels (2, 4, 1). Results reveal that the difference in array gain (SNR shift in dB) between OSTBC and MIMO BF is about 3.01 (dB), which is in line with the prediction of (28).

**VI. Conclusion**

This paper has investigated the analytical performance of MIMO BF systems in Rayleigh-product fading channels. Our results are based on a collection of new closed-form expressions which we derived for the C.D.F. and P.D.F. of the maximum eigenvalue of a product of two independent zero-mean complex Gaussian matrices. We have studied the outage probability and the SER. For the important special case of keyhole channels, we have also derived a number of insightful closed-form expressions for various key system parameters such as diversity order and array gain. Finally, we presented a comparison of MIMO BF with respect to OSTBC, demonstrating the significant benefits due to transmitter chan-
nel knowledge in Rayleigh-product MIMO channels.

**APPENDIX I**

**PROOF OF THEOREM 1**

Here we present the proof for the case $s > N_s$. The proof for $s \leq N_s$ follows similarly; for full details see [22]. In both cases we achieve the same result if we exchange $N_r$ and $N_t$ due to the reciprocity of the MIMO channel. Hence, here we shall assume that $N_r \geq N_t$, without loss of generality.

It is easily observed that $HH^H$ and $H^HH$ have $N_s$ non-zero eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{N_s} < \infty$ and $0 < \phi_1 < \phi_2 < \cdots < \phi_{N_s} < \infty$, respectively. Noting that [23]

$$\lambda_{\max} \left( \frac{1}{N_s} H_s^H H_s^H H_s^H \right) = \lambda_{\max} \left( \frac{1}{N_s} H_s^H H_s^H H_s^H H_s^H \right) \tag{29}$$

and utilizing the results in [24], we can derive the C.D.F. of $\lambda_{\max}$ of $HH^H$, conditioned on $H^H$, as

$$F_{\lambda_{\max,1}}(x|H^H) = \frac{\det \Psi_1(x)}{\det V_1 \prod_{N_t=1}^{N_t} \Gamma(N_t - i + 1)}, \tag{30}$$

where $V_1$ is an $N_s \times N_s$ matrix, with determinant

$$\det V_1 = \prod_{N_t=1}^{N_t} \prod_{1 \leq l \leq k \leq N_s} (\phi_k - \phi_l). \tag{31}$$

Also, $\Psi_1(x)$ is an $N_s \times N_s$ matrix with entries given by

$$\{\Psi_1(x_{N_s}), i, j\} = \phi_{i,j}^{N_i-N_i+1} \gamma(N_t - i + 1, x N_s/\phi_j), \tag{32}$$

where $\gamma(\cdot, \cdot)$ is the lower incomplete gamma function. To obtain the unconditional C.D.F. of $\lambda_{\max}$, we must further average (30) over the P.D.F. of $H^H$, or equivalently, over the joint P.D.F. of $\phi_1, \cdots, \phi_{N_s}$. To this end, let $W = \text{diag}(\phi_1, \cdots, \phi_{N_s})$, then the joint P.D.F. $f_1(W)$ is given by [25–27]

$$f_1(W) = \frac{e^{-\sum_{i=1}^{N_s} \phi_i N_i - N_s} \prod_{i<j} (\phi_j - \phi_i)^2}{\prod_{N_t=1}^{N_t} \Gamma(N_t - i + 1) \Gamma(N_s - i + 1)}. \tag{33}$$

The unconditional C.D.F. of $\lambda_{\max}$ can be obtained as

$$F_{\lambda_{\max,1}}(x) = \int_W F_{\lambda_{\max,1}}(x|H^H) f_1(W) dW. \tag{34}$$

Now, substituting (33) into (34), using a Vandermonde determinant identity from [8, Eq. (56)], and applying [28, Corr. 2], it can be shown that (34) can be written as

$$F_{\lambda_{\max,1}}(x) = \frac{\det \left( \int_0^\infty e^{-t N_r - i + 1, x/\phi_i} dt \right) \prod_{N_t=1}^{N_t} \Gamma(N_s - i + 1) \Gamma(N_s - i + 1) \Gamma(N_s - i + 1)}{\prod_{N_t=1}^{N_t} \Gamma(N_s - i + 1) \Gamma(N_s - i + 1) \Gamma(N_s - i + 1)}. \tag{35}$$

Finally, integrating this yields the desired result.

**APPENDIX II**

**PROOF OF THEOREM 2**

Noting $f_{\lambda_{\max}}(x) = dF_{\lambda_{\max}}(x)/dx$, and making use of (17) and a classical formula for the derivative of a determinant, the P.D.F. of $\lambda_{\max}$ of $HH^H$ can be written as

$$f_{\lambda_{\max}}(\lambda_{\max}) = \sum_{i=m-s+1}^{m} \frac{\det \Delta_i(\lambda_{\max})}{\prod_{N_t=1}^{N_t} (m - i + 1) \Gamma(n - i + 1)}, \tag{36}$$

where $\Delta_i(\lambda_{\max})$ is an $m \times m$ matrix with the $(i,j)$th entry expressed by (37) (at the top of the next page). Considering the differential in (37) carefully, we find

$$\frac{d}{dx} \left[ \sum_{l=0}^{i-m-s-1} \frac{2(x_{N_s})^{\tau_{l+i}}}{\Gamma(l+1)} K_{\tau_{l+i-1}} \left( 2\sqrt{x_{N_s}} \right) \right]$$

$$= \frac{d}{dx} \left[ \sum_{l=1}^{i-m-s-1} \frac{2(x_{N_s})^{\tau_{l+i}}}{\Gamma(l+1)} K_{\tau_{l+i-1}} \left( 2\sqrt{x_{N_s}} \right) \right]$$

$$+ \sum_{l=1}^{i-m-s-1} \frac{1}{\Gamma(l+1)} \frac{d}{dx} \left[ 2(x_{N_s})^{\tau_{l+i}} K_{\tau_{l+i-1}} \left( 2\sqrt{x_{N_s}} \right) \right]. \tag{38}$$

Utilizing the following recurrence formula for modified Bessel functions of the second kind [15]

$$K_{\nu}'(x) = -K_{\nu-1}(x) - \frac{\nu}{x} K_{\nu}(x), \tag{39}$$

we get

$$\frac{d}{dx} \left[ 2(x_{N_s})^{\tau_{l+i}} K_{\tau_{l+i-1}} \left( 2\sqrt{x_{N_s}} \right) \right] =$$

$$-2N_s(x_{N_s})^{\tau_{l+i-1}} K_{\tau_{l+i-1}} \left( 2\sqrt{x_{N_s}} \right) \tag{40}$$

and (41) (see top of the next page). Substituting (40) and (41) into (38) yields the desired result.

**REFERENCES**


\[ \{ \Delta_i(\lambda_{\text{max}}) \}_{i,j} = \begin{cases} \frac{d}{dx} \left[ \sum_{l=0}^{i-m+s-1} \frac{2(xN_s)^{\tau_{i,j+l}}}{\Gamma(l+1)} K_{\tau_{i,j+l} - l} \left( \frac{2}{xN_s} \right)^{\frac{1}{2}} \right] \bigg|_{x = \lambda_{\text{max}}} & \text{for } i = l, \\ - \frac{d}{dx} \left[ \sum_{l=0}^{i-m+s-1} \frac{2(xN_s)^{\tau_{i,j+l}}}{\Gamma(l+1)} K_{\tau_{i,j+l} - l} \left( \frac{2}{xN_s} \right)^{\frac{1}{2}} \right] \bigg|_{x = \lambda_{\text{max}}} & \text{for } i \neq l, \end{cases} \]

\[ \sum_{l=1}^{i-m+s-1} \frac{1}{\Gamma(l+1)} \frac{d}{dx} \left[ 2(xN_s)^{\tau_{i,j+l}} K_{\tau_{i,j+l}} \left( \frac{2}{xN_s} \right)^{\frac{1}{2}} \right] = 2N_s \left[ (xN_s)^{\frac{\tau_{i,j-1}}{2}} K_{\tau_{i,j} - l} \left( \frac{2}{xN_s} \right)^{\frac{1}{2}} - \frac{(xN_s)^{\frac{\tau_{i,j}}{2}} K_{\tau_{i,j} - l}}{\Gamma(i-m+s)} K_{n-s+j-1} \left( \frac{2}{xN_s} \right)^{\frac{1}{2}} \right] \]